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THE CONDITIONAL LAW OF A M/M/1 QUEUE WITH RESPECT TO ITS DEPARTURE PROCESS AND IDLE TIME

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Abstract: A filtering problem is considered in the case when the state process is a M/M/1 queue Q_t and the observation process is the pair of its idle time and departure process. An explicit construction of the filter is given.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{F}, P)$ and define on it a M/M/1 queue Q_t. Consider its standard representation, that is:

$$Q_t = Q_0 + A_t - D_t, \tag{1}$$

where

- Q_0 is a $\mathbb{N}^+$-valued random variable;
- A_t is a Poisson process with with P-intensity λ;
- $D_t = \int_0^t \mathbb{I}\,(Q_{s-} > 0)\, dN_s$, N Poisson process with P-intensity μ, where $\mathbb{I}(x > 0) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0; \end{cases}$
- Q_0, A_t and N_t mutually independent.

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Suppose to observe at any fixed time t, the pair (D_t, C_t), where C_t is the *idle time* of Q_t, that is the amount of time spent by Q_t at level 0, i.e. the process

$$C_t = \int_0^t \mathbb{I}\,(Q_s = 0)\, ds. \tag{2}$$

We are interested in the filtering problem arising when, at any fixed time t, we want to estimate the unknown state Q_t by observing the process (D_s, C_s) up to time t.

This work follows the same ideas and techniques as in [3], where authors derive the filter on a queue with respect to its idle time, and, up to a scaling, also its weak limit. Here we add to the observation of the idle time, that of the departure process. This kind of problem falls in the setting of the so called *singular filtering*, since the observation process can be expressed as deterministic functional of the state process via the equality $D_t = \sum_{s\le t} \mathbb{I}\,(Q_{s-} > Q_s)$.

In order to solve the singular filtering problems need alternative procedures to those usually used for the computation of the filter, as the innovation theory approach ore the method of the reference probability measure. In this paper we derive an explicit expression for the filter based on some regenerative properties of some processes involved and on the structure of the filtration generated by the observation process. Denote by π_t the filter of the state with respect to the history generated by the observation process up to time t. As well known, the filter is uniquely defined by the rule

$$\pi_t(f) = E\big[f(Q_t)/\mathcal{F}_t^{(D,C)}\big] \tag{3}$$

for f in a sufficiently large class of functions.

For our purposes it will be useful to consider the *elapsed time from the last visit to* 0 for the process Q_t, i.e. the process

$$\xi_t = t - \tau_t(Q), \tag{4}$$

where the functional $\tau_t : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is defined by

$$\tau_t(x) = \sup\{s < t : x_s = 0\}. \tag{5}$$

2. The Representation Result

In order to prove the main result of this paper, we need to recall that observing the continuous process C_t is the same as observing the pair of point processes (I_t, B_t), where I_t is the process that counts the times when the queue starts an idle period, and B_t is the process that counts the times when the queue starts a busy period, that is:

$$I_t = \int_0^t \mathbb{I}(Q_{s-} = 1) dN_s; \tag{6}$$

$$B_t = \int_0^t \mathbb{I}(Q_{s-} = 0) dA_s. \tag{7}$$

So we can express the observed history as follows:

$$\mathcal{G}_t = \mathcal{F}_t^D \vee \mathcal{F}_{t+}^C = \mathcal{F}_t^D \vee \mathcal{F}_t^I \vee \mathcal{F}_t^B.$$

The equality between the filtration generated by the idle time C_t and the filtration generated by the pair of processes (I_t, B_t) together with some regenerative properties of the point processes I_t, B_t, have been widely studied in [3], Section 6. For the sake of completeness, we recall in the sequel the properties that are used in this paper.

Let $\{\sigma_k^B,\ k \geq 1\}$ and $\{\sigma_k^I,\ k \geq 1\}$ be the jump times of the process I_t and the process B_t, defined by Eq. (6) and by Eq. (7), respectively. For notational convenience, also consider for $k = 0$, $\sigma_0^B = \sigma_0^I = 0$.

Under the assumption $Q_0 = 0$, it easy to verify that

$$\sigma_k^B < \sigma_k^I < \sigma_{k+1}^B < \sigma_{k+1}^I, \qquad \text{for each } k \geq 1,$$

and that $Q_t = 0$ when $\sigma_k^I \leq t < \sigma_{k+1}^B$, for $k \geq 0$, while $Q_t > 0$ otherwise.

We start by observing some regenerative properties of the above jump times, which are fundamental in the sequel.

Lemma 1. (Lemma 6.1 in [3]) *For each $k \in \mathbb{N}$ the processes $Q_{k,t}^I = Q_{t+\sigma_k^I} - Q_{\sigma_k^I}$ and $Q_{k,t}^B = Q_{t+\sigma_k^B} - Q_{\sigma_k^B}$ are independent of $\mathcal{F}_{\sigma_k^I}^Q$ and $\mathcal{F}_{\sigma_k^B}^Q$ respectively. Moreover, the process $Q_{k,t}^I$ has the same law as the process Q_t.*

Proof. Clearly $Q_{k,0}^I = Q_0 = 0$, moreover it is easy to see that $Q_{k,t}^I$ solves the same martingale problem as Q_t, hence these processes share the same law.

The independence property follows by the strong Markov property of Q_t, since $Q_{\sigma_k^I} = 0$:

$$E\left[\exp\left(iu\left(Q_{t+\sigma_k^I} - Q_{\sigma_k^I}\right)\right)/\mathcal{F}^Q_{\sigma_k^I}\right] = E\left[\exp\left(iu\left(Q_{t+\sigma_k^I} - Q_{\sigma_k^I}\right)\right)/Q_{\sigma_k^I}\right]$$
$$= E\left[\exp\left(iu\left(Q_{t+\sigma_k^I} - Q_{\sigma_k^I}\right)\right)\right] = E\left[\exp\left(iu\left(Q_t\right)\right)\right].$$

Similar arguments apply to show that the process $Q^B_{k,t}$ is independent of $\mathcal{F}^Q_{\sigma_k^B}$. $\square$

Remark 2. The process $Q^B_{k,t}$ can be written as $Q^B_{k,t} = A^B_{k,t} - D^B_{k,t}$, where

$$\begin{cases} A^B_{k,t} &= A_{t+\sigma_k^B} - A_{\sigma_k^B}; \\ D^B_{k,t} &= D_{t+\sigma_k^B} - D_{\sigma_k^B}. \end{cases} \tag{8}$$

Moreover, the process $D^B_{k,t}$ can be expressed as a functional of the process $Q^B_{k,t}$, that is $D^B_{k,t} = \int_0^t \mathbb{I}(Q^B_{k,s-} > Q^B_{k,s}) dQ^B_{k,s}$. Then it turns out to be independent of $\mathcal{F}^Q_{\sigma_k^B}$.

Remark 3. The process I_t is a renewal process, and B_t is a delayed renewal process, i.e. the random variables $\sigma^B_{k+1} - \sigma^B_k$ are mutually independent for $k \geq 0$ and identically distributed for $k \geq 1$. (see, for example, [2] VI.7.3 page 187).

Also $\{\sigma^I_k - \sigma^B_k\}_{k\geq 1}$ is a sequence of mutually independent random variables.

In the sequel, for the sake of clarity we state two results useful for our purpose. The first one is about a characterization of a filtration generated by a point processes (see [1]). As far as the second result is concerned, it is a very easy result on the conditional expectation. Since the author cannot find a source to refer to, the statement and the proof.

Lemma 4. (Theorem 5, Chap. III in [1]) *Let $\mathcal{F}_t$ the natural filtration generated by a (possibly multidimensional) point process K and let S be a finite $\mathcal{F}_t$-stopping time. Denote by $\{T_n\}_{n\in N}$ the jump times of the process K. Then for all $n \in \mathbb{N}$*

$$\mathcal{F}_S \cap \{T_n \leq S < T_{n+1}\} = \mathcal{F}_{T_n} \cap \{T_n \leq S < T_{n+1}\}. \tag{9}$$

Lemma 5. *Let X and Y be two random variables and $\mathcal{H}$, $\mathcal{K}$ be two σ-algebras. Suppose that X and Y are independent, $\sigma(X) \vee \mathcal{K}$ independent of $\mathcal{H}$ and Y $\mathcal{H}$-measurable.*

If $f : Im(X) \times Im(Y) \to \mathbb{R}$ is a measurable function, then

$$E[f(X,Y)|\mathcal{H} \vee \mathcal{K}] = E[f(X,y)|\mathcal{K}]_{y=Y}. \tag{10}$$

Proof. We show the statement when $f(X,Y) = XY$. The general case follows by using the fact that the linear space generated by the product functions is dense on the space of measurable functions.

Let $A \in \mathcal{H}$ and $B \in \mathcal{K}$. Then

$$\int_\Omega E[XY|\mathcal{H} \vee \mathcal{K}]\mathbb{I}_A\mathbb{I}_B dP = \int_\Omega XY\mathbb{I}_A\mathbb{I}_B dP = \int_\Omega X\mathbb{I}_B dP \int_\Omega Y\mathbb{I}_A dP,$$

where the last equality easily follows by the hypotheses. Moreover

$$\int_\Omega X\mathbb{I}_B dP = \int_\Omega E[X|\mathcal{K}]\mathbb{I}_B dP$$

and $Y = E[Y|\mathcal{H}]$ so that

$$\int_\Omega X\mathbb{I}_B dP \int_\Omega Y\mathbb{I}_A dP = \int_\Omega X\mathbb{I}_B dP = \int_\Omega YE[X|\mathcal{K}]\mathbb{I}_A\mathbb{I}_B dP,$$

that is

$$E[XY|\mathcal{H} \vee \mathcal{K}] = YE[X|\mathcal{K}] = E[yX|\mathcal{K}]_{y=Y}.$$

□

For every $t > 0$ denote by $\mathcal{M}\left(D_{\mathbb{R}}[0,t]\right)$ the space of the probability measures on the space $D_{\mathbb{R}}[0,t]$ endowed with the Skorohod topology and by $\mathcal{M}\left(\mathbb{R}\right)$ the space of the probability measures on $\mathbb{R}$.

Denote also $\mathcal{D}^B_{k,r} = \sigma(D^B_{k,u}, \quad 0 \le u < r)$, where $D^B_{k,u}$ is the process defined by Eq. (8).

We can finally state the representation theorem for the filter of Q_t when the observed history is

$$\mathcal{G}_t = \mathcal{F}^D_t \vee \mathcal{F}^C_{t+} = \mathcal{F}^D_t \vee \mathcal{F}^I_t \vee \mathcal{F}^B_t.$$

Theorem 6. *For every $t > 0$ there exists $\pi_t : \Omega \to \mathcal{M}(\mathbb{R})$ such that, if $f \in C_b^0(\mathbb{R})$, then*

$$\pi_t(f) = E\big[f(Q_t)/\mathcal{G}_t\big] = \mathbb{I}(Q_t = 0)\, f(0) + \sum_{j=1}^{\infty} \frac{E\big[f(A^B_{j,s} - y_s + 1)\mathbb{I}_{(0,\infty)}(A^B_{j,u} - y_u + 1, u \in [0,s))\big]}{E\big[\mathbb{I}_{(0,\infty)}(A^B_{j,u} - y_u + 1, u \in [0,s))\big]}\Big|_{s=t-\sigma^B_j, y_s = D^B_{j,s}} \times I_{[\sigma^B_j, \sigma^I_j)}(t). \quad (11)$$

Proof. We begin by noting that $\mathbb{I}(Q_t > 0)$ is $\mathcal{G}_t$-adapted. Then

$$E\big[f(Q_t)/\mathcal{G}_t\big] = \mathbb{I}(Q_t = 0)\, f(0) + \mathbb{I}(Q_t > 0)\, E\big[f(Q_t)/\mathcal{G}_t\big].$$

It is useful to note that, when $Q_t > 0$, there exists some $j \in \mathbb{N}$ such that $\{\sigma^B_j \le t < \sigma^I_j\}$. Moreover, for each $t > 0$ the set $\{\sigma^B_j \le t < \sigma^I_j\}$ is $\mathcal{G}_t$-measurable. Then we can write previous equality as

$$E\big[f(Q_t)/\mathcal{G}_t\big] = \mathbb{I}(Q_t = 0)\, f(0) + \sum_{j=1}^{\infty} E\big[f(Q_t)/\mathcal{G}_t\big]\mathbb{I}\{\sigma^B_j \le t < \sigma^I_j\}.$$

Now, on the intervals $[\sigma^B_j, \sigma^I_j)$ the departure process $D^B_{j,t}$ of $Q^B_{j,t} = Q_{t+\sigma^B_j} - Q_{\sigma^B_j}$ behaves just like a Poisson process of intensity μ. In the sequel we formalize and prove this intuition, by using argument similar as in Lemma 4, arising when S is deterministic.

More precisely, for each $j \in \mathbb{N}$, the trace of σ-algebra $\mathcal{G}_t$ on the set $\{\sigma^B_j \le t < \sigma^I_j\}$ can be written as

$$\mathcal{G}_t \cap \{\sigma^B_j \le t < \sigma^I_j\} = \{\mathcal{G}_{\sigma^B_j} \vee \mathcal{D}^B_{j,t-\sigma^B_j}\} \cap \{\sigma^B_j \le t < \sigma^I_j\}. \quad (12)$$

In particular, Eq. (12), implies that for each $\mathcal{G}_t$-measurable random variable X there exists a $\{\mathcal{G}_{\sigma^B_j} \vee \mathcal{D}^B_{j,t-\sigma^B_j}\}$-measurable random variable Y such that following equality holds:

$$X\ I_{[\sigma^B_j, \sigma^I_j)}(t) = Y\ I_{[\sigma^B_j, \sigma^I_j)}(t). \quad (13)$$

Our goal is to find the $\{\mathcal{G}_{\sigma^B_j} \vee \mathcal{D}^B_{j,t-\sigma^B_j}\}$-measurable random variable Y satisfying Eq. (13) when $X = E\big[f(Q_t)/\mathcal{G}_t\big]$, that is

$$E\big[f(Q_t)/\mathcal{G}_t\big]\ I_{[\sigma^B_j, \sigma^I_j)}(t) = Y\ I_{[\sigma^B_j, \sigma^I_j)}(t). \quad (14)$$

To this end it is useful to note that $I_{[\sigma_j^B,\sigma_j^I)}(t) = I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B)$. Then by conditioning on $\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\}$ both sides of equality 13 it follows

$$\begin{aligned} &E\left[E\left[f(Q_t)/\mathcal{G}_t\right] I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B)/\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\}\right] \\ =&YE\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B)/\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\}\right]. \end{aligned} \tag{15}$$

Consider the left hand side of previous equality. We can rewrite it as

$$\begin{aligned} &E\left[E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q_{(t-\sigma_j^B)+\sigma_j^B} - Q_{\sigma_j^B} + Q_{\sigma_j^B})/\mathcal{G}_t\right]/\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\}\right] \\ =&E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q^B_{j,(t-\sigma_j^B)}+1)/\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\right]. \end{aligned}$$

Now, observe that Lemma 1 and Remark 2 are applied to prove that the processes $Q^B_{j,u}$ and $D^B_{j,u}$ are independent of $\mathcal{G}_{\sigma_j^B}$. Moreover,

$$\sigma_j^I - \sigma_j^B = \inf\left\{u \geq 0 : \; Q^B_{j,u} + 1 = 0\right\}, \tag{16}$$

so that also the stopping time $\sigma_j^I - \sigma_j^B$ is independent of $\mathcal{G}_{\sigma_j^B}$ while $t-\sigma_j^B$ is measurable w.r.t. $\mathcal{G}_{\sigma_j^B}$. Then by conditioning on the bigger filtration $\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,\sigma_j^I-\sigma_j^B}$, we rewrite the right hand side of previous equality as

$$\begin{aligned} &E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q^B_{j,(t-\sigma_j^B)}+1)/\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\right] \\ =&E\left[E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q^B_{j,(t-\sigma_j^B)}+1)/\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,\sigma_j^I-\sigma_j^B}\right]/\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,t-\sigma_j^B}\right]. \end{aligned}$$

Moreover setting $\mathcal{H} := \mathcal{G}_{\sigma_j^B}$, $\mathcal{K} := \mathcal{D}^B_{j,\sigma_j^I-\sigma_j^B}$, $X := (\sigma_j^I - \sigma_j^B, Q_j^B)$, $Y := t-\sigma_j^B$ and $f(X,Y) = I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q^B_{j,(t-\sigma_j^B)}+1)$, the hypotheses of Lemma 5 are fulfilled, and consequently

$$\begin{aligned} &E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(t-\sigma_j^B) f(Q^B_{j,(t-\sigma_j^B)}+1)/\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}^B_{j,\sigma_j^I-\sigma_j^B}\right] \\ =&E\left[I_{[0,\sigma_j^I-\sigma_j^B)}(s) f(Q^B_{j,s}+1)/\mathcal{D}^B_{j,\sigma_j^I-\sigma_j^B}\right]_{s=t-\sigma_j^B}. \end{aligned}$$

Observe that the process $Q^B_{j,s}+1 = A^B_{j,s} - D^B_{j,s} + 1$ for $s < \sigma_j^I - \sigma_j^B$ behaves like a continuous time random walk where the processes $A^B_{k,s}$and $D^B_{k,s}$ are two independent Poisson processes with intensities λ and μ, respectively. In fact, by Eq. (16), it follows that, if $0 < u < \sigma_j^I - \sigma_j^B$, then $\mathbb{I}(Q_{j,u}^B + 1 > 0) = 1$ and so,

$$D_{k,u}^B = \int_{\sigma_j^B}^{\sigma_j^B + u} \mathbb{I}(Q_s > 0) dN_s = \int_{\sigma_j^B}^{\sigma_j^B + u} \mathbb{I}(Q_{j,s-\sigma_j^B}^B + 1 > 0) dN_s = N_{\sigma_j^B + u} - N_{\sigma_j^B}$$

and $N_{\sigma_j^B + u} - N_{\sigma_j^B}$ is a Poisson process with intensity μ independent of $\mathcal{G}_{\sigma_j^B}$. Finally let us observe that $\mathbb{I}_{[0,\sigma_j^I - \sigma_j^B)}(s) = \mathbb{I}_{(0,\infty)}(Q_{j,u}^B + 1, u \in [0,s))$.

As a consequence,

$$\begin{aligned} &E\big[I_{[0,\sigma_j^I - \sigma_j^B)}(s) f(Q_{j,s}^B + 1)/\mathcal{D}_{j,\sigma_j^I - \sigma_j^B}^B\big]_{s=t-\sigma_j^B} \\ &\quad = E\big[\mathbb{I}_{(0,\infty)}(A_{j,u}^B - y_u + 1, u \in [0,s)) f(A_{j,s}^B - y_s + 1)\big]_{s=t-\sigma_j^B, y_s = D_{j,s}^B}. \end{aligned}$$

Now, the latter conditional expectation is measurable with respect to the σ-algebra $\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}_{j,t-\sigma_j^B}^B\}$ so we can write

$$\begin{aligned} &E\left[E\left[f(Q_t)/\mathcal{G}_t\right] I_{[0,\sigma_j^I - \sigma_j^B)}(t - \sigma_j^B)/\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}_{j,t-\sigma_j^B}^B\}\right] \\ &\quad = E\big[\mathbb{I}_{(0,\infty)}(A_{j,u}^B - y_u + 1, u \in [0,s)) f(A_{j,s}^B - y_s + 1)\big]_{s=t-\sigma_j^B, y_s = D_{j,s}^B}. \end{aligned}$$

As far as the right hand side of Eq. (15) is concerned, by using similar tools we find

$$\begin{aligned} &E\left[I_{[0,\sigma_j^I - \sigma_j^B)}(t - \sigma_j^B)/\{\mathcal{G}_{\sigma_j^B} \vee \mathcal{D}_{j,t-\sigma_j^B}^B\}\right] \\ &\quad = E\big[\mathbb{I}_{(0,\infty)}(A_{j,u}^B - y_u + 1, u \in [0,s))\big]_{s=t-\sigma_j^B, y_s = D_{j,s}^B}, \end{aligned}$$

so that

$$Y = \left.\frac{E\big[\mathbb{I}_{(0,\infty)}(A_{j,u}^B - y_u + 1, u \in [0,s)) f(A_{j,s}^B - y_s + 1)\big]}{E\big[\mathbb{I}_{(0,\infty)}(A_{j,u}^B - y_u + 1, u \in [0,s))\big]}\right|_{s=t-\sigma_j^B, y_s = D_{j,s}^B},$$

and the statement is achieved. □

Observe that by Eq. (4) we can write the following equality

$$\xi_t = t - \tau_t(Q) = t - \sup\{s < t : Q_s = 0\} = \sum_{j=1}^{\infty} (t - \sigma_j^B) \mathbb{I}\{\sigma_j^B \le t < \sigma_j^I\}. \quad (17)$$

In the light of the previous equality, we can give an unifying notation for the processes $(Q^B_{j,s}, A^B_{j,s}, D^B_{j,s})$, $j \in \mathbb{N}$. To this end for each $t > 0$ we define the processes

$$\begin{cases} Q^t_s & := Q_{s+\tau_t(Q)} - Q_{\tau_t(Q)}, \\ A^t_s & := A_{s+\tau_t(Q)} - A_{\tau_t(Q)}, \\ D^t_s & := D_{s+\tau_t(Q)} - D_{\tau_t(Q)}, \end{cases} \tag{18}$$

where the superscript t stands to remember these processes starting from the random time $\tau_t(Q) = \sup\{s < t : Q_s = 0\}$.

Then, if $t \in [\sigma^B_j, \sigma^I_j)$ $Q^t_s = Q^B_{j,s}$, $A^t_s = A^B_{j,s}$ and $D^t_s = D^B_{j,s}$.

By using the notations given by Eq. (17) and Eq. (18), we can restate Theorem 6 as follows.

Theorem 7. *For every $t > 0$ there exists $\pi_t : \Omega \to \mathcal{M}(\mathbb{R})$ such that, if $f \in C^0_b(\mathbb{R})$, then*

$$\begin{aligned} \pi_t(f) =& E\big[f(Q_t)/\mathcal{G}_t\big] = \mathbb{I}(Q_t = 0) f(0) \\ &+ \mathbb{I}(Q_t > 0) \frac{E\big[f(A^t_s - y_s + 1)\mathbb{I}_{(0,\infty)}(A^t_u - y_u + 1, u \in [0,s))\big]}{E\big[\mathbb{I}_{(0,\infty)}(A^t_u - y_u + 1, u \in [0,s))\big]}\Big|_{s=\xi_t, y_s = D^t_s}. \end{aligned} \tag{19}$$

References

[1] P. Brémaud, *Point Process and Queues*, Springer, New York (1981).

[2] M. Abundo, *An Introduction to Probability Theory and Its Applications*, Wiley (1971).

[3] G. Nappo and B. Torti, Continuous time random walks and queues: explicit forms and approximations of the conditional law with respect to local times, *Stochastic Process. Appl.* **116** (2006), 585–610.