International Journal of Applied Mathematics

Volume 26 No. 1 2013, 9-17
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v26i1.2

ON THE GLOBAL CHARACTER OF THE SOLUTIONS OF $X_{n+1} = \frac{\alpha+\beta X_n+\gamma X_{n-k}}{A+X_{n-k}}$

E. Chatterjee

Department of Mathematics
Indian Institute of Science
Bangalore, Karnataka, 560012, INDIA

Abstract: We investigate the global character of solutions of the difference equation in the title with nonnegative parameters, nonnegative initial conditions, and where k is a positive integer. In particular, for the case $k = 2$ we will address the Open Problem 5.173.1 posed in E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, 2008.

AMS Subject Classification: 39A10, 39A11
Key Words: difference equation, global attractor, boundedness

1. Introduction and Preliminaries

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + x_{n-k}}, \qquad n = 0, 1, \dots, \tag{1}$$

where k is a positive integer, the parameters α, β, γ, and A are non-negative real numbers, and the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0$ are arbitrary non-negative real numbers such that the denominator of Eq. (1) is never zero.

Received: August 6, 2012

When $k = 1$, Eq. (1) was investigated in [4], where the global asymptotic stability of the positive equilibrium was proved for certain ranges of the parameters.

Our goal is to investigate the global stability of the solutions of Eq. (1) when $k \in \{2, 3, \ldots\}$. For the remainder of the sequel we will assume that k is a positive integer greater than one.

Note that, when $k = 2$ we address the Open Problem 5.173.1 posed in [3].

We now present some known results which will be useful in our investigation of Eq. (1).

The following theorem, which was given by C. Clark in [1], provides a sufficient condition for local asymptotic stability of the equilibria of Eq. (1).

Theorem A. (Clark's Theorem) *Consider the difference equation*

$$z_{n+1} + a_k z_n + \ldots + a_0 z_{n-k} = 0, \quad n = 0, 1, \ldots, \tag{2}$$

where $k \in \{1, 2, \ldots\}$ and a_i real numbers for all i. Then

$$\sum_{i=0}^{k} |a_i| < 1$$

is a sufficient condition for the asymptotic stability of Eq. (2).

The following general global asymptotic stability result presented in [3] apply to several special cases of the $(k+1)^{st}$ - order rational difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \beta_i x_{n-i}}{A + \sum_{i=0}^{k} B_i x_{n-i}} \tag{3}$$

with $A > 0$, the remaining parameters nonnegative, with

$$\sum_{i=0}^{k} \beta_i \text{ and} \sum_{i=0}^{k} B_i \in (0, \infty),$$

and with arbitrary nonnegative initial conditions such that the denominator is always positive.

Theorem B. *Assume that*

$$\beta = \sum_{i=0}^{k} \beta_i < A.$$

Then the following statements are true:

(a) *If $\alpha = 0$ the zero equilibrium of Eq. (3) is globally asymptotically stable.*

(b) *If $\alpha > 0$ the positive equilibrium of Eq. (3) is globally asymptotically stable.*

The next convergence result about the global attractivity of the equilibrium has been established in [4].

Theorem C. *Let $[a, b]$ be a closed and bounded interval of real numbers and let $F \in C([a, b]^2, [a, b])$ satisfy the following conditions:*

(a) *The function $F(z_1, z_2)$ is monotonic in each of its arguments.*

(b) *For each $m, M \in [a, b]$ and for each $i \in \{1, 2\}$, we define*

$$M_i(m, M) = \begin{cases} M, & \text{if } F \text{ is increasing in } z_i, \\ m, & \text{if } F \text{ is decreasing in } z_i, \end{cases}$$

and

$$m_i(m, M) = M_i(M, m)$$

and assume that if (m, M) is a solution of the system:

$$M = F(M_1(m, M), M_2(m, M))$$

$$m = F(m_1(m, M), m_2(m, M))$$

then $M = m$.

Then there exists exactly one equilibrium $\bar{y} \in [a, b]$ of the equation

$$y_{n+1} = F(y_n, y_{n-k}), \quad n = 0, 1, \ldots \tag{4}$$

and every solution of Eq. (4) converges to the unique equilibrium.

The next result about the global attractivity of the positive equilibrium has been established in [6] and [7].

Theorem D. *Consider the difference equation*

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}}, \quad n = 0, 1, \ldots, \tag{5}$$

where

$$a, b \in [0, \infty) \quad \textit{with} \quad a + b > 0, \; A \in (0, \infty), \quad \textit{and} \quad k \in \{1, 2, \ldots\}.$$

Then the positive equilibrium $\bar{x}$ of Eq. (5) is globally asymptotically stable if

$$a > 0, \quad k \geq 2 \quad \text{and} \quad A > b > 0.$$

2. Special Cases

When $\alpha = 0$, Eq. (1) was investigated in [2], where with out loss of generality it was reduced to the difference equation

$$y_{n+1} = \frac{py_n + y_{n-k}}{q + y_{n-k}}, \qquad n = 0, 1, \ldots,$$

every solution of which was shown to converge to the zero equilibrium when $q \geq p + 1$. When $q < p + 1$ it was shown that the zero equilibrium is unstable while the unique positive equilibrium is globally asymptotically stable if either $q > p - 1$ or $q \geq \frac{(k-1)p-1}{k}$.

When $\beta = 0$, Eq. (1) is of the Riccati type. See [3] for the character of solutions of Riccati equations.

When $\gamma = 0$ Eq. (1) was investigated in [6],where the global stability of the equilibria were proved in certain regions of the parameter space. Theorem D gives the global stability results as presented in [6]. See [3] for important conjectures regarding the global stabilty of the equilibrium of this equation when $k = 1$ and $k = 2$. Note that, this equation contains Pielou's discrete delay logistic model as a special case when further $\alpha = 0$.

Thus, we may assume that the parameters α, β, and γ are all positive hereafter. The change of variables

$$x_n = \gamma y_n$$

reduces Eq. (1) to the difference equation

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{q + y_{n-k}}, \qquad n = 0, 1, \ldots, \tag{6}$$

with $k \in \{2, 3, \ldots\}$, the parameters $r, p \in (0, \infty)$, $q \in [0, \infty)$ and non-negative initial conditions such that the denominator is never zero.

Our goal is to investigate the global stability of the solutions of Eq. (6).

3. Invariant Intervals

In this section we show that when $q > p + r$, the interval $[0, 1]$ is an invariant interval for all solutions of Eq. (6). We also show that when $q < p + r$, the interval $(1, \infty)$ is an invariant interval for all solutions of Eq. (6).

The equilibrium point of Eq. (6) is the positive solution of the equation

$$\bar{y} = \frac{r + p\bar{y} + \bar{y}}{q + \bar{y}}.$$

Clearly, Eq. (6) has an unique equilibrium point

$$\bar{y} = \frac{1}{2}(p + 1 - q + \sqrt{(p + 1 - q)^2 + 4r}).$$

The following lemma, the proof of which is straightforward and will be omitted, exhibits an identity that will be useful in the study of Eq. (6).

Lemma 3.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6). Then for $n \geq 0$ the following identity holds:*

$$y_{n+1} - 1 = \frac{p(y_n - \frac{q-r}{p})}{q + y_{n-k}}. \tag{7}$$

The following lemma establishes the existence of invariant intervals for solutions of Eq. (6).

Theorem 3.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6). Then the following statements are true:*

(a) Suppose that $q > p + r$ and that there exists $N \geq 0$ such that $y_N \leq 1$. Then $y_n < 1$ for all $n > N$.

(b) Suppose that $q = p + r$.

(i) If $y_0 < 1$, then $y_n < 1$ for all $n \geq 0$.

(ii) If $y_0 = 1$, then $y_n = 1$ for all $n \geq 0$.

(iii) If $y_0 > 1$, then $y_n > 1$ for all $n \geq 0$.

(c) Suppose that $q < p + r$ and that there exists $N \geq 0$ such that $y_N \geq 1$. Then $y_n > 1$ for all $n > N$.

Proof. We will prove (a). The proofs of (b) and (c) are similar and will be omitted. From Eq. (7) we see that $y_N \leq 1 < \frac{q-r}{p}$ and so $y_{N+1} < 1$. The result now follows by induction. □

4. Boundedness

In this section we show that the solutions of Eq. (6) are bounded for all values of parameters. For $k \in \{1, 2\}$ the boundedness result has been proved in [3].

Theorem 4.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (6). Then y_n is bounded for all $n > 0$.*

Proof. Note that

$$\begin{aligned} y_{n+(k+1)} =& \frac{r + py_{n+k} + y_n}{q + y_n} \\ =& \frac{r + y_n}{q + y_n} + \frac{p}{q + y_n}\left[\frac{r + py_{n+(k-1)} + y_{n-1}}{q + y_{n-1}}\right] \\ =& \dots \\ =& \frac{r + y_n}{q + y_n} + \frac{p}{q + y_n}\left[\frac{r + y_{n-1}}{q + y_{n-1}}\right] + \dots \\ &+ \frac{p^k}{(q + y_n)(q + y_{n-1}) \dots (q + y_{n-(k-1)})}\left[\frac{r + y_{n-k}}{q + y_{n-k}} + \frac{py_n}{q + y_{n-k}}\right], \end{aligned}$$

from which the boundedness follows.

5. Local Stability of the Equilibrium

In this section we establish the local stability character of the unique equilibrium $\bar{y}$ of Eq. (6). The linearized equation of Eq. (6) with respect to the equilibrium is

$$z_{n+1} - \frac{p}{q + \bar{y}} z_n + \frac{\bar{y} - 1}{q + \bar{y}} z_{n-k} = 0, \qquad n = 0, 1, \dots,$$

with associated characteristic equation

$$\lambda^{k+1} - \frac{p}{q + \bar{y}} \lambda^k + \frac{\bar{y} - 1}{q + \bar{y}} = 0.$$

The following lemma is a direct consequence of Theorem A and gives us a sufficient condition for local asymptotic stability of the equilibrium $\bar{y} = \frac{1}{2}(p + 1 - q + \sqrt{(p + 1 - q)^2 + 4r})$ of Eq. (6).

Lemma 5.1. *The unique equilibrium $\bar{y}$ of Eq. (6) is locally asymptotically stable when $q > p - 1$.*

6. Global Stability of the Equilibrium

We now discuss the global stability of the unique equilibrium of Eq. (6).

Note that by Theorem B, when $q \geq p+1$ the unique equilibrium point $\bar{y}$ of Eq. (6) is globally asymptotically stable. We now investigate the global asymptotic stability of the equilibrium $\bar{y}$ of Eq. (6) in the region $q < p+1$.

Theorem 6.1. *The unique equilibrium $\bar{y}$ of Eq. (6) is globally asymptotically stable when $q > p-1$.*

Proof. It suffices to show that the unique equilibrium is a global attractor of all solutions of Eq. (6).

Case (i) Suppose that $p+r < q < p+1$. Clearly from Theorem 3.2, we obtain that either

$$y_n > 1, \text{ for all } n \geq 0, \tag{8}$$

or eventually

$$y_n \leq 1 < \frac{q-r}{p}. \tag{9}$$

When(8) holds, from Eq. (7), we see that, for all $n \geq 1$,

$$y_n > \frac{q-r}{p}.$$

Clearly the function

$$f(y_n, y_{n-k}) = \frac{r + py_n + y_{n-k}}{q + y_{n-k}}$$

is strictly increasing in y_n and strictly decreasing in y_{n-k}. By employing Theorem C the result follows.

When (9) holds, clearly the function

$$f(y_n, y_{n-k}) = \frac{r + py_n + y_{n-k}}{q + y_{n-k}}$$

is strictly increasing in y_n and y_{n-k}. By employing Theorem C the result follows.

Case (ii) Suppose that $r < q = p + r$.

Observe that, for all $n \geq 0$,

$$y_{n+1} - y_n = \frac{(r + y_{n-k})(1 - y_n)}{r + p + y_{n-k}}.$$

From this and from Theorem 3.2, we obtain that, for all $n \geq 0$,

$$y_n \leq y_{n+1} \leq 1 \text{ or } 1 \leq y_{n+1} \leq y_n$$

from which the result follows.

Case (iii) Suppose that $r < q < p+r$. Clearly from Theorem 3.2, we obtain that either

$$y_n \leq 1, \text{ for all } n \geq 0 \tag{10}$$

or eventually

$$y_n > 1 > \frac{q-r}{p}. \tag{11}$$

When (10) holds, from Eq. (7), we see that, for all $n \geq 1$,

$$y_n < \frac{q-r}{p}.$$

Clearly the function

$$f(y_n, y_{n-k}) = \frac{r + py_n + y_{n-k}}{q + y_{n-k}}$$

is strictly increasing in y_n and y_{n-k}. By employing Theorem C the result follows. When (11) holds by using the change of variables $y_n = pw_n + 1$, Eq. (6)takes the form

$$w_{n+1} = \frac{\frac{r+p-q}{p^2} + w_n}{\frac{q+1}{p} + w_{n-k}}, \qquad n = 0, 1, \ldots. \tag{12}$$

The result is a consequence of Theorem D.

Case (iv) Suppose that $q < r$. Clearly from Theorem 3.2, we obtain that

$$y_n > 1, \text{ for all } n \geq 0$$

and for all $n \geq 1$,

$$y_n > \frac{q-r}{p}.$$

By using the change of variables $y_n = pw_n + 1$, Eq. (6) takes the form of Eq. (12). The result is a consequence of Theorem D. The proof is complete. □

Acknowledgments

This work was supported by DST Grant (D.O.No SR/FTP/MS-013/2011). Also supported by UGC.

References

[1] C.W. Clark, A delayed recruitment model of population dynamics with an application to baleen whale populations, *J. Math. Biol.*, **3** (1976), 381-391.

[2] E. Chatterjee, R. DeVault, G. Ladas, On the global character of $x_{n+1} = \dfrac{\beta x_n + \delta x_{n-k}}{A + x_{n-k}}$, *International Journal of Applied Mathematical Sciences*, **2**, No. 1 (2005), 39-46.

[3] Camouzis, G. Ladas, *Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press (2008).

[4] M.R.S. Kulenovič, G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press (2001).

[5] M.R.S. Kulenovič, G. Ladas, N.R. Prokup, A rational difference equation, *Computers and Mathematics with Applications*, **41** (2001), 671-678.

[6] V.L. Kocic, G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, Holland (1993).

[7] V.L. Kocic, G. Ladas, I.W. Rodrigues, On rational recursive sequences, *J. Math. Anal. Appl.*, **173** (1993), 127-157.