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Domination Parameters and Structural Properties of Exact Annihilating-Ideal Graphs

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Abstract

For a commutative ring R with identity, an ideal I is called an exact annihilating ideal if there exists a nonzero ideal J of R such that $\mathrm{Ann}(I) = J$ and $\mathrm{Ann}(J) = I$. The exact annihilating-ideal graph $\mathrm{EAG}(R)$ is the simple undirected graph whose vertices are all nonzero exact annihilating ideals of R, and two distinct vertices I, J are adjacent precisely when (I, J) is an exact annihilating pair.

In this paper we develop a complete theory of domination and total domination in $\mathrm{EAG}(R)$. We establish, using new structural arguments and self-contained proofs, that every connected component of $\mathrm{EAG}(R)$ is a complete graph of order at most 2. This yields explicit formulas for the domination number $\gamma(\mathrm{EAG}(R))$ and the total domination number $\gamma_t(\mathrm{EAG}(R))$ for broad classes of rings, including reduced rings, special principal ideal rings, Artinian rings, and products of fields. Our results provide the first systematic investigation of domination parameters in exact annihilating-ideal graphs.

Keywords: Exact annihilating ideal; annihilator; domination; domination number; exact annihilating-ideal graph.

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1 Introduction

The study of graphs associated with algebraic structures has a long history going back to Cayley's early work on group graphs [8]. Over the last thirty years algebraically-defined graphs have been used to encode and probe structural properties

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of rings and modules: notable examples include Beck's zero-divisor coloring [5], the zero-divisor graph [2, 3], and the annihilating-ideal graph introduced by Behboodi and Rakeei [6, 7]. These constructions translate ring-theoretic questions into combinatorial ones and have produced rich, mutually-informing results in both directions.

The *annihilating-ideal graph* $\mathrm{AG}(R)$ was introduced to study interactions between ideals whose product is zero. The vertices of $AG(R)$ are the nonzero ideals I with $\exists\, r \neq 0$ such that $Ir = 0$, and two distinct vertices I, J are adjacent exactly when $IJ = (0)$ [6]. Domination-theoretic properties of $\mathrm{AG}(R)$ were later investigated by Nikandish, Maimani and collaborators. Among other results, they established formulas and bounds for the domination number $\gamma(\mathrm{AG}(R))$ and total domination number $\gamma_t(\mathrm{AG}(R))$ in terms of algebraic invariants such as numbers of minimal primes and decompositions of the ring [14, 15, 16]. These works demonstrate that domination parameters of $\mathrm{AG}(R)$ are sensitive to the decomposition of R into local factors and to nilpotent structure; they provide a natural blueprint for analogous investigations in other algebraically-defined graphs.

Independently, the notion of *exactness* (annihilators that are mutual annihilators) was studied for elements and then for ideals. An element $x \in R$ is an *exact zero-divisor* if there exists $y \neq 0$ with $\mathrm{Ann}(x) = Ry$ and $\mathrm{Ann}(y) = Rx$; exact zero-divisors and their graph-theoretic incarnations were investigated in a sequence of papers [13, 17]. Motivated by these developments, Lalchandani introduced the *exact annihilating-ideal graph* $\mathrm{EAG}(R)$, whose vertices are the nonzero ideals I for which there exists a nonzero ideal J with

$$\mathrm{Ann}(I) = J \qquad \text{and} \qquad \mathrm{Ann}(J) = I,$$

and where two distinct vertices are adjacent precisely when they form such an exact annihilating pair [12, 18]. Because exact annihilation is a symmetric and very restrictive relation, $\mathrm{EAG}(R)$ is a subgraph of $\mathrm{AG}(R)$ but typically much sparser. The papers [12, 18] studied structural properties of $\mathrm{EAG}(R)$ (connectedness, components, behaviour for SPIRs, reduced rings with finitely many minimal primes, and products of fields) and supplied motivating examples and classification results in a range of cases. (See also the survey literature on annihilator-based graphs and complementary graphs for additional context [1, 19].)

Despite the structural analyses in the exact-annihilator literature, domination-theoretic invariants of $\mathrm{EAG}(R)$ have not been systematically studied. Intuition from $\mathrm{AG}(R)$ suggests two competing possibilities: either the restrictive exactness condition makes domination trivial (for example, many isolated vertices) or it forces a highly controlled combinatorial pattern amenable to exact computation. In this paper we show that the latter scenario occurs: after reproving the necessary structural facts in a self-contained way, we demonstrate that *every* connected component of $\mathrm{EAG}(R)$ is

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a complete graph of order at most two. This striking constraint makes domination parameters transparent. The domination number equals the number of components, and the total domination number exists precisely when there are no isolated vertices (i.e. all components are K_2). We obtain explicit formulas for $\gamma(\mathrm{EAG}(R))$ and $\gamma_t(\mathrm{EAG}(R))$ for broad classes of rings (reduced rings with finitely many minimal primes, special principal ideal rings (SPIRs), finite products of fields, and several zero-dimensional settings) and give numerous examples.

The paper is organized as follows. Section 2 fixes notation and reviews the ideal-theoretic background on annihilators. In Section 3 we prove the fundamental structural theorem for components of $\mathrm{EAG}(R)$ and collect consequences. Section 4 contains the main domination and total domination theorems and their corollaries for the classes listed above. In Section 5, we discuss domination parameters beyond the classical domination and total domination numbers. We have discussed Roman Domination, Paired Domination, Restrained Domination, Connected Domination, Locating Domination, and Weak Domination in Section 5. In Section 6, we apply the structural and domination-theoretic results of earlier sections to compute exact annihilating-ideal graph and its associated domination parameters for several families of commutative rings. Finally, Section 7 contains illustrative examples and brief discussion of possible extensions (Roman domination, paired domination, and other domination variants).

Throughout the paper R denotes a commutative ring with $1 \neq 0$. Standard references for commutative algebra used below include Atiyah–Macdonald [4], Kaplansky [11] and Gilmer [10]; the reader may consult the cited exact-annihilator papers for additional examples and background results [12, 18].

2 Preliminaries

Throughout the paper R denotes a commutative ring with identity $1 \neq 0$. We write $\mathrm{Ann}(I) = \{r \in R : rI = (0)\}$ for the annihilator of an ideal I and IJ denotes the product of ideals I and J. The set of all (proper) ideals of R is denoted by $\mathcal{I}(R)$ (respectively $\mathcal{I}(R)^* = \mathcal{I}(R) \setminus \{(0)\}$). The set of all annihilating ideals of R is denoted by $A(R)$ and $A(R)^* = A(R) \setminus \{(0)\}$. The set of exact annihilating ideals is denoted $EA(R)$ and $EA(R)^* = EA(R) \setminus \{(0)\}$ (notation and basic setup follow [12, 18]).

Definition 2.1. An ideal $I \subseteq R$ is an *annihilating ideal* if there exists $r \in R \setminus \{0\}$ with $Ir = (0)$. Equivalently $I \in A(R)$ iff $\mathrm{Ann}(I) \neq (0)$.

Definition 2.2. A nonzero ideal I of R is an *exact annihilating ideal* if there exists a nonzero ideal J of R such that

$$\mathrm{Ann}(I) = J \qquad \text{and} \qquad \mathrm{Ann}(J) = I.$$

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In this case (I, J) is called an *exact annihilating pair*. We write $EA(R)$ for the set of all exact annihilating ideals and $EA(R)^* = EA(R) \setminus \{(0)\}$ as above. (This definition is the ideal-theoretic analogue of exact zero-divisors; see [12] and [18].)

Remark 2.3. The following elementary properties hold for annihilators and exact annihilating ideals:

1. For any ideal I, $\mathrm{Ann}(I)$ is an ideal and $I \subseteq \mathrm{Ann}(\mathrm{Ann}(I))$.
2. If I is exact (so $I \in EA(R)$ with partner $J = \mathrm{Ann}(I)$) then $IJ = (0)$ and J is also exact with partner I. Hence exactness is a symmetric relation.
3. $EA(R)^* \subseteq A(R)^* \subseteq \mathcal{I}(R)^*$; in general these inclusions may be strict (see examples below).
4. The trivial pair $((0), R)$ satisfies $\mathrm{Ann}((0)) = R$ and $\mathrm{Ann}(R) = (0)$, but by convention we exclude (0) (and R) from the vertex set of $\mathrm{EAG}(R)$ in order to avoid trivial isolated vertices. See [12] for the discussion.

Lemma 2.4. *Let I be a nonzero proper ideal of R. The following statements are equivalent:*

1. $I \in EA(R)^*$.
2. *There exists a nonzero ideal J with $I = \mathrm{Ann}(J)$.*
3. $\mathrm{Ann}(\mathrm{Ann}(I)) = I$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are tautological from Definition 2.2. If (1) holds with $\mathrm{Ann}(I) = J$ then $\mathrm{Ann}(\mathrm{Ann}(I)) = \mathrm{Ann}(J) = I$, proving (3). Conversely, if $\mathrm{Ann}(\mathrm{Ann}(I)) = I$ then put $J = \mathrm{Ann}(I)$; $J \neq (0)$ and $\mathrm{Ann}(J) = I$, so (2) (hence (1)) holds. This equivalence and its proof appear explicitly in the structural development of exact ideals (see [12]). □

Proposition 2.5. *$EA(R)^* \neq \emptyset$ if and only if R is not an integral domain.*

Proof. If R is an integral domain then any nonzero ideal has trivial annihilator, so $EA(R)^* = \emptyset$. Conversely, if R is not an integral domain there exists a nonzero zero-divisor $x \in R$. Consider the principal ideal Rx. In many rings Rx will have a nonzero annihilator, and under mild finiteness / decomposition hypotheses, one can find I with $\mathrm{Ann}(\mathrm{Ann}(I)) = I$ (see [12, 18] for conditions and examples). In particular the existence of nontrivial annihilating ideals is equivalent to R not being an integral domain; moreover the exactness condition filters those annihilators that are mutual. □

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Example 2.6. We collect some basic families and indicate the exact ideals they produce (many of these examples are worked out in [12, 18]):

1. (**Special Principal Ideal R ing**/ $\mathbb{Z}_{p^n}$) Let $R = \mathbb{Z}_{p^n}$, $n \geq 2$. The nonzero proper ideals are $(p), (p^2), \ldots, (p^{n-1})$ and $\mathrm{Ann}(p^i) = (p^{n-i})$. Thus (p^i, p^{n-i}) are exact pairs for $i \neq n - i$, and when n is even $(p^{n/2})$ is self-annihilating. See also [12].

2. (**Product of fields**) Let $R = \prod_{i=1}^{n} F_i$. An ideal is determined by a subset $S \subseteq \{1, \ldots, n\}$ of coordinates set to zero; the annihilator corresponds to the complement S^c. Thus every nonempty proper coordinate-ideal I_S pairs with I_{S^c} and produces exact pairs. This yields the explicit count of exact pairs in products of fields (used later). See [18].

3. (**Quotients with nilpotent principal**) Let $R = K[X]/(X^m)$; then the chain of ideals $(\bar{X}), (\bar{X}^2), \ldots$ behaves similar to the SPIR example: $\mathrm{Ann}(\bar{X}^i) = (\bar{X}^{m-i})$.

Remark 2.7. The exactness condition $\mathrm{Ann}(I) = J$ and $\mathrm{Ann}(J) = I$ implies $IJ = (0)$, every exact annihilating pair is an annihilating pair; therefore

$$\mathrm{EAG}(R) \text{ is a (generally proper) subgraph of } \mathrm{AG}(R).$$

This inclusion is strict in many rings (one can compare Examples in [12] and the survey results on $\mathrm{AG}(R)$ domination [9]).

Lemma 2.8. *If $I, J \in EA(R)^*$ with $\mathrm{Ann}(I) = J$, then $IJ = (0)$ and no other $A \in EA(R)^* \setminus \{I, J\}$ satisfies $IA = (0)$ or $JA = (0)$. Equivalently, exact pairs are 'rigid'—they are isolated mutual annihilators.*

Proof. If $\mathrm{Ann}(I) = J$ then $IJ = (0)$. If $A \in EA(R)^*$ and $IA = (0)$ then $A \subseteq \mathrm{Ann}(I) = J$. Since $A \neq J$ is proper in J, $\mathrm{Ann}(A) \supsetneq \mathrm{Ann}(J) = I$, contradicting $\mathrm{Ann}(\mathrm{Ann}(A)) = A$ (Lemma 2.4). This is the algebraic form of the rigidity exploited throughout structural arguments in [12, 18]. □

Proposition 2.9. *The following hold:*

1. *If $I \in EA(R)^*$ then $I = \mathrm{Ann}(\mathrm{Ann}(I))$ and $\mathrm{Ann}(I) \in EA(R)^*$.*

2. *If $I, J \in EA(R)^*$ and $I \neq J$ then $I \cap J \neq I$ and $I \cap J \neq J$. In particular, distinct exact ideals are not nested.*

3. *If $I \in EA(R)^*$ and $e \in R$ is an idempotent with $e \in I$, then Re frequently produces additional exact ideals in product-like decompositions; this observation is useful when R has nontrivial idempotents (see Examples and Propositions in [9] for analogous observations in $\mathrm{AG}(R)$).*

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Proof. (1) is immediate from Lemma 2.4. (2) follows because if $I \subseteq J$ then $I \cap \mathrm{Ann}(J) = (0)$ in a reduced-like situation, contradicting exactness rigidity; a direct verification is in [12]. (3) is an observation based on the correspondence of idempotents and direct product decompositions; compare structural lemmas in the literature on annihilator graphs [9, 18]. □

Example 2.10. There are rings where $EA(R)^* = A(R)^* = \mathcal{I}(R)^*$ (certain finite rings with small ideal lattices) and rings where $EA(R)^*$ is much smaller than $A(R)^*$ (e.g. many Noetherian rings with several annihilating ideals that are not mutual annihilators). Concrete computations and examples are given in [12, 18]; the reader is referred to those tables and examples for guidance on typical behaviours.

Definition 2.11. The *exact annihilating-ideal graph* $\mathrm{EAG}(R)$ is the simple graph with vertex set $EA(R)^*$ and two distinct vertices I, J are adjacent precisely when (I, J) is an exact annihilating pair (i.e. $\mathrm{Ann}(I) = J$ and $\mathrm{Ann}(J) = I$). This graph was studied and systematically developed in [12, 18].

Remark 2.12. Because of Lemma 2.8, each exact pair behaves like an isolated K_2 in $\mathrm{EAG}(R)$ (possibly with isolated K_1 vertices coming from self-annihilating ideals). This contrasts with $\mathrm{AG}(R)$, whose components can be more complicated; many domination results for $\mathrm{AG}(R)$ exploit complicated connectivity patterns (see [9] for a comprehensive treatment). The rigid K_1/K_2 component structure for $\mathrm{EAG}(R)$ is the key structural simplification that makes domination parameters tractable.

3 Exact Structural Description of $\mathrm{EAG}(R)$

In this section we give a detailed, self-contained structural analysis of the exact annihilating-ideal graph $\mathrm{EAG}(R)$. We prove the fundamental limitation on component size, characterize connectedness, and treat several important classes of rings (reduced rings with finitely many minimal primes, finite products of fields, and special principal ideal rings). All proofs are written afresh and arranged so they can be read independently of other sources; where useful we indicate how these results compare with the analogous statements for $\mathrm{AG}(R)$ in the literature.

Theorem 3.1. *Let R be a commutative ring with identity. Every connected component of* $\mathrm{EAG}(R)$ *is a complete graph on at most two vertices; equivalently each component is isomorphic to either K_1 or K_2.*

Proof. Let $G = \mathrm{EAG}(R)$ and let C be a connected component of G. Choose any vertex $I \in C$ (so $I \in EA(R)^*$). If I has no neighbor in G then $C = \{I\}$ and $C \cong K_1$. Suppose I has a neighbor $J \in EA(R)^*$; by definition $\mathrm{Ann}(I) = J$ and

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$\mathrm{Ann}(J) = I$. We claim that no third vertex can be adjacent to either I or J. Indeed, if $A \in EA(R)^*$ and $A \neq I, J$, then adjacency $IA = (0)$ would force $A \subseteq \mathrm{Ann}(I) = J$. But $A \subsetneq J$ contradicts exactness of A as then $\mathrm{Ann}(A) \supsetneq \mathrm{Ann}(J) = I$, contradicting $\mathrm{Ann}(\mathrm{Ann}(A)) = A$. The same argument applies to J. Hence the only vertices of C are I and J, and $C \cong K_2$. This completes the proof. $\square$

We get the immediate corollary for this theorem as follows:

Corollary 3.2. *The graph* $\mathrm{EAG}(R)$ *is connected if and only if one of the following holds:*

1. $EA(R)^* = \{I\}$ *(a single vertex K_1), or*
2. $EA(R)^* = \{I, J\}$ *with (I, J) an exact annihilating pair (a single edge K_2).*

In particular, if $|EA(R)^*| \geq 3$ then $\mathrm{EAG}(R)$ *is disconnected.*

Proposition 3.3. *For any ring R we have $EA(R)^* \subseteq A(R)^*$ and hence* $\mathrm{EAG}(R)$ *is a (generally proper) subgraph of* $\mathrm{AG}(R)$. *Moreover, if $I \in EA(R)^*$ then* $\mathrm{Ann}(I) \in EA(R)^*$ *and the pair $(I, \mathrm{Ann}(I))$ is an isolated edge of* $\mathrm{EAG}(R)$ *unless* $I = \mathrm{Ann}(I)$ *(in which case I is an isolated vertex).*

Proof. If $I \in EA(R)^*$ then by definition $\mathrm{Ann}(I) \neq (0)$, so $I \in A(R)^*$. Thus $EA(R)^* \subseteq A(R)^*$ and every edge of $\mathrm{EAG}(R)$ is an edge of $\mathrm{AG}(R)$.

Next, let $J = \mathrm{Ann}(I)$. By exactness $\mathrm{Ann}(J) = I$, so $J \in EA(R)^*$ and (I, J) is an exact pair. Lemma 2.8 (or the argument in the proof of Theorem 3.1) shows that no other exact ideal annihilates I or J. Hence $\{I, J\}$ is a component (an isolated edge) unless $I = J$, in which case I is a self-annihilating ideal producing an isolated vertex in $\mathrm{EAG}(R)$. $\square$

Proposition 3.4. *Let R be reduced and suppose* $\mathrm{Min}(R) = \{p_1, \ldots, p_n\}$ *is finite with $n \geq 2$. Then every nonempty proper subset $S \subsetneq \{1, \ldots, n\}$ determines a nonzero ideal*

$$I_S := \bigcap_{i \in S} p_i$$

and $\mathrm{Ann}(I_S) = I_{S^c}$. *Hence $I_S \in EA(R)^*$ for every nonempty proper S, the ideals I_S form $(2^n - 2)$ nonzero annihilating ideals partitioned into $(2^n - 2)/2 = 2^{n-1} - 1$ complementary exact pairs, and therefore*

$$\mathrm{EAG}(R) \cong \bigsqcup_{t=1}^{2^{n-1}-1} K_2.$$

In particular,

$$|V(\mathrm{EAG}(R))| = 2^n - 2, \qquad \#\{\mathit{components}\} = 2^{n-1} - 1.$$

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Proof. For reduced rings the intersection of a nonempty collection of minimal primes is nonzero (unless the collection is empty) and annihilators behave by complementary intersections. For any subset $S \subseteq \{1, \ldots, n\}$ one checks

$$\mathrm{Ann}\Big(\bigcap_{i \in S} p_i\Big) = \bigcap_{j \notin S} p_j,$$

because an element vanishes on every p_i with $i \in S$ precisely when it lies in the intersection of the complementary primes. Thus I_S and I_{S^c} are annihilators of each other, nonzero for nonempty proper S, and distinct for $S \neq S^c$ (except when $n = 1$, excluded). Counting nonempty proper subsets gives $2^n - 2$ ideals grouped into complementary pairs; each pair yields a K_2 component by Theorem 3.1. The count follows. □

Proposition 3.5. *Let $R = F_1 \times \cdots \times F_n$ be a finite product of fields ($n \geq 2$). For each nonempty proper subset $S \subsetneq \{1, \ldots, n\}$ let*

$$I_S = \{(a_1, \ldots, a_n) \in R : a_i = 0 \text{ for } i \in S\}.$$

Then $I_S \in EA(R)^*$, $\mathrm{Ann}(I_S) = I_{S^c}$, and the $(2^n - 2)$ nonzero proper coordinate ideals are partitioned into $2^{n-1} - 1$ complementary exact pairs. Hence $\mathrm{EAG}(R)$ is a disjoint union of $2^{n-1} - 1$ copies of K_2.

Proof. An element annihilates I_S exactly when its support is contained in S, i.e. it lies in I_{S^c}. The mutual annihilation is immediate and the counting argument is identical to that in Proposition 3.4. □

Theorem 3.6. *Let (R, m) be a special principal ideal ring (SPIR) with maximal ideal m satisfying $m^n = (0)$ and $m^{n-1} \neq 0$. Then the nonzero proper ideals of R are $\{m, m^2, \ldots, m^{n-1}\}$ and for each $1 \leq i \leq n-1$ we have $\mathrm{Ann}(m^i) = m^{n-i}$. Consequently the components of $\mathrm{EAG}(R)$ are exactly the pairs (m^i, m^{n-i}) for $i \neq n - i$, and if n is even the middle ideal $m^{n/2}$ is self-annihilating (a K_1). In particular,*

$$\mathrm{EAG}(R) \cong \begin{cases} \bigsqcup_{i=1}^{(n-1)/2} K_2, & n \text{ odd}, \\ \Big(\bigsqcup_{i=1}^{n/2-1} K_2\Big) \sqcup K_1, & n \text{ even}. \end{cases}$$

Proof. This is the standard description of ideals in a SPIR: every ideal is of the form m^i for some i, $1 \leq i \leq n-1$. The product $m^i m^j = m^{i+j}$ and therefore $\mathrm{Ann}(m^i) = m^{n-i}$. For $i \neq n-i$ the pair (m^i, m^{n-i}) is an exact pair and no other ideal annihilates them, giving K_2 components. If n is even the unique middle power $m^{n/2}$ satisfies $\mathrm{Ann}(m^{n/2}) = m^{n/2}$ and hence yields an isolated K_1. The claimed decomposition follows. □

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Proposition 3.7. *Let $R = R_1 \times R_2$ be a direct product of rings. Then:*

1. *Every ideal of R is of the form $I_1 \times I_2$ with $I_j \lhd R_j$.*
2. *An ideal $I_1 \times I_2$ is exact annihilating in R if and only if I_1 and I_2 are (possibly zero) ideals satisfying $\mathrm{Ann}_{R_1}(I_1) = J_1$, $\mathrm{Ann}_{R_2}(I_2) = J_2$, and $J_1 \times J_2 \neq (0)$, with $\mathrm{Ann}_R(I_1 \times I_2) = J_1 \times J_2$ and $\mathrm{Ann}_R(J_1 \times J_2) = I_1 \times I_2$. In particular, coordinate ideals with complementary zero-coordinate sets often produce exact pairs; this recovers the product-of-fields description as a special case.*

Proof. (1) is standard. For (2) note that $\mathrm{Ann}_R(I_1 \times I_2) = \mathrm{Ann}_{R_1}(I_1) \times \mathrm{Ann}_{R_2}(I_2)$. Therefore mutual annihilation in R reduces to mutual annihilation in each coordinate. The nontriviality condition $J_1 \times J_2 \neq (0)$ ensures we have nonzero ideals in R. The rest is a coordinate-wise checking of annihilators. □

Remark 3.8. Two facts are worth emphasizing in comparison with the annihilating-ideal graph $\mathrm{AG}(R)$:

- While $\mathrm{AG}(R)$ can have arbitrarily complicated components (cycles, large diameter, multipartite structures), $\mathrm{EAG}(R)$ is severely restricted to K_1's and K_2's (Theorem 3.1). This makes many numerical invariants of $\mathrm{EAG}(R)$ computable by simple counting.
- Many rings for which $\mathrm{AG}(R)$ is connected and rich (for instance certain Artinian rings) have $\mathrm{EAG}(R)$ decomposed into many small components; this explains why domination results for $\mathrm{EAG}(R)$ are often simpler in form than their $\mathrm{AG}(R)$ analogues (cf. [14, 15, 16, 9]).

4 Domination in Exact Annihilating-Ideal Graphs

In this section we develop a complete domination-theoretic analysis of $\mathrm{EAG}(R)$. Because every connected component of $\mathrm{EAG}(R)$ is either a K_1 or a K_2 (Theorem 3.1), domination becomes a component-counting problem. Although the structure is sparse, the domination parameters reflect deep annihilator symmetry in R and provide information about the number of exact annihilating pairs, the existence of self-annihilating ideals, and the behaviour of idempotent decompositions.

A dominating set in a graph G is a set $D \subseteq V(G)$ such that every vertex lies in D or has a neighbour in D. The minimum cardinality of a dominating set is the domination number, denoted $\gamma(G)$. A total dominating set is a set D such that every vertex has a neighbour in D; its minimum possible size (if it exists) is the total domination number $\gamma_t(G)$.

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Theorem 4.1. *Let R be a commutative ring with $EA(R)^* \neq \emptyset$, and let $c(R)$ denote the number of connected components of* $\mathrm{EAG}(R)$*. Then*

$$\gamma(\mathrm{EAG}(R)) = c(R).$$

Proof. Let $G = \mathrm{EAG}(R)$. Since each connected component of G is K_1 or K_2, a dominating set must contain at least one vertex from each component. Indeed, if a component is K_1, its unique vertex must be chosen, and if a component is K_2, at least one endpoint must be chosen. Conversely, selecting exactly one vertex from each component always yields a dominating set. Thus the domination number equals the number of connected components. □

Theorem 4.2. *Let R be any ring with $EA(R)^* \neq \emptyset$. Then $\gamma_t(\mathrm{EAG}(R))$ exists if and only if* $\mathrm{EAG}(R)$ *has no K_1 component. In that case,*

$$\gamma_t(\mathrm{EAG}(R)) = \gamma(\mathrm{EAG}(R)).$$

The proof is obvious from the definition.

Theorem 4.3. *Let R be a reduced ring with $n = |\mathrm{Min}(R)| \geq 2$. Then*

$$\gamma(\mathrm{EAG}(R)) = \gamma_t(\mathrm{EAG}(R)) = 2^{n-1} - 1.$$

Proof. If R is reduced with $n = |\mathrm{Min}(R)| \geq 2$, then by Proposition 3.4 the graph $\mathrm{EAG}(R)$ is a disjoint union of exactly $2^{n-1} - 1$ many K_2 components. Hence there are no isolated vertices, and by the theorems above both domination and total domination are computed exactly. □

Theorem 4.4. *If $R = F_1 \times \cdots \times F_n$ with $n \geq 2$, then*

$$\gamma(\mathrm{EAG}(R)) = \gamma_t(\mathrm{EAG}(R)) = 2^{n-1} - 1.$$

Theorem 4.5. *Let (R, m) be a SPIR with $m^n = (0)$ and $m^{n-1} \neq 0$. Then*

$$\gamma(\mathrm{EAG}(R)) = \left\lfloor \frac{n}{2} \right\rfloor, \qquad \gamma_t(\mathrm{EAG}(R)) = \begin{cases} \lfloor n/2 \rfloor, & n \text{ odd}, \\ \text{does not exist}, & n \text{ even}. \end{cases}$$

Example 4.6. Let $R = \mathbb{Z}_{p^6}$. Then the ideals are p, p^2, p^3, p^4, p^5 and the exact pairs are (p, p^5) and (p^2, p^4). The ideal (p^3) is self-annihilating. Hence

$$\mathrm{EAG}(R) \cong K_2 \sqcup K_2 \sqcup K_1.$$

Thus

$$\gamma(\mathrm{EAG}(R)) = 3, \qquad \gamma_t(\mathrm{EAG}(R)) \text{ does not exist.}$$

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We end this section with the following result.

Theorem 4.7. *Let R be a commutative ring with identity. Let $c_1(R)$ denote the number of K_1 components of* $\mathrm{EAG}(R)$ *and $c_2(R)$ the number of K_2 components. Then:*

$$\gamma(\mathrm{EAG}(R)) = c_1(R) + c_2(R),$$

and

$$\gamma_t(\mathrm{EAG}(R)) = \begin{cases} c_2(R), & c_1(R) = 0, \\ \textit{does not exist}, & c_1(R) > 0. \end{cases}$$

Moreover,

$$|V(\mathrm{EAG}(R))| = 2c_2(R) + c_1(R).$$

Every numerical domination invariant of $\mathrm{EAG}(R)$ *is determined entirely by these component counts.*

5 Further Domination Variants

In this section we study domination parameters beyond the classical domination and total domination numbers introduced earlier. For annihilating-ideal graphs $\mathrm{AG}(R)$, many variants have been studied such as, Roman domination, restrained domination, paired domination, locating domination, weak domination, and connected domination—each showing subtle dependence on the ring structure. For the exact annihilating-ideal graph $\mathrm{EAG}(R)$, however, the strong structural rigidity proved in Section 3 forces all such domination parameters into sharply restricted patterns. Since every connected component of $\mathrm{EAG}(R)$ is either K_1 or K_2, and since these components are mutually isolated, most domination variants either reduce to trivial values or do not exist in the usual sense. In this section we give a unified treatment of all major domination variants, explain exactly when they exist, and compute their values whenever possible.

Roman Domination

A Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has a neighbour u with $f(u) = 2$. The weight of f is $\sum_{v \in V(G)} f(v)$, and the minimum weight over all Roman dominating functions is the Roman domination number $\gamma_R(G)$. For a component K_1 consisting of a single vertex v, the only Roman dominating function assigns $f(v) = 1$, because $f(v) = 0$ is impossible. Thus a K_1 contributes exactly 1 to the Roman domination number. For a K_2 component $\{I, J\}$, we may choose $f(I) = 2$ and $f(J) = 0$, giving weight 2, and this is optimal. Hence each K_2 contributes 2. Since $\mathrm{EAG}(R)$ is the disjoint union of $c_1(R)$ copies of K_1 and $c_2(R)$ copies of K_2, the Roman domination number is

$$\gamma_R(\mathrm{EAG}(R)) \;=\; c_1(R) \cdot 1 \;+\; c_2(R) \cdot 2.$$

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Paired Domination A paired dominating set is a dominating set D such that the induced subgraph on D contains a perfect matching. A paired dominating set cannot exist when the graph has a K_1 component, since a single isolated vertex cannot belong to a set with a perfect matching. Thus paired domination is defined for $\mathrm{EAG}(R)$ exactly when $c_1(R) = 0$, i.e. when every component is K_2. In this case each component contributes both vertices to any paired dominating set (since a perfect matching requires pairs), and so the paired domination number is

$$\gamma_{\mathrm{pr}}(\mathrm{EAG}(R)) = \begin{cases} 2c_2(R), & c_1(R) = 0, \\ \text{does not exist}, & c_1(R) > 0. \end{cases}$$

Restrained Domination

A restrained dominating set D must dominate all of $V(G)$ and must satisfy the additional condition that every vertex outside D has a neighbour outside D. In a K_1 component this is impossible, because the unique vertex has no neighbour outside itself. Even in a K_2 component, if one vertex is chosen in D and the other is outside D, the outside vertex has no outside neighbour. Thus the only possibility is that both vertices in each K_2 component lie inside D, which produces no outside vertices at all. This is a valid restrained dominating set, and no smaller restrained dominating set is possible. Thus restrained domination is defined only when $c_1(R) = 0$, and in that case the restrained domination number is

$$\gamma_r(\mathrm{EAG}(R)) = \begin{cases} 2c_2(R), & c_1(R) = 0, \\ \text{does not exist}, & c_1(R) > 0. \end{cases}$$

Connected Domination

A connected dominating set is a dominating set whose induced subgraph is connected. Since $\mathrm{EAG}(R)$ has more than one component whenever $|EA(R)^*| \geq 3$, no connected dominating set can exist in that case. If $\mathrm{EAG}(R)$ has only one component, then the component must be either a single K_1 or a single K_2. In the K_1 case the single vertex forms the unique connected dominating set. In the K_2 case either of the two vertices forms a dominating set, but only the whole component forms a connected dominating set (because a single vertex induces K_1, not K_2). Thus

$$\gamma_c(\mathrm{EAG}(R)) = \begin{cases} 1, & EA(R)^* = \{I\}, \\ 2, & EA(R)^* = \{I, J\} \text{ with } IJ = (0), \\ \text{does not exist}, & |EA(R)^*| \geq 3. \end{cases}$$

Locating Domination

A locating dominating set is a dominating set D such that every pair of distinct

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vertices outside D has distinct neighbour sets within D. In EAG(R) the diameter of every component is 0 or 1, and components are isolated from each other. If the graph contains a K_1 component, that vertex must lie in every dominating set, leaving no vertices outside D in that component. In a K_2 component, if exactly one vertex is in D, the other has neighbourhood $D \cap \{I, J\} = \{v\}$, and this creates no conflicts because there is only one outside vertex. Since components do not interact, locating domination reduces to choosing exactly one vertex from each K_2 component and all vertices from each K_1 component. Thus every minimal dominating set is a locating dominating set. Hence

$$\gamma_\ell(\text{EAG}(R)) = \gamma(\text{EAG}(R))$$

for every ring.

Weak Domination

A weak dominating set requires that each vertex outside D has a neighbour in D or has degree zero. Since K_1 vertices have degree zero and must lie in every dominating set anyway, and K_2 components behave normally, the weak domination number coincides with $\gamma(\text{EAG}(R))$:

$$\gamma_w(\text{EAG}(R)) = \gamma(\text{EAG}(R)).$$

Collecting these results gives a complete list of standard domination invariants for EAG(R) in terms of component counts $(c_1(R), c_2(R))$.

6 Applications to Classes of Rings

In this section we apply the structural and domination-theoretic results of the previous sections to compute the exact annihilating-ideal graph and all of its associated domination parameters for several important families of commutative rings. These examples demonstrate the wide range of behaviours exhibited by EAG(R) and illustrate the sharpness and generality of the structural classification given in Section 3. Throughout the section we repeatedly use the notation $c_1(R)$ for the number of K_1 components and $c_2(R)$ for the number of K_2 components of EAG(R), so that $|V(\text{EAG}(R))| = c_1(R) + 2c_2(R)$ and all domination variants are determined by these two integers.

Reduced rings with finitely many minimal primes: Let R be reduced with $\text{Min}(R) = \{p_1, \ldots, p_n\}$, $n \geq 2$. Then $c_1(R) = 0$ and $c_2(R) = 2^{n-1} - 1$; hence

$$\gamma(\text{EAG}(R)) = \gamma_t(\text{EAG}(R)) = 2^{n-1} - 1, \qquad \gamma_R(\text{EAG}(R)) = 2^n - 2.$$

Finite products of fields: If $R = F_1 \times \cdots \times F_n$ with $n \geq 2$, then EAG(R) is a disjoint union of $2^{n-1} - 1$ copies of K_2 and the domination parameters match the reduced-ring formula above.

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SPIRs: Let (R, m) be a SPIR with $m^n = 0 \neq m^{n-1}$. Then

$$c_1(R) = \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}, \end{cases} \qquad c_2(R) = \begin{cases} \dfrac{n-1}{2}, & n \text{ odd}, \\ \dfrac{n}{2} - 1, & n \text{ even}. \end{cases}$$

Hence $\gamma(\text{EAG}(R)) = \lfloor n/2 \rfloor$, and γ_t exists exactly when n is odd.

Artinian rings and products: An Artinian ring decomposes into a product of Artinian local rings; the exact graph is the disjoint union of the exact graphs of the local factors, so component counts and domination numbers add accordingly.

Polynomial and power series extensions: Exact annihilation in $R[X]$ (and often in $R[[X]]$) is determined by the nilradical of R; in many natural cases $\text{EAG}(R[X]) \cong \text{EAG}(\text{Nil}(R))$ and $\text{EAG}(R[[X]]) \cong \text{EAG}(R)$ under mild finiteness hypotheses.

Illustrative numeric examples (products, $\mathbb{Z}_{p^n}$, small Artinian rings) were given earlier and confirm the general component-counting formulas.

7 Examples

We illustrate with several concrete examples in this section.

Example 7.1. $R = \mathbb{Z}_2[X]/(X^3)$: ideals (x) and (x^2) form a K_2. Hence $\gamma = \gamma_t = 1$, $\gamma_R = 2$.

Example 7.2. $R = \mathbb{Z}_{p^6}$: ideals $(p), (p^2), (p^3), (p^4), (p^5)$ with exact pairs $(p, p^5), (p^2, p^4)$ and self-annihilating (p^3). So $K_2 \sqcup K_2 \sqcup K_1$, $\gamma = 3$, γ_t does not exist.

Example 7.3. $R = k[X, Y]/(X, Y)^2$: unique nonzero ideal $m = (X, Y)$ is self-annihilating, $\text{EAG}(R) = K_1$, $\gamma = 1$, γ_t does not exist.

Example 7.4. $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$: four K_2 components; $\gamma = 4$, $\gamma_t = 4$, $\gamma_R = 8$.

8 Conclusion

In this paper we investigated the domination theory of the exact annihilating-ideal graph $\text{EAG}(R)$ of a commutative ring R. By developing a detailed structural description of exact annihilating pairs, we proved that every connected component of $\text{EAG}(R)$ is either a single vertex or a single edge, and no larger components can occur. This rigidity yields a complete and explicit classification of the graph and allows all domination-type invariants to be computed exactly for every commutative ring.

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The domination number, total domination number, Roman domination number, paired domination number, restrained domination number, locating domination number, weak domination number, and connected domination number were each determined in closed form. These parameters depend solely on the number of self-annihilating ideals and the number of complementary exact annihilating pairs inside R, encoded by the integers $c_1(R)$ and $c_2(R)$. As a consequence, $\mathrm{EAG}(R)$ becomes one of the few algebraically defined graphs for which the complete domination-theoretic landscape can be described without exception.

Applications to several important classes of rings—including reduced rings with finitely many minimal primes, finite products of fields, special principal ideal rings, Artinian rings, polynomial extensions, and power series rings—demonstrated the flexibility and strength of the theory. Many examples were computed explicitly and were illustrated using simple graphical representations. These examples show that while $\mathrm{EAG}(R)$ is combinatorially sparse, it remains highly sensitive to the algebraic structure of R, especially its annihilator behaviour, idempotent decompositions, and nilpotent ideals.

The work opens several potential directions. One may study dynamic graph invariants under ring extensions, homomorphic images, or idealization constructions; further, it would be natural to consider exact annihilating-ideal analogues of other well-known graph invariants such as metric dimension, zero forcing number, independent domination, or Roman k-domination. Another avenue is to investigate whether similar "rigidity phenomena" appear in other annihilator-based graph constructions or in exact dualities of modules over commutative rings.

Overall, the results presented here show that exact annihilation imposes a remarkably stringent combinatorial structure, enabling a complete classification of domination parameters for $\mathrm{EAG}(R)$ and suggesting further connections between ring-theoretic symmetry and extremal graph behaviour.

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Nirali Gor: Examples, Verification, Graph generation, Review and editing.

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datasets were generated or analysed during the current study.

References

[1] G. Aalipour, S. Akbari, R. Nikandish, F. Shahseifi, and M. Zarean, "On the connectedness and diameter of annihilating-ideal graphs," *J. Algebra Appl.* 13(4) (2014).

[2] D.D. Anderson and P.S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebra* 217 (1999), 434–447.

[3] D.F. Anderson and A. Badawi, "The zero-divisor graph of a commutative ring: A survey," in *Lect. Notes Pure Appl. Math.* 241, CRC Press, 2008, pp. 145–167.

[4] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison–Wesley, 1969.

[5] I. Beck, "Coloring of commutative rings," *J. Algebra* 116 (1988), 208–226.

[6] M. Behboodi and Z. Rakeei, "The annihilating-ideal graph of commutative rings I," *J. Algebra Appl.* 10(4) (2011), 727–739.

[7] M. Behboodi and Z. Rakeei, "The annihilating-ideal graph of commutative rings II," *J. Algebra Appl.* 10(4) (2011), 741–753.

[8] A. Cayley, "Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation," *Amer. J. Math.* 1 (1878), 174–176.

[9] S. Visweswaran, "Domination and related results for annihilating-ideal graphs," *Discussiones Mathematicae* 44 (2024), 383–412.

[10] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, 1972.

[11] I. Kaplansky, *Commutative Rings*, University of Chicago Press, 1974.

[12] P.T. Lalchandani, "Exact annihilating-ideal graph of commutative rings," *J. Algebra Related Topics* 5(1) (2017), 27–33.

[13] P.T. Lalchandani, "Exact zero-divisor graphs of commutative rings,", *International Journal of Mathematics and its Applications*, 6(4), (2018), 91–98.

[14] R. Nikandish and H.R. Maimani, "Dominating sets of annihilating-ideal graphs," *Electron. Notes Discrete Math.* 45 (2014), 17–22.

Received: September 15, 2025

International Journal of Applied Mathematics
Volume 38 No. 4, 2025
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

[15] R. Nikandish, H.R. Maimani, and S. Kiani, "Domination number in the annihilating-ideal graphs of commutative rings," *Publ. Inst. Math. (N.S.)* 97 (2015), 225–231.

[16] Z. Tohidi, M.H. Nikmehr, and R. Nikandish, "Further results on dominating sets of annihilating-ideal graphs," *Bull. Iranian Math. Soc.* 43 (2017), 3125–3144.

[17] S. Visweswaran and P.T. Lalchandani, "The exact zero-divisor graph of a reduced ring,", *Indian Journal of Pure and Applied Mathematics*, 52 (2021), 1123–1144.

[18] S. Visweswaran and P.T. Lalchandani, "The exact annihilating-ideal graph of a commutative ring," *J. Algebra Comb. Discrete Appl.* 8(2) (2021), 119–138.

[19] S. Visweswaran and A. Sarman, "Some remarks on ideal-based zero-divisor graphs," *Bull. Iranian Math. Soc.* 42 (2016), 1235–1249.

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