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ON DETERMINING HIGHER COEFFICIENT OF A SECOND ORDER HYPERBOLIC EQUATION BY THE VARIATIONAL METHOD

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Abstract

The paper deals with an inverse problem of determining a higher coefficient of a second order hyperbolic equation. This problem is reduced to an optimal control problem and the new problem is studied by the methods of optimal control theory. It is proved existence theorem for optimal control and obtained necessary condition of optimality in the form integral inequality.

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1. **Introduction**

In direct problems of theory of partial differential equations or in mathematical physics problems the functions that describe various physical phenomena as propagation of heat, sound, various vibrations, electromagnetic waves, etc. are sought. This time, the features of the medium under consideration or coefficients of equations are assumed to be known. However, just the features of medium in great majority of cases are unknown. Then there arise inverse problems in which on the information on the solution of the direct problem it is required to determine the coefficients of equations. As is known, these problems in many cases are ill-posed. But, at the same time, the desired coefficients of the equations characterize the medium under consideration. Therefore, solving inverse problems is very important both from a practical and theoretical point of view [1], [2], [3], [8], [9].

2. **Problem statement**

Let Ω be a bounded domain in the space R^n with a smooth boundary Γ, $T > 0$ be a given number, $Q = \{(x,t) : x \in \Omega, t \in (0,T)\}$ be a cylinder in $R^{n+1}, S = \{(x,t) : x \in \Gamma, t \in (0,T)\}$ be a lateral surface of the cylinder Q.

It is required to determine a pair of functions $(u(x,t), \upsilon(x))$ from the conditions

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)u = f(x,t), (x,t) \in Q, \tag{1}$$

$$u(x,0) = u_0(x),\ \frac{\partial u(x,0)}{\partial t} = u_1(x),\ x \in \Omega,\ u|_S = 0, \tag{2}$$

$$\int_0^T K(x,t)u(x,t)dt = \varphi(x), \tag{3}$$

$$\upsilon = \upsilon(x) \in V, \text{ where}$$
$$V = \{\upsilon(x) \in W_2^1(\Omega) : \nu_0 \le \upsilon(x) \le \mu_0, \left|\frac{\partial \upsilon}{\partial x_i}\right| \le \mu_i,$$
$$i = 1, ..., n, \text{ a.e. on } \Omega\} \tag{4}$$

is a given set, $\nu_0, \mu_0, \mu_1, ..., \mu_n$ are given positive numbers, $a_0(x) \ge 0$, $a_0 \in L_\infty(\Omega)$, $f \in L_2(Q)$, $u_0 \in \overset{0}{W}{}_2^1(\Omega)$, $u_1 \in L_2(Q)$, $K \in L_\infty(Q)$, $\varphi \in L_2(Q)$ are the given functions.

For the given function $\upsilon(x)$ the problem (1), (2) is a direct problem in the domain Q, for the unknown function $\upsilon(x)$ the problem (1)-(4) is said to be an inverse problem to the problem (1), (2). Note that for each fixed function $\upsilon(x) \in V$ the solution of the boundary value problem (1), (2) understood as a generalized solution from the space $W_{2,0}^1(Q)$, [4].

Under the solution from $W_{2,0}^1(Q)$ of the boundary value problem (1), (2) for the given function $\upsilon \in V$ we will understand the function $u = u(x,t)$, equal to $u_0(x)$ for $t = 0$ and satisfying the integral identity

$$\int_Q \left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \upsilon(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} + a_0(x)u\eta\right] dxdt$$
$$-\int_\Omega u_1(x)\eta(x,0)dx = \int_Q f\eta dxdt \tag{5}$$

for all $\eta = \eta(x,t)$ from $W_{2,0}^1(Q)$, equal to zero for $t = T$.

From the results of [[4], p.209-215] follows that under the above assumptions the boundary value problem (1), (2) for each fixed function $\upsilon \in V$ has a unique generalized solution from $W_{2,0}^1(Q)$ and the estimation

$$\|u\|_{W_2^1(Q)} \le c\left[\|u_0\|_{W_2^1(\Omega)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)}\right] \tag{6}$$

is valid. Here and in the sequel, by c we will denote various constants independent of the estimated quantities and admissible controls.

To the problem (1)-(4) we associate the following optimal control problem: it is required to minimize the functional

$$J(\upsilon) = \frac{1}{2}\int_\Omega \left[\int_0^T K(x,t)u(x,t;\upsilon)dt - \varphi(x)\right]^2 dx \tag{7}$$

under the conditions (1), (2), (4), where $u = u(x,t) = u(x,t;\upsilon)-$ is the solution of the boundary value problem (1), (2) corresponding to the function $\upsilon = \upsilon(x) \in V$.

We call the function $\upsilon(x)$ a control, the class $V-$ a set of admissible controls. There is close connection between the problems (1)-(4) and (1), (2), (4), (7) if in the problem (1), (2), (4), (7) $\min\limits_{\upsilon\in V} J(\upsilon) = 0$, then additional integral condition (3) is fulfilled.

In further, in order to avoid possible degeneration in the obtained we consider the following functional condition for optimality:

$$J_\alpha(\upsilon) = J(\upsilon) + \frac{\alpha}{2}\int\limits_\Omega \left[\upsilon^2(x) + \sum_{i=1}^n \left(\frac{\partial \upsilon}{\partial x_i}\right)^2\right] dx$$
$$= J(\upsilon) + \frac{\alpha}{2}\|\upsilon\|^2_{W_2^1(\Omega)}, \tag{8}$$

where $\alpha > 0$ is a given number.

3. On the existence of the solution to problem (1), (2), (4), (8)

Theorem 3.1. *Let the conditions assumed in the statement of problem (1)-(4) be fulfilled. Then the set of optimal controls in the problem (1), (2), (4), (8) $V_* = \{\upsilon_* \in V : J(\upsilon_*) = J_* = \inf\limits_{\upsilon\in V} J(\upsilon)\}$ is non-empty, weakly compact in $W_2^1(\Omega)$ and any minimizing sequence $\{\upsilon^{(m)}\}$ weakly in $W_2^1(\Omega)$ converges to the set V_*.*

P r o o f. It is easy to be convinced that the set V, determined by the relation (4) is weakly compact in $W_2^1(\Omega)$. Show that the functional (8) is weakly, in $W_2^1(\Omega)$ continuous on the set V. Let $\upsilon = \upsilon(x) \in V$ be some element and $\{\upsilon^{(m)}\} \in V$ be such an arbitrary sequence that $\upsilon^{(m)} \to \upsilon$ weakly in $W_2^1(\Omega)$ as $m \to \infty$. Hence, and from the compactness of the embedding $W_2^1(\Omega) \to L_2(\Omega)$ [[5], p.153], it follows that

$$\upsilon^{(m)} \to \upsilon \text{ strongly in } L_2(\Omega) \text{ as } m \to \infty. \tag{9}$$

Due to unique solvability of the boundary value problem (1), (2) each control $\upsilon^{(m)} \in V$ corresponds to a unique generalized solution $u^{(m)} = u(x,t;\upsilon^{(m)})$ of the problem (1), (2) and the estimation $\|u^{(m)}\|_{W_2^1(Q)} \leq c,\ \forall m = 1,2,...,$ valid, i.e. the sequence $\{u^{(m)}\}$ is uniformly bounded by the norm of the space $W_2^1(Q)$. Then from the embedding theorem

[[6], p.116] it follows that from the sequence $\{u^{(m)}\}$ we can select such a subsequence $\{u^{(m_k)}\}$ that as $k \to \infty$

$$u^{(m_k)} \to u \text{ strongly in } L_2(\Omega), \tag{10}$$

$$\frac{\partial u^{(m_k)}}{\partial x_i} \to \frac{\partial u}{\partial x_i},\ i = 1, ..., n,\ \frac{\partial u^{(m_k)}}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L_2(\Omega), \tag{11}$$

where $u = u(x,t) \in W_2^1(Q)$ is some element.

Show that $u(x,t) = u(x,t;\upsilon)$, i.e. the function $u(x,t)$ is a generalized solution of the problem (1), (2), corresponding to the control $\upsilon \in V$. It is clear that for $t = 0$, $u^{(m_k)}(x,0) = u_0(x)$ and the following identities

$$\int\limits_Q \left[-\frac{\partial u^{(m_k)}}{\partial t}\frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \upsilon_i^{(m_k)} \frac{\partial u^{(m_k)}}{\partial x_i}\frac{\partial \eta}{\partial x_i} + a_0 u^{(m_k)} \eta \right] dxdt$$

$$- \int\limits_\Omega u_1(x)\eta(x,0)dx = \int\limits_Q f\eta dxdt \tag{12}$$

are valid for all $\eta = \eta(x,t)$ from $C^1(\bar{Q})$, $\eta|_S = 0$, that equal to zero for $t = T$.

Passing to limit (12) as $k \to \infty$ and using (9)-(11), we get that the function $u(x,t)$ is equal to $u_0(x)$ for $t = 0$ and satisfies the identity (5) for all η from $C^1(\bar{Q})$, $\eta|_S = 0$, $\eta(x,T) = 0$. Since the set of functions $\eta(x,t)$ from $C^1(\bar{Q})$, $\eta|_S = 0$, $\eta(x,T) = 0$ is everywhere dense in the space $W_{2,0}^1(Q)$, that equal to zero for $t = T$, it follows that identity (5) is valid for all the functions $\eta \in W_{2,0}^1(Q)$, $\eta(x,T) = 0$. Hence and from the uniqueness of the solution to problem (1), (2) corresponding to the control $\upsilon \in V$ it follows that $u(x,t) = u(x,t;\upsilon)$.

Now, using the uniqueness of the solution to problem (1), (2), corresponding to the control $\upsilon \in V$, it is easy to verify that relations (10), (11) are valid not only for the subsequence $\{u^{(m_k)}\}$, but also for all the sequence $\{u^{(m)}\}$. Consequently, in particular, the following limit relation is valid

$$u^{(m)} \to u \text{ strongly in } L_2(Q) \text{ as } m \to \infty.$$

Using this relation and the fact that $\upsilon^{(m)}$ weakly in $W_2^1(Q)$ converges to $\upsilon \in V$, from (8) we obtain $\underline{\lim}_{m\to\infty} J_\alpha(\upsilon^{(m)}) \geq J_\alpha(\upsilon)$ as $m \to \infty$ i.e. $J_\alpha(\upsilon)$ weakly in $W_2^1(Q)$ is lower semi-continuous on the set V. Then by virtue of

Theorems 2 and 4 from [[7], p.49, p.51] it follows that all the statements of the theorem are valid and the theorem is proved. □

4. Differentiability of the functional (8) and necessary condition for optimality

Let $\psi = \psi(x,t;\upsilon)$ be a generalized solution from $W_{2,0}^1(Q)$ of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\upsilon(x)\frac{\partial\psi}{\partial x_i}\right) + a_0\psi$$

$$= -K(x,t)\left[\int_0^T K(x,\tau)u(x,\tau;\upsilon)d\tau - \varphi(x)\right],\ (x,t)\in Q, \tag{13}$$

$$\psi(x,T) = 0,\ \frac{\partial\psi(x,T)}{\partial t} = 0,\ x\in\Omega,\ \psi|_S = 0. \tag{14}$$

Under the generalized solution of the boundary value problem (13), (14) for each fixed control $\upsilon\in V$ we will understand the function $\psi = \psi(x,t;\upsilon)$ from $W_{2,0}^1(Q)$, that equal to zero for $t = T$ and satisfying the integral identity

$$\int_Q\left[-\frac{\partial\psi}{\partial t}\frac{\partial g}{\partial t} + \sum_{i=1}^{n}\upsilon(x)\frac{\partial\psi}{\partial x_i}\frac{\partial g}{\partial x_i} + a_0\psi g\right]dxdt$$

$$= -\int_\Omega K(x,t)\left[\int_0^T K(x,\tau)u(x,\tau;\upsilon)d\tau - \varphi(x)\right]g(x,t)dxdt \tag{15}$$

for all $g = g(x,t)$ from $W_{2,0}^1(Q)$ that equal to zero for $t = 0$.

From the results of [4], p.209-215, it follows that the boundary value problem (13), (14) for each fixed control $\upsilon(x)\in V$ has a unique generalized solution from $W_{2,0}^1(Q)$ and the estimation

$$\|\psi\|_{W_2^1(Q)} \le c\left\lfloor\|u\|_{L_2(Q)} + \|\varphi\|_{L_2(\Omega)}\right\rfloor$$

is valid. Taking into account estimation (6), hence we have

$$\|\psi\|_{W_2^1(Q)} \le c\left\lfloor\|u_0\|_{W_2^1(Q)} + \|u_1\|_{L_2(\Omega)} + \|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\Omega)}\right\rfloor. \tag{16}$$

Let the generalized solutions $u = u(x,t;\upsilon)$ and $\psi = \psi(x,t;\upsilon)$ from $W_2^1(Q)$ of problem (1), (2) and (13), (14), respectively, have the derivatives

$$\frac{\partial^2 u}{\partial x_i^2},\ \frac{\partial^2 \psi}{\partial x_i^2},\ i = 1,...,n, \text{ that belong to the space } L_2(Q). \tag{17}$$

THEOREM 4.1. *Let the conditions of Theorem 3.1 and condition (17) be fulfilled. Then the functional (8) is continuously Frechet differentiable on V and its differential at the point $\upsilon \in V$ for the increment $\delta\upsilon \in W_\infty^1(Q)$ is determined by the expression*

$$\langle J_\alpha^{'}(\upsilon), \delta\upsilon\rangle = \int\limits_\Omega \left[\int\limits_0^T \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}dt\right]\delta\upsilon(x)dx$$

$$+ \alpha\int\limits_\Omega \left[\upsilon\delta\upsilon + \sum_{i=1}^n \frac{\partial \upsilon}{\partial x_i}\frac{\partial \delta\upsilon}{\partial x_i}\right]dx. \tag{18}$$

P r o o f. Let us calculate the increment of the functional (8). Let $\delta\upsilon \in W_\infty^1(Q)$ be such an increment of the control on the element $\upsilon \in V$ that $\upsilon + \delta\upsilon \in V$. Denote $\delta u(x,t) = u(x,t;\upsilon+\delta\upsilon) - u(x,t;\upsilon)$. It is clear that the function $\delta u(x,t)$ is the generalized solution from $W_{2,0}^1(Q)$ of the boundary value problem

$$\frac{\partial^2 \delta u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left((\upsilon+\delta\upsilon)\frac{\partial \delta u}{\partial x_i}\right) + a_0\delta u$$

$$= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\delta\upsilon\frac{\partial u}{\partial x_i}\right), \quad (x,t) \in Q, \tag{19}$$

$$\delta u(x,0) = 0,\ \frac{\partial \delta u(x,0)}{\partial t} = 0,\ x \in \Omega,\ \delta u|_S = 0. \tag{20}$$

The generalized solution from $W_{2,0}^1(Q)$ of the problem (19), (20) equals zero for $t = 0$ and satisfies the identity

$$\int\limits_{Q}\left[\frac{\partial\delta u}{\partial t}\frac{\partial\eta}{\partial t}-\sum_{i=1}^{n}(\upsilon+\delta\upsilon)\frac{\partial\delta u}{\partial x_i}\frac{\partial\eta}{\partial x_i}-a_0\delta u\eta\right]dxdt$$
$$=\int\limits_{Q}\sum_{i=1}^{n}\delta\upsilon\frac{\partial u}{\partial x_i}\frac{\partial\eta}{\partial x_i}dxdt \tag{21}$$

for all $\eta=\eta(x,t)\in W_{2,0}^{1}(Q)$, equal to zero for $t=T$.

Let us consider the increment of the functional (8):

$$\triangle J_\alpha(\upsilon)=J_\alpha(\upsilon+\delta\upsilon)-J_\alpha(\upsilon)$$
$$=\frac{1}{2}\int\limits_{\Omega}\Big[\int\limits_{0}^{T}K(x,t)[(u+\delta u)dt-\varphi(x)]^2dx$$
$$-\frac{1}{2}\int\limits_{\Omega}\left[\int\limits_{0}^{T}K(x,t)udt-\varphi(x)\right]^2dx\Big]$$
$$+\frac{\alpha}{2}\int\limits_{\Omega}\left[(\upsilon+\delta\upsilon)^2-\upsilon^2+\sum_{i=1}^{n}\left(\frac{\partial(\upsilon+\delta\upsilon)}{\partial x_i}\right)^2-\sum_{i=1}^{n}\left(\frac{\partial\upsilon}{\partial x_i}\right)^2\right]dx$$
$$=\int\limits_{\Omega}\left[\int\limits_{0}^{T}K(x,\tau)ud\tau-\varphi(x)\right]\int\limits_{0}^{T}K(x,t)\delta udtdx$$
$$+\frac{1}{2}\int\limits_{\Omega}\left[\int\limits_{0}^{T}K(x,t)\delta udt\right]^2dx+\alpha\int\limits_{\Omega}\left(\upsilon\delta\upsilon+\sum_{i=1}^{n}\frac{\partial\upsilon}{\partial x_i}\frac{\partial\delta\upsilon}{\partial x_i}\right)dx$$
$$+\frac{\alpha}{2}\int\limits_{\Omega}\left[(\delta\upsilon)^2+\sum_{i=1}^{n}\left(\frac{\partial\delta\upsilon}{\partial x_i}\right)^2\right]dx. \tag{22}$$

If in (15) we put $g=\delta u(x,t)$, and in (21) $\eta=\psi(x,t;\upsilon)$ and sum the obtained relations, we have

$$-\int_Q \sum_{i=1}^n \frac{\partial \delta u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta\upsilon(x)dxdt = \int_Q \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta\upsilon(x)dxdt$$

$$-\int_Q K(x,t)\left[\int_0^T K(x,\tau)u(x,\tau;\upsilon)d\tau-\varphi(x)\right]\delta u(x)dxdt.$$

Taking this equality into account in (22), we obtain

$$\triangle J_\alpha(\upsilon) = \int_\Omega \left(\int_0^T \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}dt\right)\delta\upsilon(x)dx$$

$$+\alpha\int_\Omega \left[\upsilon\delta\upsilon+\sum_{i=1}^n \frac{\partial \upsilon}{\partial x_i}\frac{\partial \delta\upsilon}{\partial x_i}\right]dx + R, \tag{23}$$

where

$$R = \int_Q \sum_{i=1}^n \frac{\partial \delta u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta\upsilon(x)dxdt$$

$$+\frac{1}{2}\int_\Omega \left[\int_0^T K(x,t)\delta u\delta t\right]^2 dx + \frac{\alpha}{2}\int_\Omega \left[(\delta\upsilon)^2+\sum_{i=1}^n\left(\frac{\partial\delta\upsilon}{\partial x_i}\right)^2\right]dx$$

is a residual term.

It is clear that the sum of the first and second addents in the right hand side of (23) for the given $\upsilon \in V$ determines the linear bounded functional from $\delta\upsilon$ on $W^1_\infty(\Omega)$.

We now estimate the residual term R. For solution of problem (19), (20), as in [4], p.209-215, we can obtain the following estimation:

$$\|\delta u\|_{W^1_2(Q)}$$

$$\leq c\left[\sum_{i=1}^n \left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{L_2(Q)}\|\delta\upsilon\|_{L_\infty(\Omega)}+\sum_{i=1}^n\left\|\frac{\partial \upsilon}{\partial x_i}\right\|_{L_\infty(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L_2(Q)}\right].$$

Here, taking into account (6) and conditions (17) we have

$$\|\delta u\|_{W^1_2(Q)} \leq c\|\delta\upsilon\|_{W^1_\infty(\Omega)}. \tag{24}$$

Taking into account this estimation from the expression of R we obtain

$$|R| \leq \sum_{i=1}^{n} \left\| \frac{\partial \delta u}{\partial x_i} \right\|_{L_2(Q)} \left\| \frac{\partial \psi}{\partial x_i} \right\|_{L_2(Q)} \|\delta \upsilon\|_{L_\infty(\Omega)} + c\|\delta u\|^2_{L_2(Q)}$$
$$+ c\|\delta \upsilon\|^2_{W^1_\infty(\Omega)} \leq c\|\delta \upsilon\|^2_{W^1_\infty(\Omega)}.$$

Then it follows from formula (23) that the functional (8) is Frechet differentiable on V and formula (18) is valid. Show that the mapping $\upsilon \to J'(\upsilon)$, determined by the equality (18) continuously acts from V to the adjoint space $(W^1_\infty(\Omega))^*$ of $W^1_\infty(\Omega)$. Let $\delta\psi = \psi(x,t;\upsilon+\delta\upsilon) - \psi(x,t;\upsilon)$. Then it follows from (13), (14) that $\delta\psi$ is a generalized solution from $W^1_{2,0}(Q)$ of the boundary value problem

$$\frac{\partial^2 \delta\psi}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left((\upsilon+\delta\upsilon)\frac{\partial \delta\psi}{\partial x_i}\right) + a_0\delta\psi = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\delta\upsilon \frac{\partial \psi}{\partial x_i}\right)$$
$$- K(x,t)\int_0^T K(x,\tau)\delta u(x,\tau)d\tau, \ (x,t) \in Q,$$
$$\delta\psi|_{t=T} = 0, \ \frac{\partial \delta\psi}{\partial t}|_{t=T} = 0, \ x \in \Omega, \ \delta\psi|_S = 0.$$

Reasoning in the same way as estimation (24) was obtained, we can show that for solution of this problem the estimation

$$\|\delta\psi\|_{W^1_2(Q)} \leq c\|\delta\upsilon\|_{W^1_\infty(\Omega)}. \tag{25}$$

is valid.

Furthermore, using (18) and the Cauchy-Bunyakovskii inequality, it is not difficult to verify the validity of the inequality

$$\|J'_\alpha(\upsilon+\delta\upsilon) - J'_\alpha(\upsilon)\|_{(W^1_\infty(\Omega))^*}$$
$$\leq \sum_{i=1}^{n} \Big[\left\| \frac{\partial \delta u}{\partial x_i} \right\|_{L_2(Q)} \left\| \frac{\partial \psi}{\partial x_i} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(Q)} \left\| \frac{\partial \delta\psi}{\partial x_i} \right\|_{L_2(Q)}$$
$$+ \left\| \frac{\partial \delta u}{\partial x_i} \right\|_{L_2(Q)} \left\| \frac{\partial \delta\psi}{\partial x_i} \right\|_{L_2(Q)} \Big] + c\|\delta\upsilon\|_{W^1_\infty(\Omega)}.$$

By virtue of (24) and (25) the right hand side of this inequality tends to zero as $\|\delta\upsilon\|_{W^1_\infty(\Omega)} \to 0$. Hence it follows that $\upsilon \to J'(\upsilon)$ is a continuous mapping from V to $(W^1_\infty(\Omega))^*$. Theorem 4.1 is proved. □

Theorem 4.2. *Let the conditions of Theorem 4.1 be fulfilled. Then for the optimality of the control $\upsilon_* = \upsilon_*(x) \in V$ in the problem (1), (2), (4), (8) it is necessary for the inequality*

$$\int_\Omega \left[\int_0^T \sum_{i=1}^n \frac{\partial u_*}{\partial x_i}\frac{\partial \psi_*}{\partial x_i} dt\right](\upsilon(x) - \upsilon_*(x))dx$$
$$+\alpha\int_\Omega \left[\upsilon_*(x)(\upsilon(x)-\upsilon_*(x)) + \sum_{i=1}^n \frac{\partial \upsilon_*}{\partial x_i}\left(\frac{\partial \upsilon(x)}{\partial x_i} - \frac{\partial \upsilon_*(x)}{\partial x_i}\right)\right]dx \geq 0 \tag{26}$$

to hold for any $\upsilon = \upsilon(x) \in V$ where $u_* = u(x,t;\upsilon_*)$ and $\psi_* = \psi(x,t;\upsilon_*)-$ are the solutions of the problem (1), (2) and (13), (14), respectively for $\upsilon = \upsilon_*(x)$.

P r o o f. The set V determined by the relation (4) is convex in $W_\infty^1(\Omega)$. Furthermore, according to Theorem 4.1, the functional $J(\upsilon)$ is continuously Frechet differentiable on V and its differential at the point $\upsilon \in V$ is determined by the expression (18). Then by virtue of Theorem 5 from [7], p.28, on the element $\upsilon_* \in V_*$ it is necessary for the inequality $\langle J'(\upsilon_*), \upsilon - \upsilon_*\rangle \geq 0$ to be fulfilled for all $\upsilon \in V$. Hence and from (18) we have the validity of inequality (26). Theorem 4.2 is proved. □

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