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THE FROBENIUS NUMBER OF A CLASS OF NUMERICAL SEMIGROUPS WITH DIMENSION 4

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Abstract

In this paper we present a formula for Frobenius number of numerical semigroup G with emmbeding dimension 4 such that $G = <n, x, y, z>$, $n < x < y < z$, $x \equiv 1(modn)$, $y \equiv j(modn)$, $z \equiv k(modn)$ and $x+y+z = tn$, for $t \geq 2$ and $t \in \mathbf{N}$.

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1. Introduction

Numerical semigroups play an important role in the algebraic theory of semigroups. Formally, a nonempty subset G of $\mathbf{N}_0$ is called a numerical semigroup if G is closed under addition, contains 0 and $\mathbf{N}_0 \setminus G$ is a finite set. A nonempty set $S = \{n_1, ..., n_m\}$ is called a set of generators for G, denoted by $G =< S >$ or $G =< n_1, ..., n_m >$, if $S \subseteq G$ and

$$G = \{\alpha_1 n_1 + ... + \alpha_1 n_m \mid \alpha_1, ..., \alpha_m \in \mathbf{N}_0\}.$$

Let S be a set of generators for G. The set S is called a minimal set of generators for G if no proper subset of S is a set of generators for G. The cardinality of a minimal set of generators is called embedding dimension of G, denoted by $ed(G)$. The smallest nonzero element of G is called the multiplicity of G and it is denoted by n. The set $\mathbf{N}_0 \setminus G$ is called the set of gaps of G. Its elements cannot be presented as a linear combination of its minimal set of generators with nonnegative integer coefficients. The largest gap of G is called Frobenius number and it is denoted by $F(G)$. The cardinality of the set of gaps of G is called genus of G, denoted by $g(G)$. The Apéry set of G with respect to a nonzero element $a \in G$ is defined by

$$Ap(G, a) = \{s \in G \mid s - a \notin G\}.$$

The Apéry set corresponding to a nonzero element $a \in G$ is a finite set with a elements, one for each congruence class modulo a. This set plays a significant role in the study of numerical semigroups, as knowing the Apéry set allows the determination of other invariants of the semigroups, regardless of its dimension. In this context, for any nonzero element $a \in G$ the following holds

$$F(G) = \max(Ap(G, a)) - a.$$

The results in this paper are based on the algorithm presented in [5], which for given natural numbers n, j and k, where $1 < j, k < n$ and $k \neq j$ generates all the (n, j, k)-good matrices $M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$, corresponding to a given (n, j)-good 2×2 matrix $K_0 = \begin{bmatrix} a_0 & -u_0 \\ -b_0 & v_0 \end{bmatrix}$, such that $a \leq a_0$ and $v \leq v_0$. This algorithm is a summary of the results obtained in [1], [2], [3], [4], [6], [7] and [9]. For the purposes of this paper, the following definitions and theorems will be presented.

Definition 1.1. For an integer 2×2 matrix $M = \begin{bmatrix} a & -u \\ -b & v \end{bmatrix}$ we say that it is (n, j)-good if:

(1) $a, b, u, v \in \mathbf{N}_0$, $a > 1, v > 1, a > b, a > u, v > u$ and $a + v \leq n + 1$;
(2) $det M = av - bu = n$, and
(3) The commutative group $H(M)$ with the presentation $\langle g, h \mid g^a = h^u, h^v = g^b \rangle$ is the cyclic group of order n, generated by g and $h = g^j$ in $H(M)$.

For an (n, j)-good matrix M, let

$$P(M) = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbf{N}_0, 0 \leq \alpha < a - b, 0 \leq \beta < v\}$$
$$\cup \{(\alpha, \beta) \mid \alpha, \beta \in \mathbf{N}_0, 0 \leq \alpha < a, 0 \leq \beta < v - u\}.$$

Definition 1.2. Let $n, j, k \in \mathbf{N}_0, n > k$ and $j \neq k$. We say that a matrix $M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ is (n, j, k)-nice, if:

(1.1) $a, b, c, v, u, w, r, p, q$ are nonnegative integers smaller than n;
(1.2) $a, v, r > 1$; $a + v + r \leq n + 2$;
(1.3) $a > b, c$; $a > u + p$; $v > u, w$; $v > q$; $a + v > u + p + b + q$; $r > p, q$;
(1.4) if $a - b - c \leq 0$ then $0 < v - u - w$, $0 < r - p - q$ and $(0 < v - 2u - w$ or $0 < r - 2p - q)$;
(1.5) if $v - u - w \leq 0$ then $0 < a - b - c, 0 < r - p - q$ and $(0 < a - 2b - c$ or $0 < r - p - 2q)$;
(1.6) if $r - p - q \leq 0$ then $0 < a - b - c$, $40 < v - u - w$ and $(0 < a - b - 2c$

or $0 < v - u - 2w$);
(1.7) $ar - pc > ur + pw$, $av - ub > uq + vp$ and $av - ub > bp + a$;
(1.8) if d is a divisor of a, u, p, then d is a divisor of n;
(1.9) if d is a divisor of b, v, q, then d is a divisor of n;
(1.10) if d is a divisor of c, w, r, then d is a divisor of n;
(1.11) n is a divisor of $detM$;
(1.12) If $r \geq c + w$ then for each $r' < r$ with $r'k \equiv c' + w'j(modn)$, $c' + w' > r'$, and
(1.13) The commutative group $H(M)$ with presentation

$$\langle g, h, f | g^a = h^u f^p, h^v = g^b f^q, f^r = g^c h^w, g^n = 1 \rangle$$

is the cyclic group of order n, generated by $g, h = g^j$ in $H(M)$ and $f = g^k$ in $H(M)$.

Definition 1.3. Let M be an (n, j, k)-nice matrix. Define $P(M) \subseteq \mathbf{N}_0^3$ to be the maximal set of triples of nonnegative integers (α, β, γ) satisfying the following conditions:

(1) if $(\alpha, \beta, \gamma) \in P(M)$,$\alpha > 0$ then $(\alpha - 1, \beta, \gamma) \in P(M)$;
(2) if $(\alpha, \beta, \gamma) \in P(M)$,$\beta > 0$ then $(\alpha, \beta - 1, \gamma) \in P(M)$;
(3) if $(\alpha, \beta, \gamma) \in P(M)$,$\gamma > 0$ then $(\alpha, \beta, \gamma - 1) \in P(M)$;
(4) $(a - 1, 0, 0), (0, v - 1, 0), (0, 0, r - 1) \in P(M)$;
(5) $(a, 0, 0), (0, v, 0), (0, 0, r) \notin P(M)$;
(6) if $g^\alpha = h^\beta f^\gamma$ in $H(M)$ and $0 \leq \alpha < a$, then $(0, \beta, \gamma) \notin P(M)$;
(7) if $h^\beta = g^\alpha f^\gamma$ in $H(M)$ and $0 \leq \beta < v$, then $(\alpha, 0, \gamma) \notin P(M)$;
(8) if $f^\gamma = g^\alpha h^\beta$ in $H(M)$ and $0 \leq \gamma < r$, then $(\alpha, \beta, 0) \notin P(M)$, and
(9) if $1 = g^\alpha h^\beta f^\gamma$ in $H(M)$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, then $(\alpha, \beta, \gamma) \notin P(M)$.

If the (n, j, k)-nice matrix M from the previous definition satisfies $|P(M)| = n$, then we say that M is an (n, j, k)-good matrix.

Theorem 1.1. *Let $G = <n, x, y>$ be a numerical semigroup with $ed(G) = 3$, $n < x < y$, $x \equiv 1(modn)$ and $y \equiv j(modn)$. Then, there is a unique (n, j)-good matrix $M = \begin{bmatrix} a & -u \\ -b & v \end{bmatrix}$ with $vy > bx$, so that the Apéry set of G with respect to n is:*

$$Ap(G, n) = \{\alpha x + \beta y \mid (\alpha, \beta) \in P(M)\}.$$

We note that for numerical semigroup G from the above theorem, the following also hold: $ax \equiv uy(modn)$ and $vy \equiv bx(modn)$.

Theorem 1.2. *Let* G=<n, x, y, z> *be a numerical semigroup with* $ed(G) = 4$, $n < x < y < z$, $x \equiv 1(modn)$,$y \equiv j(modn)$, $z \equiv k(modn)$ *and* $GCD(n, y) = 1$. *Then, there is a unique* (n, j, k)*-good matrix* $M = M(G) = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ *with* $rz > cx + wy$ *and* $vy > bx + qz$, *so that the Apéry set of* G *with respect to* n *is:*

$$Ap(G, n) = \{\alpha x + \beta y + \gamma z \mid (\alpha, \beta, \gamma) \in P(M)\}.$$

So, the Apéry set of G = <n, x, y, z> *with respect to* n *is*

$$Ap(G, n) = \{\alpha x + \beta y + \gamma z \mid (\alpha, \beta, \gamma) \in P(M)\}$$

and $|P(M)| = n$.

2. The main result

In continuation we will focus on determining the Frobenius number of a class of numerical semigroups with dimension 4, using the algorithm presented in [5]. First, we note that if G is a numerical semigroup with embedding dimension equal to 4, there exists a correspondence between $Ap(G, a)$ and a staircase shaped body of lattice points in the space with volume equal to a, where a is a nonzero element of G. The shape of this body is not uniquely determined and depends on the inequalities between the elements of the minimal set of generators of G.

Let G be a numerical semigroup such that G = <n, x, y, z>, $ed(G) = 4$, $x \equiv 1(modn)$, $y \equiv j(modn)$, $z \equiv k(modn)$, $n < x < y < z$ and $x+y+z = tn$, for $t \geq 2$ and $t \in \mathbf{N}$. Applying Theorem 1.1 we have that:

− for numerical semigroup G_1 = <n, x, y>, there is a unique (n, j)-good matrix $\begin{bmatrix} a_1 & -u_1 \\ -b_1 & v_1 \end{bmatrix}$ with $v_1 y > b_1 x$,$a_1 x \equiv u_1 y(modn)$ and $v_1 y \equiv b_1 x(modn)$;
− for numerical semigroup G_2 = <n, x, z> there is a unique (n, k)-good matrix $\begin{bmatrix} a_2 & -u_2 \\ -b_2 & v_2 \end{bmatrix}$ with $v_2 z > b_2 x$, $a_2 x \equiv u_2 z(modn)$ and $v_2 z \equiv b_2 x(modn)$;

– for numerical semigroup $G_3 = <n, y, z>$ there is a unique (n, kj^{-1})-good matrix $\begin{bmatrix} a_3 & -u_3 \\ -b_3 & v_3 \end{bmatrix}$ with $v_3z > b_3y$,$a_3y \equiv u_3z(modn)$ and $v_3z \equiv b_3y(modn)$.

We start with the assumption that $a_1 \leq a_2$ and $v_1 \leq a_3$. For $a_1 \leq a_2$ and $a_1 - b_1 > v_1 - u_1$ we have that

$$(v_1 - u_1)z \equiv (a_1 - b_1 - 1 - (v_1 - u_1 - 1))x(modn) \text{ and}$$
$$0 < (a_1 - b_1 - 1 - (v_1 - u_1 - 1))x < (a_1 - b_1)x < a_2x.$$

Hence, $(a_1 - b_1 - 1 - (v_1 - u_1 - 1))x \in Ap(G_2, n)$ and $(v_1 - u_1)z \geq (a_1 - b_1 - 1 - (v_1 - u_1 - 1))x$. Similarly, it can be shown that if $v_1 \leq a_3$ and $a_1 - b_1 \leq v_1 - u_1$ we obtain that

$$(a_1 - b_1)z \equiv (v_1 - u_1 - 1 - (a_1 - b_1 - 1))y(modn) \text{ and}$$
$$(a_1 - b_1)z \geq (v_1 - u_1 - 1 - (a_1 - b_1 - 1))y,$$

i.e., $(v_1 - u_1 - 1 - (a_1 - b_1 - 1))y \in Ap(G_3, n)$.

Since $(a_1 - b_1)x + (v_1 - u_1)y \equiv 0(modn)$ and $z \equiv -x - y(modn)$ we conclude that

$$z \equiv (a_1 - b_1 - 1)x + (v_1 - u_1 - 1)y(modn) \text{ i.e.,}$$
$$rz \equiv (a_1 - b_1 - r)x + (v_1 - u_1 - r)y(modn)$$

for $r = 1, \ldots, a_1 - b_1$ and $a_1 - b_1 > v_1 - u_1$ (or, for $r = 1, \ldots, v_1 - u_1$ and $a_1 - b_1 \leq v_1 - u_1$). Hence, the elements of $Ap(G, n)$ will be points lying in the xy, yz and xz planes. Thus, to determine the $Ap(G, n)$ and subsequently the Frobenius number of G it suffices to consider the elements of the sets:

$D_{yz} = \{(0, \beta, \gamma)|1 \leq \beta < v, 1 \leq \gamma < r, (0, \beta, \gamma) \notin P(M), (0, \beta - 1, \gamma), (0, \beta, \gamma - 1) \in P(M)\}$,

$D_{xz} = \{(\alpha, 0, \gamma)|1 \leq \alpha < a, 1 \leq \gamma < r, (\alpha, 0, \gamma) \notin P(M), (\alpha-1, 0, \gamma), (\alpha, 0, \gamma - 1) \in P(M)\}$ and

$D_{xy} = \{(\alpha, \beta, 0)|1 \leq \alpha < a, 1 \leq \beta < v, (\alpha, \beta, 0) \notin P(M), (\alpha - 1, \beta, 0), (\alpha, \beta - 1, 0) \in P(M)\}$,

where M is (n, j, k)-good matrix corresponding to numerical semigroup G. These set can also be obtained by the algorithm presented in [5].

Since for numerical semigroup $G_1 = <n, x, y>$ we have that $a_1x \equiv u_1y(modn)$ and $v_1y \equiv b_1x(modn)$ together with $x + y + z \equiv 0(modn)$ we obtain that

$$(a_1 - 1)x \equiv ((u_1 + 1)y + z)(modn) \text{ and}$$
$$(v_1 - 1)y \equiv ((b_1 + 1)x + z)(modn).$$

Hence, four cases are possible, which can also be obtained by the algorithm presented in [5]. Namely, we consider the following cases:

i) Let $K_0 = \begin{bmatrix} a_1 & -u_1 \\ -b_1 & v_1 \end{bmatrix}$ Then $c' = (a_1 - b_1 - 1)$ and $w' = (v_1 - u_1 - 1)$. From

$$\lceil \tfrac{a_1-b_1}{a_1-b_1-1} \rceil = \lceil \tfrac{v_1-u_1}{v_1-u_1-1} \rceil = 2,\ \lceil \tfrac{a_1}{a_1-b_1-1} \rceil \geq 2 \text{ and } \lceil \tfrac{v_1}{v_1-u_1-1} \rceil \geq 2,$$

we conclude that

$$r_0 = \min\{\lceil \tfrac{a_1}{a_1-b_1-1} \rceil, \lceil \tfrac{v_1}{v_1-u_1-1} \rceil, \max\{\lceil \tfrac{(a_1-b_1}{(a_1-b_1-1} \rceil, \lceil \tfrac{v_1-u_1}{v_1-u_1-1} \rceil\}\} = 2.$$

Thus, we obtain the first semigroup determined by following (n, j, k)-good matrix

$$\begin{bmatrix} a_1 & -u_1 & 0 \\ -b_1 & v_1 & 0 \\ -(a_1 - b_1 - 2) & -(v_1 - u_1 - 2) & 2 \end{bmatrix}$$

and the sets $D_{yz} = \{(0, u_1 + 1, 1)\}, D_{xz} = \{(b_1 + 1, 0, 1)\}$ and $D_{xy} = \{(a_1 - b_1 - 1, v_1 - u_1 - 1, 0)\}$. Hence

$$Ap(G, n) = \{\alpha x + \beta y \mid 0 \leq \alpha \leq a_1 - b_1 - 2, 0 \leq \beta \leq v_1 - 1\} \cup \{\alpha x + \beta y \mid a_1 - b_1 - 1 \leq \alpha \leq a_1 - 1, 0 \leq \beta \leq v_1 - u_1 - 2\} \cup \{\alpha x + z \mid 0 \leq \alpha \leq b_1\} \cup \{\alpha y + z \mid 0 \leq \alpha \leq u_1\}.$$

In fact, this numerical semigroup is obtained when $(a_1-1)x \leq (u_1+1)y+z$ and $(v_1 - 1)y \leq (b_1 + 1)x + z$.

ii) Let $(a_1 - 1)x > (u_1 + 1)y + z$ and $(v_1 - 1)y \leq (b_1 + 1)x + z$. These imply that the y border of the staircase shaped body corresponding to G is determined by the inequality $v_1 y > b_1 x$. Furthermore, for $r \geq 2$ the inequality

$$(a_1 - (r - 2))x > (u_1 + (r - 2))y + (r - 2)z$$

holds, because otherwise, from the inequalities

$$(a_1 - (r - 2))x \leq (u_1 + (r - 2))y + (r - 2)z \text{ and}$$
$$(r - 1)z < (a_1 - b_1 - 1 - ((r - 1) - 1))x + (v_1 - u_1 - 1 - ((r - 1) - 1))y$$

we have that

$$(b_1 + 1)x + z < (v_1 - 1)y,$$

which contradicts the condition. Therefore, according to the algorithm presented in [5], for this case the (n, j, k)-good matrix and the sets D_{yz}, D_{xz} and D_{xy} corresponding to G depend on which of the following two conditions is satisfied:

$-$ If $(a_1 - (r - 1))x > (u_1 + (r - 1)y + (r - 1)z$, then

$$\begin{bmatrix} a_1-r+1 & -(u_1+r-1) & -(r-1) \\ -b_1 & v_1 & 0 \\ -(a_1-b_1-r) & -(v_1-u_1-r) & r) \end{bmatrix},$$

$D_{yz}=\emptyset, D_{xz}=\{(b_1+1,0,1)\}$ and $D_{xy}=\{(a_1-b_1-r+1,v_1-u_1-r+1,0)\}$. Hence

$Ap(G,n)=\{\alpha x+\beta y \mid 0\le\alpha\le a_1-b_1-r, 0\le\beta\le v_1-1\}\cup\{\alpha x+\beta y \mid a_1-b_1-r+1\le\alpha\le a_1-r, 0\le\beta\le v_1-u_1-r\}\cup\{\alpha x+\beta z \mid, 0\le\alpha\le b_1, 0\le\beta\le r-1\}\cup\{\alpha y+\beta z \mid\le\alpha\le v_1-1, 0\le\beta\le r-1\}$.

– If $(a_1-(r-1))x\le(u_1+(r-1)y+(r-1)z$ then

$$\begin{bmatrix} a_1-r+2 & -(u_1+r-2) & -(r-2) \\ -b_1 & v_1 & 0 \\ -(a_1-b_1-r) & -(v_1-u_1-r) & r) \end{bmatrix},$$

$D_{yz}=\{(0,u_1+r-1,r-1)\}, D_{xz}=\{(b_1+1,0,1)\}$ and $D_{xy}=\{(a_1-b_1-r+1,v_1-u_1-r+1,0)\}$. Hence,

$Ap(G,n)=\{\alpha x+\beta y \mid 0\le\alpha\le a_1-b_1-r, 0\le\beta\le v_1-1\}\cup\{\alpha x+\beta y \mid a_1-b_1-r+1\le\alpha\le a_1-r+1, 0\le\beta\le v_1-u_1-r\}\cup\{\alpha x+\beta z \mid 0\le\alpha\le b_1, 0\le\beta\le r-1\}\cup\{\alpha y+\beta z \mid 0\le\alpha\le v_1-1, 0\le\beta\le r-2\}\cup\{\alpha y+(r-1)z \mid 0\le\alpha\le u_1-1+(r-1)\}$.

iii) Let $(a_1-1)x\le(u_1+1)y+z$ and $(v_1-1)y>(b_1+1)x+z$. These imply that the x border of the staircase shaped body corresponding to G is determined by the inequality $a_1x>u_1y$. Furthermore, for $r\ge 2$ the inequality

$$(v_1-(r-2))y>(b_1+(r-2))x+(r-2)z$$

holds, because otherwise, from the inequalities

$$(v_1-(r-2))y\le(b_1+(r-2))x+(r-2)z \text{ and}$$
$$(r-1)z<(a_1-b_1-1-((r-1)-1))x+(v_1-u_1-1-((r-1)-1))y,$$

we obtain that $(u_1+1)y+z<(a_1-1)x$, which contradicts the condition. Therefore, according to the algorithm presented in [5], for this case the (n,j,k)-good matrix and the sets D_{yz}, D_{xz} and D_{xy} corresponding to G depend on which of the following two conditions is satisfied:

– If $(v_1-(r-1))y\le(b_1+(r-1))x+(r-1)z$, then

$$\begin{bmatrix} a_1 & -u_1 & 0 \\ -(b_1+(r-2)) & (v_1-(r-2)) & -(r-2) \\ -(a_1-b_1-r) & -(v_1-u_1-r) & r \end{bmatrix},$$

$D_{yz}=\{(0,u_1+1,1)\}, D_{xz}=\{(b_1+r-1,0,r-1)\}$ and $D_{xy}=\{(a_1-b_1-r+1,v_1-u_1-r+1,0)\}$. Hence,

$$Ap(G,n) = \{\alpha x + \beta y \mid 0 \le \alpha \le a_1 - b_1 - r, 0 \le \beta \le v_1 - (r-1)\} \cup \{\alpha x + \beta y \mid a_1 - b_1 - r + 1 \le \alpha \le a_1 - 1, 0 \le \beta \le v_1 - u_1 - r\} \cup \{\alpha x + \beta z \mid 0 \le \alpha \le a_1 - 1, 0 \le \beta \le r-2\} \cup \{\alpha y + \beta z \mid 0 \le \alpha \le u_1, 1 \le \beta \le r-1\} \cup \{\alpha x + (r-1)z \mid 0 \le \alpha \le b_1 + r - 2\}.$$

– If $(v_1 - (r-1))y > (b_1 + (r-1))x + (r-1)z$ then

$$\begin{bmatrix} a_1 & -u_1 & 0 \\ -(b_1 + (r-1)) & v_1 - (r-1) & -(r-1) \\ -(a_1 - b_1 - r) & -(v_1 - u_1 - r) & r \end{bmatrix},$$

$D_{yz} = \{(0, u_1 + 1, 1)\}, D_{xz} = \emptyset$ and $D_{xy} = \{(a_1 - b_1 - r + 1, v_1 - u_1 - r + 1, 0)\}$. Hence,

$$Ap(G,n) = \{\alpha x + \beta y \mid 0 \le \alpha \le a_1 - b_1 - r, 0 \le \beta \le v_1 - r\} \cup \{\alpha x + \beta y \mid a_1 - b_1 - r + 1 \le \alpha \le a_1 - 1, 0 \le \beta \le v_1 - u_1 - r\} \cup \{\alpha x + \beta z \mid 0 \le \alpha \le a_1 - 1, 0 \le \beta \le r-1\} \cup \{\alpha y + \beta z \mid 0 \le \alpha \le u_1, 0 \le \beta \le r-1\}.$$

iv) Let $(a_1 - 1)x > (u_1 + 1)y + z$ and $(v_1 - 1)y > (b_1 + 1)x + z$. For $r > 2$, from inequalities

$$(v_1 - 1)y > (b_1 + 1)x + z \text{ and } rz \ge (a_1 - b_1 - r)x + (v_1 - u_1 - r)y$$

we have that

$$(u_1 + 1 + (r-2))y + (1 + (r-2))z > (a_1 - 1 - (r-2))x.$$

Therefore, there exists a largest nonnegative integer m such that $0 \le m < r - 2$ and

$$(a_1 - 1 - m)x \ge (u_1 + 1 + m)y + (1 + m)z.$$

From $(a_1 - 1 - m)x \ge (u_1 + 1 + m)y + (1 + m)z$ and $rz \ge (a_1 - b_1 - r)x + (v_1 - u_1 - r)y$ we obtain that

$$((r-2-m)+1)z + (b_1 + (r-m-2)+1)x \equiv (v_1 - 1 - (r-2-m))y (mod n)$$

and

$$((r-2-m)+1)z + (b_1 + (r-m-2)+1)x \ge (v_1 - 1 - (r-2-m))y$$

The last inequality will be denoted by $(*)$. Let p be the largest nonnegative integer such that $0 \le p < r - m - 2$ and for which the inequality

$$(p+1)z + (b_1 + p + 1)x < (v_1 - 1 - p)y$$

holds. Such a p exists due to the inequalities $(*)$ and $(v_1 - 1)y > (b_1 + 1)x + z$. Therefore, according to the algorithm presented in [5], we obtain that G is determined by following (n, j, k)-good matrix and the sets D_{yz}, D_{xz} and D_{xy}:

$$\begin{bmatrix} (a_1 - 1 - m & -(u_1 + 1 + m) & -(1 + m) \\ -(b_1 + p + 1) & (v_1 - 1 - p) & -(p+1) \\ -(a_1 - b_1 - k) & -(v_1 - u_1 - k) & k \end{bmatrix},$$

$D_{yz} = \{(0, u_1 + m + 2, m + 2)\}$, $D_{xz} = \{(b_1 + p + 2, 0, p + 2)\}$ and $D_{xy} = \{(a_1 - b_1 - r + 1, v_1 - u_1 - r + 1, 0)\}$. Hence,

$$Ap(G, n) = \{\alpha x + \beta y \mid 0 \leq \alpha \leq a_1 - m - 2, 0 \leq \beta \leq v_1 - u_1 - r\} \cup \{\alpha x + \beta y \mid 0 \leq \alpha \leq a_1 - b_1 - r, 0 \leq \beta \leq v_1 - p - 2\} \cup \{\alpha x + \beta z \mid 0 \leq \alpha \leq a_1 - m - 2, 0 \leq \beta \leq p + 1\} \cup \{\alpha x + \beta z \mid 0 \leq \alpha \leq b_1 + p + 1, p + 2 \leq \beta \leq r - 1\} \cup \{\alpha y + \beta z \mid 0 \leq \alpha \leq v_1 - p - 2, 0 \leq \beta \leq m + 1\} \cup \{\alpha y + \beta z \mid 0 \leq \alpha \leq u_1 + m + 1, m + 2 \leq \beta \leq r - 1\}.$$

From the discussion above we obtain the following result:

THEOREM 2.1. *Let G be a numerical semigroup such that $G = <n, x, y, z>$, $ed(G) = 4$, $x \equiv 1(mod n)$, $y \equiv j(mod n)$, $z \equiv k(mod n)$, $n < x < y < z$ and $x + y + z = tn$, for $t \geq 2$ and $t \in \mathbf{N}$. Let $\begin{bmatrix} a_1 & -u_1 \\ -b_1 & v_1 \end{bmatrix}$, $\begin{bmatrix} a_2 & -u_2 \\ -b_2 & v_2 \end{bmatrix}$ and $\begin{bmatrix} a_3 & -u_3 \\ -b_3 & v_3 \end{bmatrix}$ be the (n, j)-good, (n, k)-good and (n, kj^{-1})-good matrices cornresponding to nurmerical semigroups $<n, x, y>$, $<n, x, z>$ and $<n, y, z>$, respectively, such that $a_1 \leq a_2$ and $v_1 \leq a_3$. Then:*

i) If $(a_1 - 1)x \leq (u_1 + 1)y + z$ and $(v_1 - 1)y \leq ((b_1 + 1)x + z)$ then

$$F(G) = \max\{(a_1 - b_1 - 2)x + (v_1 - 1)y, (a_1 - 1)x + (v_1 - u_1 - 2)y, b_1 x + z, u_1 y + z\} - n;$$

ii) If $(a_1 - 1)x > (u_1 + 1)y + z$ and $(v_1 - 1)y \leq (b_1 + 1)x + z$ then there exists $r \in \mathbf{N}$ such that, if $(a_1 - (r - 1))x > (u_1 + (r - 1)y) + (r - 1)z$ then

$$F(G) = \max\{(a_1 - b_1 - r)x + (v_1 - 1)y, (a_1 - r)x + (v_1 - u_1 - r)y, b_1 x + (r - 1)z, (v_1 - 1)y + (r - 1)z\} - n,$$

and, if $(a_1 - (r - 1))x \leq (u_1 + (r - 1)y + (r - 1)z$ then

$$F(G) = \max\{(a_1 - b_1 - r)x + (v_1 - 1)y, (a_1 - r + 1)x + (v_1 - u_1 - r)y, b_1 x + (r - 1)z, (v_1 - 1)y + (r - 2)z, (u_1 + r - 2)y + (r - 1)z\} - n;$$

iii) If $(a_1 - 1)x \leq (u_1 + 1)y + z$ and $(v_1 - 1)y > (b_1 + 1)x + z$ then there exists $r \in \mathbf{N}$ such that, if $(v_1 - (r - 1))y \leq (b_1 + (r - 1))x + (r - 1)z$ then

$$F(G) = \max\{(a_1 - b_1 - r)x + (v_1 - (r - 1))y, (a_1 - 1)x + (v_1 - u_1 - r)y, (a_1 - 1)x + (r - 2)z, u_1 y + (r - 1)z, (b_1 + r - 2)x + (r - 1)z\} - n,$$

and if $(v_1 - (r - 1))y > (b_1 + (r - 1))x + (r - 1)z$ then

$$F(G) = \max\{(a_1 - b_1 - r)x + (v_1 - r)y, (a_1 - 1)x + (v_1 - u_1 - r)y, (a_1 - 1)x + (r - 1)z, u_1 y + (r - 1)z\} - n;$$

iv) If $(a_1-1)x > (u_1+1)y+z$ *and* $(v_1-1)y > (b_1+1)x+z$ *then there exist* $r \in \mathbf{N}$ *and* $p, m \in \mathbf{N}_0$, *and*

$$F(G) = \max\{(a_1-b_1-r)x+(v_1-p-2)y, (a_1-m-2)x+(v_1-u_1-r)y, (a_1-m-2)x+(p+1)z, (b_1+p+1)x+(r-1)z, (v_1-p-2)y+(m+1)z, (u_1+m+1)y+(r-1)z\} - n.$$

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