International Journal of Applied Mathematics

Volume 38 No. 2 2025, 161–183
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i2.1

HIGHER-ORDER HARMONIC OSCILLATOR PERTURBED BY ALMOST PERIODIC POTENTIAL

Mohamed El Hammaji [1,§] **, Ilias Aarab** [2]

and Mohamed Ali Tagmouti [3]

[1,2,3] Abdelmalek Essaadi University

Faculty of Sciences

B.P 2121, Tetuan, MOROCCO

[1] e-mail: mohamedelhammaji@gmail.com (§ corresponding author)

[2] e-mail: ilias.aarab1989@gmail.com

[3] e-mail: tagmoutimohamedali@gmail.com

Abstract

We study the perturbation $L = H+V$ in $L^2(\mathbb{R})$, where $H = (-1)^l \frac{d^{2l}}{dx^{2l}} + x^{2l}$, $l \in \mathbb{N}^*$, and V is an almost periodic potential with uniformly continuous derivatives $V^{(n)}$. We assume that the eigenvalues of L around λ_k can be written in the form $\lambda_k + \mu_k$. We establish an asymptotic formula for the fluctuations $\{\mu_k\}$, which are determined by a transformation of V.

Math. Subject Classification: 47G30, 47A10

Key Words and Phrases: pseudo-differential operator, averaging method, perturbation theory, spectrum, eigenvalue asymptotics

Received: 29 August 2024

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1. **Introduction and main result:**

We consider in $L^2(\mathbb{R})$ the operator H defined by:

$$H = (-1)^l \frac{d^{2l}}{dx^{2l}} + x^{2l}, \quad l \in \mathbb{N}^*. \tag{1}$$

We recall that H is essentially self-adjoint in $C_0^\infty(\mathbb{R})$ with compact resolvent [1]. Its spectrum is the increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 0}$ of finite multiplicity, such that there exists a positive integer k_0, for $k \geq k_0$, λ_k is simple and has the following asymptotic expansion:

$$\lambda_k^{\frac{1}{l}} = \frac{2\pi}{T}\left(k + \frac{1}{2}\right) + O\left(\frac{1}{k}\right) \quad (k \to +\infty), \tag{2}$$

and

$$T = \frac{1}{l} B\left(\frac{1}{2l}, \frac{1}{2l}\right), \tag{3}$$

where B is the beta function. Let V be a smooth function from $\mathbb{R}$ to $\mathbb{R}$, that is almost periodic, with uniformly continuous derivatives,

$$V(x) = \sum_{n=1}^{+\infty} a_n e^{i\nu_n x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}. \tag{4}$$

We suppose that:

$$\nu_n > 0, \quad \sum_{n=1}^{+\infty}\sum_{i=0}^{k} \nu_n^i \left(\frac{1}{\nu_n^{\frac{1}{2l}}} + \frac{1}{\nu_n}\right)|a_n| < +\infty. \tag{5}$$

The operator $L = H + V$ is essentially self-adjoint with compact resolvent [12]. The Min-Max theorem [7] shows that the spectrum of L around λ_k, for $k \geq k_0$ can be written in the form $\lambda_k + \mu_k$. The goal is to study the asymptotic behavior of the fluctuation μ_k when $\lambda_k \to +\infty$. Let us state the main result of this paper.

THEOREM 1.1. *The asymptotic behavior of μ_k is:*

$$\mu_k = \frac{l}{B(\frac{1}{2l}, \frac{1}{2l})} \int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}} dy + O\left(\lambda_k^{-(\frac{1}{l} - \frac{1}{4l^2})}\right), \quad l \geq 2.$$

For the case $l = 1$ (harmonic oscillator), various mathematicians have investigated such problems. Notably, the authors in [3] treated the case:

$V(x) \sim |x|^{-\alpha} \sum_m a_m \cos \omega_m x$. In a scenario similar to ours, authors in [8] Studied the case where V is periodic. Additionally, in [9] authors explored the perturbation:

$$-\frac{d^{2m}}{dx^{2m}} + x^{2m} + V(x),$$

where V satisfies $\left|V^{(n)}(x)\right| \leq (1+x^2)^{\frac{-s}{2}}, s \in\ ]0,1[\ \cup\ ]1,+\infty[$. In [2] authors studied the case where V is is a polynomial of degree $< 2q$. Our objective is to utilize the averaging method of Weinstein (see [5], [10], [11]). However, this method is not directly applicable here because the operator H, considered as a pseudo-differential operator (ΨDO), lacks a periodic flow. Instead, it is the operator $H^{\frac{1}{l}}$, that exhibits this property [1]. We begin by considering a perturbation of the operator $H^{\frac{1}{l}}$:

$$L_l = H^{\frac{1}{l}} + B, \quad l \in \mathbb{N}^*, \tag{6}$$

where

$$B = \frac{1}{l} H^{\frac{1}{l}-1} V. \tag{7}$$

We apply the averaging method by replacing B in the perturbation (6) with its average:

$$\overline{B} = \frac{1}{T} \int_0^T e^{-itH^{\frac{1}{l}}} B e^{itH^{\frac{1}{l}}} dt, \tag{8}$$

where T is the period of the flow of $H^{\frac{1}{l}}$, as given by (3). The main advantage of this method is that $\overline{B}$ is a compact operator, and the operators L_l, $\overline{L}_l = H^{\frac{1}{l}} + \overline{B}$ are almost unitarily equivalent. This means there exists a unitary operator U such that $UL_lU^{-1} - \overline{L}_l$ is compact. Note that both L_l and $\overline{L}_l$ have a compact resolvents [12]. Using Min-Max theorem, their spectrum near $\lambda_k^{\frac{1}{l}}$ are of the form $\lambda_k^{\frac{1}{l}} + v_k$ and $\lambda_k^{\frac{1}{l}} + \overline{v}_k$, respectively. Then we study $\overline{v}_k$ by using a functional calculus of the operator H. We begin by establishing the link between v_k and μ_k.

Proposition 1.1. *For $l \geq 1$, we have:*

$$\mu_k = l \lambda_k^{1-\frac{1}{l}} v_k + O\left(\lambda_k^{-1}\right), \quad (\lambda_k \to +\infty).$$

Using a functional calculus for H, we obtain:

Proposition 1.2. *For $l \geq 1$, we have:*

$$l\lambda_k^{1-\frac{1}{l}}\bar{v}_k = \frac{l}{B(\frac{1}{2l},\frac{1}{2l})}\int_{-1}^{1}\frac{V(y\lambda_k^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}}dy + O(\lambda_k^{-(\frac{1}{l}-\frac{1}{4l^2})}).$$

The following proposition gives the relation between v_k and $\overline{v}_k$:

Proposition 1.3. *For $l \geq 1$, we have:*

$$v_k = \overline{v}_k + O(\lambda^{-(\frac{1}{2l^2}+2-\frac{2}{l})}).$$

The rest of this paper is organised as follows. In Section 2, this section provides additional details about specific properties of Weyl pseudo-differential operators and their functional calculus. In Section 3, we examine the spectrum of the spectrum of L and show the relation between μ_k and v_k. Section 4 is devoted to the study of the asymptotic behavior of $\overline{v}_k$ and we establish Proposition 1.2. In Section 5, we study the relation between the spectrum of L_l and $\overline{L}_l$ which will help in proving Proposition 1.3, a key step in demonstrating the main theorem.

2. Weyl pseudo-differential operator and functional calculus

Let $\rho \in [0,1]$ and $m \in \mathbb{R}$. We consider the temperate weight function [14]

$$(x,\xi) \longrightarrow (1+\sigma_H)^{\frac{m}{2}}, \qquad (x,\xi) \in \mathbb{R}^2,$$

where $\sigma_H(x,\xi)$ is the Weyl symbol of the operator H defined in the space phase $T^*\mathbb{R} = \mathbb{R}_x \times \mathbb{R}_\xi$ by:

$$\sigma_H(x,\xi) = x^{2l} + \xi^{2l}.$$

We denote by $\Gamma_\rho^m(\mathbb{R}\times\mathbb{R})$ the space of symbols associated with the temperate weight function, precisely:

$$\begin{aligned}\Gamma_\rho^m \;=\;& \{a \in C^\infty(\mathbb{R}^2) : \forall\alpha,\beta \in \mathbb{N}, \exists\, c_{\alpha,\beta} > 0 \,/\, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \\ & \leq c_{\alpha,\beta}(1+\sigma_H)^{\frac{m-\rho(\alpha+\beta)}{2}}\}.\end{aligned}$$

We will use the standard Weyl quantization of the symbols. To be precise, if $a \in \Gamma_\rho^m$, then for $u \in S(\mathbb{R})$, the operator associated is defined by:

$$op^w(a)\,u(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi} a\left(\frac{x+y}{2},\xi\right)u(y)\,dyd\xi.$$

We note by G_ρ^m the operator class whose symbol belongs to Γ_ρ^m, for example $H \in G_{\frac{1}{l}}^2$ and $V \in G_0^0$. Let us now introduce the notion of the asymptotic expansion of symbols.

Definition 2.1. Let $a_j \in \Gamma_\rho^{m_j}$, $j \in \mathbb{N}$, we suppose that m_j is a decreasing sequence tending towards $-\infty$. We say that $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ has an asymptotic expansion and we write:

$$a = \sum_{j=0}^{+\infty} a_j,$$

if

$$a - \sum_{j=0}^{r-1} a_j \in \Gamma_\rho^{m_r}, \qquad \forall r \geq 1.$$

We require the symbolic calculation of these classes of operators, therefore, we present the following proposition, which will be proven in the Appendix, Section 6.

Proposition 2.1. *i) If $A \in G_\rho^{m_1}$, $\rho \in]0,1]$ and $B \in G_0^{m_2}$ then the operator $AB \in G_0^{m_1+m_2}$. Its Weyl symbol admits the following asymptotic development:*

$$c = \sum_{j=0}^{+\infty} c_j, \quad c_j \in \Gamma_0^{m_1+m_2-\rho j},$$

where

$$c_j = \frac{1}{2^j} \sum_{\alpha+\beta=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b),$$

ii) If $(B_i)_{i\in\{1,\cdots,n\}}$ is the family of operators such as $B_i \in G_0^{m_i}$. Then the operator

$$B_1 B_2 \cdots B_n H^{-\frac{m_1+\cdots m_n}{2}},$$

is bounded.

Theorem 2.1. *(Calderon-Vaillancourt Theorem)*
If $a \in \Gamma_0^0$ then the operator $op^w(a)$ is bounded on $L^2(\mathbb{R})$.

THEOREM 2.2. *(Compactness)*
If $a \in \Gamma_\rho^m, m < 0$ and $\rho \in]0,1]$, then the operator $op^w(a)$ is compact on $L^2(\mathbb{R})$.

In our work, we need the functional calculus of operators H and we use the properties of a function f that satisfies, for all $r \in \mathbb{R}, k \in \mathbb{N}$ and $\rho \in [1-\frac{1}{2l}, 1]$,

$$|f^{(k)}(x)| \leq C_k(1+|x|)^{r-\rho k}.$$

PROPOSITION 2.2. *$f(H)$ is a (ΨDO) included in $G^{2r}_{\frac{1}{l}-2(1-\rho)}$ and its weyl symbol admits the following development:*

$$\sigma_{f(H)} = \sum_{j\geq 0} \sigma_{f(H),2j},$$

$$\sigma_{f(H),2j} = \sum_{k=2}^{3j} \frac{d_{jk}}{k!} f^{(k)}(\sigma_H), \quad \forall j \geq 1,$$

where

$$d_{j,k} \in \Gamma^{2k-j\frac{4}{l}}_{\frac{1}{l}}, \quad \sigma_{f(H),2j} \in \Gamma^{2r-j\left(\frac{4}{l}-6(1-\rho)\right)}_{\frac{1}{l}-2(1-\rho)}, \tag{9}$$

in particular

$$\sigma_{f(H),0} = f(\sigma_H).$$

P r o o f. For studying $f(H)$ we follow the same strategy in [13], using the Mellin transformation, the latter consists of the following steps:

1) We prove by induction that $(H-\lambda)^{-1}$, $\lambda \in \mathbb{C}$, is a (ΨDO) and its Weyl symbol admits the development $b_\lambda = \sum_{j=0}^{+\infty} b_{j,\lambda}$ where

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1}, \\ b_{2j+1,\lambda} = 0, \\ b_{2j,\lambda} = \sum_{k=2}^{3j} (-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{j,k} \in \Gamma_1^{2lk-4j}. \end{cases}$$

2) We study the operator H^s using the Cauchy integral formula:

$$H^s = \frac{1}{2\pi i}\int_{\Delta} \lambda^s (H-\lambda)^{-1} d\lambda,$$

Δ is the same domain defined in the article [13]. H^s is a (ΨDO) and its Weyl symbol is given by:

$$\sigma_s = \sum_{j=0}^{+\infty} \sigma_{s,2j},$$

with

$$\sigma_{s,0} = \sigma_H^s, \quad \sigma_{s,2j} = \sum_{k=2}^{3j} d_{j,k}.\frac{s(s-1)\cdots(s-k+1)}{k!}\sigma_H^{s-k},$$

$$\sigma_{s,2j} \in \Gamma_1^{2ls-4j}.$$

3) We study $f(H)$ using the representation formula

$$f(H) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M\left[f\right](s) H^{-s} ds,$$

$\sigma \in [0,-r[$, $r<0$ and $M\left[f\right]$ is the Mellin transformation of f. □

3. Reduction to a perturbation of $H^{\frac{1}{l}}$

If we translate H by a sufficiently large positive constant, we can assume that L is positive and $\|H^{-1}V\| < 1$. This allows us to reduce the problem to a perturbation of $H^{\frac{1}{l}}$ by expressing

$$(H+V)^{\frac{1}{l}} = H^{\frac{1}{l}} + W,$$

consequently,

$$W = B + H^{\frac{1}{l}}\left(H^{-1}V\right)^2 \sum_{k=0}^{+\infty} \alpha_{k+2}\left(H^{-1}V\right)^k,$$

where

$$B = \binom{1}{l} H^{\frac{1}{l}-1}V, \quad \alpha_k = \frac{\left(\frac{1}{l}\right)\left(\frac{1}{l}-1\right)\cdots\left(\frac{1}{l}-k+1\right)}{k!}. \tag{10}$$

We can then write

$$L^{\frac{1}{l}} - L_l = H^{\frac{1}{l}}\left(H^{-1}V\right)^2 \sum_{k=0}^{+\infty} \alpha_{k+2}\left(H^{-1}V\right)^k. \tag{11}$$

Since $\|H^{-1}V\| < 1$, the operator $\sum_{k=0}^{+\infty} \alpha_{k+2} \left(H^{-1}V\right)^k$ is bounded in $L^2(\mathbb{R})$. Given that $H \in G^2_{\frac{1}{l}}$, we have $H^{-1} \in G^{-2}_{\frac{1}{l}}$. By combining the above with the fact that $V \in G^0_0$ and Proposition 2.1-(ii), the opertaor $(L^{\frac{1}{l}} - L_l)H^{2-\frac{1}{l}}$ is bounded. We thus conclude that there exists a constant $c > 0$ such that:

$$-cH^{-2+\frac{1}{l}} \leq L^{\frac{1}{l}} - L_l \leq cH^{-2+\frac{1}{l}}. \tag{12}$$

According to the Min-Max theorem, we obtain:

$$(\lambda_k + \mu_k)^{\frac{1}{l}} = \lambda_k^{\frac{1}{l}} + v_k + O\left(\lambda_k^{\frac{1}{l}-2}\right), \tag{13}$$

Using the fact that $\{\mu_k\}$ is bounded and applying Taylor's formula to the function $t \longrightarrow (1 + \frac{\mu_k}{t})^{\frac{1}{l}}$, we obtain the estimate

$$\mu_k = l\lambda_k^{1-\frac{1}{l}} v_k + O\left(\lambda_k^{-1}\right). \tag{14}$$

This completes the proof of proposition 1.1.

4. The asymptotic behavior of $\overline{v}_k$

We recall that $\overline{L}_l$ is obtained by replacing B in L_l with $\overline{B}$, and $\lambda_k^{\frac{1}{l}} + \overline{v}_k$ is the part of the spectrum of $\overline{L}_l$ around $\lambda_k^{\frac{1}{l}}$ as $\lambda_k \longrightarrow +\infty$. Setting

$$\overline{V} = \frac{1}{T}\int_0^T W(t)dt, \quad W(t) = e^{-itH^{\frac{1}{l}}} V e^{itH^{\frac{1}{l}}}, \tag{15}$$

and from (10)

$$\overline{B} = \frac{1}{l} H^{\frac{1}{l}-1}\overline{V}. \tag{16}$$

As noted in Proposition 2.2, the operator $H^{\frac{1}{l}} \in G^{\frac{2}{l}}_{\frac{1}{l}}$ possesses a Weyl symbol σ that is expressed as:

$$\sigma = \sum_{j=0}^{+\infty} \sigma_{2j},$$

where $\sigma_0 = \sigma_H^{\frac{1}{l}}$ and $\sigma_{2j} \in \Gamma^{\frac{2}{l}-\frac{4}{l}j}_{\frac{1}{l}}$.

Proposition 4.1. *For $l \geq 1$, we have:*

$$\overline{V} \in G_0^{-\frac{1}{2l^2}}, \tag{17}$$

and its Weyl symbol admits the following asymptotic development

$$\sigma_{\overline{V}} = \sum_{j=0}^{+\infty} \sigma_{\overline{V},2j}, \quad \sigma_{\overline{V},2j} \in \Gamma_0^{-\frac{1}{2l^2}-\frac{2}{l}j}, \tag{18}$$

in particular:

$$\sigma_{\overline{V},0} = \frac{1}{T}\int_0^T V(x(t))dt = \frac{l}{B(\frac{1}{2l},\frac{1}{2l})}\int_{-1}^{1} \frac{V(y\sigma_H^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}}dy. \tag{19}$$

P r o o f. We recall that $\varphi(t) = (x(t), \xi(t))$ is a solution of the dynamic system:

$$\begin{cases} \frac{dx(t)}{dt} = \frac{\partial \sigma_H^{\frac{1}{l}}}{\partial \xi} = 2E^{\frac{1}{l}-1}\xi^{2l-1}(t), \\ \frac{d\xi(t)}{dt} = \frac{-\partial \sigma_H^{\frac{1}{l}}}{\partial x} = -2E^{\frac{1}{l}-1}x^{2l-1}(t), \\ x(0) = x, \quad \xi(0) = \xi, \\ x^{2l}(t) + \xi^{2l}(t) = x^{2l} + \xi^{2l} = E. \end{cases} \tag{20}$$

To establish equation (19), we begin by recalling that $\varphi(t) = (x(t), \xi(t))$ is a solution of the dynamic system (20). We assume the initial conditions $x(0) > 0$ and $\frac{dx(0)}{dt} > 0$, the other cases can be handled similarly. Our focus now is on analyzing the properties of the function $x(t)$ over the interval $[0, T]$. From equation (20) we get the relation:

$$dt = \pm \frac{dx}{2E^{\frac{1}{l}-1}(E - x^{2l})^{1-\frac{1}{2l}}}. \tag{21}$$

Given that $x(t)$ is a smooth periodic function of period T, we can deduce from equation (21) that the function $x(t)$ reaches its maximum at t_0 where $x(t_0) = E^{\frac{1}{2l}}$, and its minimum at t_1, where $x(t_1) = -E^{\frac{1}{2l}}$. At this point, we have:

$$\sigma_{\overline{V},0} = \frac{1}{T}\left[\int_0^{t_0} V(x(t))dt + \int_{t_0}^{t_1} V(x(t))dt + \int_{t_1}^{T} V(x(t))dt\right].$$

To proceed, we perform a change of variable $x(t) = u$. Since $x(t)$ is increasing on the interval $[0, t_0]$, we obtain:

$$\int_0^{t_0} V(x(t))dt = \frac{1}{2}E^{1-\frac{1}{l}} \int_x^{E^{\frac{1}{2l}}} \frac{V(u)}{(E-u^{2l})^{1-\frac{1}{2l}}}du, \tag{22}$$

after applying a similar calculation over the intervals $[t_0, t_1]$ and $[t_1, T]$, we obtain:

$$\sigma_{\overline{V},0} = \frac{l}{B(\frac{1}{2l}, \frac{1}{2l})} E^{1-\frac{1}{l}} \int_{-E^{\frac{1}{2l}}}^{E^{\frac{1}{2l}}} \frac{V(u)}{(E-u^{2l})^{1-\frac{1}{2l}}} du. \tag{23}$$

Now, by performing the change of variable $y = \frac{u}{E^{\frac{1}{2l}}}$, we have:

$$\sigma_{\overline{V},0} = \frac{l}{B(\frac{1}{2l}, \frac{1}{2l})} \int_{-1}^{1} \frac{V(y\sigma_H^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}} dy = \frac{l}{B(\frac{1}{2l}, \frac{1}{2l})} f(\sigma_H^{\frac{1}{2l}}), \tag{24}$$

where

$$f(x) = \int_{-1}^{1} \frac{V(xy)}{(1-y^{2l})^{1-\frac{1}{2l}}} dy, \quad x > c > 0,$$

with c being a small positive constant. We can write:

$$f(x) = \int_{0}^{1} \frac{\mathcal{V}(yx)}{(1-y^{2l})^{1-\frac{1}{2l}}} dy, \quad x > c > 0, \tag{25}$$

where $\mathcal{V}(x) = V(-x) + V(x) = 2\sum_{l=1}^{+\infty} a_l \cos(\nu_l x)$. Given equations (4) and (25), and since $\mathcal{V} \in C^\infty(\mathbb{R}, \mathbb{R})$, its Fourier series converges normally, and hence uniformly on $\mathbb{R}$, with the series sum being $\mathcal{V}$. Therefore, we have:

$$f(x) = 2\sum_{l=1}^{+\infty} a_l \int_{0}^{1} \frac{Re(e^{-i\nu_l xy})}{(1-y^{2l})^{1-\frac{1}{2l}}} dy. \tag{26}$$

Let us define:

$$f_l(x) = \int_{0}^{1} \frac{e^{-i\nu_l xy}}{(1-y^{2l})^{1-\frac{1}{2l}}} dy = e^{-i\nu_l x} h_l(x), \tag{27}$$

where

$$h_l(x) = \int_{0}^{1} \frac{e^{i\nu_l x(1-y)}}{(1-y^{2l})^{1-\frac{1}{2l}}} dy.$$

By making the change of variables $z = 1 - y$, we get:

$$h_l(x) = \int_{0}^{1} \frac{e^{i\nu_l xz}}{(1-(1-z)^{2l})^{1-\frac{1}{2l}}} dz.$$

Since

$$1 - (1-z)^{2l} = z \sum_{k=0}^{2l-1} (1-z)^k,$$

we have:

$$h_l(x) = \int_0^1 \frac{e^{i\nu_l xz}}{z^{1-\frac{1}{2l}}(\sum_{k=0}^{2l-1}(1-z)^k)^{1-\frac{1}{2l}}}dz. \tag{28}$$

We define, for $z \in [0;1]$:

$$\psi(z) = \frac{1}{\left(\sum_{k=0}^{2l-1}(1-z)^k\right)^{1-\frac{1}{2l}}}.$$

It is clear that:

$$\psi(z) = \frac{1}{(2l)^{1-\frac{1}{2l}}} + z\theta(z), \tag{29}$$

where $\theta(z) = \int_0^1 \psi^{'}(tz)dt$. Substituting (29) into (28), we obtain:

$$h_l(x) = \frac{1}{(2l)^{1-\frac{1}{2l}}} \int_0^1 \frac{e^{i\nu_l xz}}{z^{1-\frac{1}{2l}}}dz + \int_0^1 z^{\frac{1}{2l}}\theta(z)e^{i\nu_l xz}dz. \tag{30}$$

Now, by making the substitution $u = \nu_l xz$ in the first integral of (30), we get:

$$\begin{aligned} & h_l(x) \\ = \ & \frac{1}{(2l)^{1-\frac{1}{2l}}}\frac{1}{(\nu_l x)^{\frac{1}{2l}}}\int_0^{+\infty}\frac{e^{iu}}{u^{1-\frac{1}{2l}}}du - \frac{1}{(2l)^{1-\frac{1}{2l}}}\frac{1}{(\nu_l x)^{\frac{1}{2l}}}\int_{\nu_l x}^{+\infty}\frac{e^{iu}}{u^{1-\frac{1}{2l}}}du \\ & + \int_0^1 z^{\frac{1}{2l}}\theta(z)e^{i\nu_l xz}dz. \end{aligned} \tag{31}$$

Note that:

$$\int_0^{+\infty}\frac{e^{iu}}{u^{1-\frac{1}{2l}}}du = e^{\frac{i\pi}{4l}}\Gamma\left(\frac{1}{2l}\right). \tag{32}$$

By substituting equations (31) and (32) into equation (27), we obtain the expression:

$$\begin{aligned} f_l(x) &= (\tfrac{1}{2l})^{1-\frac{1}{2l}}\frac{\Gamma(\frac{1}{2l})}{(\nu_l x)^{\frac{1}{2l}}}e^{-i(\nu_l x - \frac{\pi}{4l})} - (\tfrac{1}{2l})^{1-\frac{1}{2l}}\frac{e^{-i\nu_l x}}{(\nu_l x)^{\frac{1}{2l}}}\int_{\nu_l x}^{+\infty}\frac{e^{iz}}{z^{1-\frac{1}{2l}}}dz \\ &- e^{-i\nu_l x}\int_0^1 z^{\frac{1}{2l}}\theta(z)e^{i\nu_l xz}dz \\ &= f_{l,1} + f_{l,2} + f_{l,3}. \end{aligned} \tag{33}$$

A direct calculation shows that for all $x > c > 0$:

$$\begin{cases} \left|f_{l,1}^{(k)}(x)\right| \leq c_k \frac{1+\nu_n+\cdots+\nu_n^k}{\nu_n^{\frac{1}{2l}}}(1+|x|)^{-\frac{1}{2l}}, \\ \left|f_{l,2}^{(k)}(x)\right| \leq c_k \frac{1+\nu_n+\cdots+\nu_n^k}{\nu_n}(1+|x|)^{-1}, \\ \left|f_{l,3}^{(k)}(x)\right| \leq c_k \frac{\nu_n^k}{\nu_n}(1+|x|)^{-1}. \end{cases} \tag{34}$$

We will sum these results and then evaluate the estimate with respect to l. Using (26), (27), (33) and (34) it follows that:

$$\left|f^{(k)}(x)\right| \leq c_k \sum_{n=1}^{+\infty} \sum_{i=0}^{k} \nu_n^i \left(\frac{1}{\nu_n^{\frac{1}{2l}}} + \frac{1}{\nu_n} \right) |a_n| (1+|x|)^{-\frac{1}{2l}}. \tag{35}$$

From which, according to (5), we deduce that:

$$\left|f^{(k)}(x)\right| \leq C_k (1+|x|)^{-\frac{1}{2l}}. \tag{36}$$

For all $\alpha, \beta \in \mathbb{N}$, we have:

$$\begin{aligned} \partial_x^\alpha \partial_\xi^\beta (f(\sigma_H^{\frac{1}{2l}})) \quad = & \sum_{\substack{1\leq k\leq \alpha+\beta \\ \alpha=\alpha_1+\cdots+\alpha_k \\ \beta=\beta_1+\cdots+\beta_k}} C_{\beta_1,\cdots,\beta_k}^{\alpha_1,\cdots,\alpha_k} \times \partial_x^{\alpha_1} \partial_\xi^{\beta_1} (\sigma_H^{\frac{1}{2l}})) \times \cdots \\ & \times \partial_x^{\alpha_k} \partial_\xi^{\beta_k} (\sigma_H^{\frac{1}{2l}})) \times f^{(k)}(\sigma_H^{\frac{1}{2l}})). \end{aligned} \tag{37}$$

Using (5),(36), and considering that $\sigma_H^{\frac{1}{2l}} \in \Gamma_{\frac{1}{l}}^{\frac{1}{l}}$, we get:

$$\left|\partial_x^\alpha \partial_\xi^\beta (f(\sigma_H^{\frac{1}{2q}}))\right| \leq c_{\alpha,\beta} (1+\sigma_H)^{-\frac{1}{4l^2}}. \tag{38}$$

From (24) and (38), it is evident that $\sigma_{\overline{V},0} \in \Gamma_0^{-\frac{1}{2l^2}}$. To prove (18), we begin by applying Egorov's theorem. Direct calculations show that:

$$\partial_x^\alpha \partial_\xi^\beta \sigma_{2j} \in L^\infty(\mathbb{R}\times\mathbb{R}), \quad \alpha, \beta, j \in \mathbb{N},\ \alpha+\beta+j > 2,$$

and

$$\partial_x^\alpha \partial_\xi^\beta \varphi(t) \in L^\infty(\mathbb{R}\times\mathbb{R}), \quad \alpha+\beta \geq 1,$$

Applying Egorov's theorem as outlined in [14] in the context of the Heisenberg-von Neumann equation, we obtain the expression for the Weyl symbol:

$$\sigma_{W(t)} = \sum_{j=0}^{+\infty} \sigma_{W(t),2j},$$

where

$$\sigma_{W(t),2j} = \frac{1}{i}\int_0^t \sum_{\substack{\alpha'+\beta'+l'+2k'=2j+1\\ 0\le l\le 2j-1}} C_{\alpha',\beta'}(\partial_\xi^{\alpha'}\partial_x^{\beta'}\sigma_{2k})(\partial_\xi^{\beta'}\partial_x^{\alpha'}\sigma_{W,l}(\tau))|\varphi^{t-\tau}d\tau. \tag{39}$$

Here, $C_{\alpha',\beta'}$ is defined as:

$$C_{\alpha',\beta'} = \left(1-(-1)^{\alpha+\beta}\right)\Gamma\left(\alpha,\beta\right),$$

with

$$\sigma_{W(t),0}(x,\xi) = Vox(t), \quad \sigma_{W(t),1}(x,\xi) = 0.$$

For all $\alpha,\beta\in\mathbb{N}$, we have:

$$\left|\partial_x^\alpha\partial_\xi^\beta V(x(t))\right| \le C_{\alpha,\beta}\sum_{\substack{1\le l\le\alpha+\beta\\ \alpha=\alpha_1+\cdots+\alpha_l\\ \beta=\beta_1+\cdots+\beta_l}} V^{(l)}(x(t)).$$

From (39), it follows that:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{W(t),2j}\right| \le C_{\alpha,\beta}(e+\sigma_H)^{-\frac{j}{l}} \times \int_0^t \left(\sum_{m=1}^{\alpha+\beta+2j+1}\left|V^{(m)}(x(u))\right|\right)du. \tag{40}$$

Since $\left|V^{(m)}\right|$ is almost periodic for all m, we can use the previous calculation to obtain:

$$\int_0^T \left|V^{(m)}\right|(x(u))\,du \in \Gamma_0^{-\frac{1}{2l^2}}. \tag{41}$$

Applying similar calculations to (40) confirms (18). □

Proof of Proposition 1.2

P r o o f. By Proposition 4.1, we have

$$\sigma_{\overline{V}} - \sigma_{\overline{V},0} \in \Gamma_0^{-\frac{1}{2l^2}-\frac{2}{l}}, \tag{42}$$

using (19), we can write

$$\sigma_{\overline{V},0} = g(\sigma_H), \tag{43}$$

where

$$g(x) = \frac{l}{B(\frac{1}{2l},\frac{1}{2l})}\int_{-1}^1 \frac{V(y\sigma_H^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}}dy = f(x^{\frac{1}{2l}}), \tag{44}$$

by exploiting (36) and by a direct calculation, we obtain for all $k\in\mathbb{N}$

$$\left|g^{(k)}(x)\right| \le C_k(1+|x|)^{-\frac{1}{4l^2}-(1-\frac{1}{2l})k}; \quad x\in\mathbb{R}^+. \tag{45}$$

Applying Proposition 2.2, we find that $g(H) \in G_0^{-\frac{1}{2l^2}}$, and its weyl symbol $\sigma_{f(H)}$ satisfies

$$\sigma_{g(H)} - g(\sigma_H) \in \Gamma_0^{-\frac{1}{2l^2}-\frac{2}{l}}, \tag{46}$$

combining (42), (43) and (46), we get

$$\sigma_{\overline{V}} - \sigma_{g(H)} \in \Gamma_0^{-\frac{1}{2l^2}-\frac{2}{l}}. \tag{47}$$

In terms of operators, this translates to

$$\overline{V} - g(H) \in G_0^{-\frac{1}{2l^2}-\frac{2}{l}}, \tag{48}$$

we can write

$$\frac{1}{l}H^{\frac{1}{l}-1}(\overline{V} - g(H)) = \left[\overline{L}_l - (H^{\frac{1}{l}} + \frac{1}{l}H^{\frac{1}{l}-1}g(H))\right]. \tag{49}$$

From (48), Proposition 2.1, Proposition 2.2 and Theorem 2.1, we deduce that the operator

$$\left[\overline{L}_l - (H^{\frac{1}{l}} + \frac{1}{l}H^{\frac{1}{l}-1}g(H))\right] H^{1-\frac{1}{4l^2}}, \tag{50}$$

is bounded. According to Min-Max theorem, we have

$$\bar{v}_k = \frac{1}{l}\lambda_k^{\frac{1}{l}-1}g(\lambda_k) + O(\lambda_k^{-(1-\frac{1}{4l^2})}). \tag{51}$$

Thus,

$$l\lambda_k^{1-\frac{1}{l}}\bar{v}_k = \frac{l}{B(\frac{1}{2l},\frac{1}{2l})}\int_{-1}^{1}\frac{V(y\lambda_k^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}}dy + O(\lambda_k^{-(\frac{1}{l}-\frac{1}{4l^2})}). \tag{52}$$

□

5. The relation betwen the spectrum of L_l and $\overline{L}_l$

Proposition 5.1. *There exists a skew-symmetric operator* $Q \in G_0^{-(\frac{1}{2l^2}+2-\frac{2}{l})}$ *such that the operator* $(e^Q L_l e^{-Q} - \overline{L}_l)H^{\frac{1}{2l^2}+2-\frac{2}{l}}$ *is bounded.*

P r o o f. The operator Q is constructed as follows:

$$Q = Q_1 + Q_2,\ Q_1 = \frac{i}{lT} H^{\frac{1}{l}-1} \int_0^T (T-t)\, W(t)\, dt,$$
$$Q_2 = \frac{-1}{2T} \int_0^T (T-t) \int_0^t \left[\frac{1}{l} H^{\frac{1}{l}-1} W(t), \frac{1}{l} H^{\frac{1}{l}-1} W(r) \right] dr\, dt. \tag{53}$$

The following commutation formulas can be easily verified [4]:

$$\left[Q_1, H^{\frac{1}{l}} \right] = \frac{1}{l} H^{\frac{1}{l}-1} (\overline{V} - V), \tag{54}$$

$$\begin{aligned} \left[Q_2, H^{\frac{1}{l}} \right] &= \tfrac{-1}{2T} \int_0^T (T-t) \int_0^t \\ &\quad \left[\left[\tfrac{1}{l} H^{\frac{1}{l}-1} W(t), \tfrac{1}{l} H^{\frac{1}{l}-1} W(r) \right], H^{\frac{1}{l}} \right] drdt \\ &= -\overline{\overline{V}} - \tfrac{1}{2} \left[Q_1, \tfrac{1}{l} H^{\frac{1}{l}-1} V \right], \end{aligned} \tag{55}$$

where

$$\overline{\overline{V}} = \frac{1}{2Ti} \int_0^T \int_0^t \left[\frac{1}{l} H^{\frac{1}{l}-1} W(t), \frac{1}{l} H^{\frac{1}{l}-1} W(r) \right] dr\, dt. \tag{56}$$

We point out that the differential equation

$$\frac{dX(t)}{dt} = [Q, X], \quad X(0) = L_l,$$

has a unique solution given by

$$X(t) = e^{tAdQ}.L_l = e^{tQ} L_l e^{-tQ}, \tag{57}$$

where $AdQ.L_l = [Q, L_l]$.

As a consequence of (53), (54) and (55), we deduce that:

$$\begin{aligned} e^Q L_l\, e^{-Q} - \overline{L}_l &= -\overline{\overline{V}} + \tfrac{1}{2l} \left[Q_2, H^{\frac{1}{l}-1} V \right] \\ &\quad + \tfrac{1}{2l} \left[Q, H^{\frac{1}{l}-1} \overline{V} \right] + \tfrac{1}{4l} \left[Q, \left[Q_1, H^{\frac{1}{l}-1} V \right] \right] \\ &\quad + \tfrac{1}{2} \left[Q, \left[Q_2, \tfrac{1}{l} H^{\frac{1}{l}-1} V \right] \right] - \tfrac{1}{2} \left[Q, \overline{\overline{V}} \right] \\ &\quad + \sum_{n \geq 0} \frac{(AdQ)^n}{(n+3)!} \left[Q, [Q, [Q, L_l]] \right]. \end{aligned} \tag{58}$$

To continue the proof of Proposition 5.1, we will use the following lemma.

LEMMA 5.1. *$Q_1 \in G_0^{-(\frac{1}{2l^2}+2-\frac{2}{l})}$ and $\overline{\overline{V}}, Q_2 \in G_0^{-(\frac{1}{l^2}-\frac{4}{l}+4)}$.*

P r o o f. Following similar calculations as in the proof of Proposition 4.1, we obtain that: $Q_1 \in G_0^{-(\frac{1}{2l^2}+2-\frac{2}{l})}$. Now let us determine the class of $\overline{\overline{\overline{V}}}$, we can write:

$$\overline{\overline{\overline{V}}} = \frac{1}{2Ti}\int_0^T \left[AW(t), \int_0^t AW(r)dr\right] dt, \tag{59}$$

where

$$A = \frac{1}{l}H^{\frac{1}{l}-1}.$$

Let a (resp. b) denote the Weyl symbols of $AW(t)$ (resp. $\int_0^t AW(r)dr$). By applying Proposition 2.1, we have

$$a = \sum_{j=0}^{+\infty} a_j, \quad b = \sum_{j=0}^{+\infty} b_j,$$

where

$$a_j = \sum_{\alpha'+\beta'+2l'+2k'=j} C_{\alpha',\beta'}(\partial_\xi^{\alpha'}\partial_x^{\beta'}\sigma_{W(t),2l'})(\partial_x^{\alpha'}\partial_\xi^{\beta'}\sigma_{A,2k'}),$$

and

$$\begin{aligned} & b_j \\ = & \sum_{\alpha''+\beta''+2l''+2k''=j} C_{\alpha'',\beta''}(\int_0^t \partial_\xi^{\alpha''}\partial_x^{\beta''}\sigma_{W(r),2l''}dr) \\ \times & (\partial_x^{\alpha''}\partial_\xi^{\beta''}\sigma_{A,2k''}). \end{aligned}$$

The symbol of the commutator in equation (59) is expressed as:

$$c = \sum_{j=1}^{+\infty} c_j,$$

where each c_j is given by:

$$c_j = \sum_{\alpha+\beta+k+m=j} \frac{1}{\alpha!}\frac{1}{\beta!}\left(\frac{1}{2}\right)^\alpha \left(-\frac{1}{2}\right)^\beta \left(1-(-1)^{\alpha+\beta}\right) \partial_x^\alpha \partial_\xi^\beta a_m \partial_\xi^\alpha \partial_x^\beta b_k.$$

For the derivatives $\partial_x^\alpha \partial_\xi^\beta a_m$, we have:

$$\partial_x^\alpha \partial_\xi^\beta a_m = \sum_{\alpha'+\beta'+2l'+2k'=m} C_{\alpha',\beta'} \sum_{\substack{i_1+i_2=\alpha\\ j_1+j_2=\beta}} C_{\alpha,\beta}(\partial_\xi^{\alpha'+j_1}\partial_x^{\beta'+i_1}\sigma_{W(t),2l'}) \times(\partial_x^{\alpha'+i_2}\partial_\xi^{\beta'+j_2}\sigma_{A,2k'}).$$

By applying (9) and (39), we obtain:

$$\left|\partial_x^\alpha \partial_\xi^\beta a_m\right| \leq C_{\alpha,\beta}(1+\sigma_H)^{\frac{1}{l}-1-\frac{m}{2l}} \times \int_0^t \left(\sum_{i=1}^{\alpha+\beta+m+1} \left|V^{(i)}(x(u))\right|\right) du. \tag{60}$$

Similarly:

$$\left|\partial_\xi^\alpha \partial_x^\beta b_k\right| \leq C(1+\sigma_H)^{\frac{1}{l}-1-\frac{k}{2l}} \int_0^t \left(\sum_{i=1}^{\alpha+\beta+k+1} \left|V^{(i)}(x(u))\right|\right) du, \tag{61}$$

thus, for all j, we have:

$$|c_j| \leq C(1+\sigma_H)^{\frac{2}{l}-2}\left(\int_0^t \left(\sum_{i=1}^{j+1} \left|V^{(i)}(x(u))\right|\right) du\right)^2, \tag{62}$$

by applying the Leibniz formula, we obtain the following result for $j \geq 1$:

$$\left|\partial_\xi^\alpha \partial_x^\beta c_j\right| \leq C(1+\sigma_H)^{\frac{2}{l}-2} \times \left(\int_0^t \left(\sum_{i=1}^{\alpha+\beta+j+1} \left|V^{(i)}(x(u))\right|\right) du\right)^2. \tag{63}$$

The symbol of $\overline{\overline{V}}$ is obtained by integrating the symbol c with respect to t. For the integral given in equation (63), we perform the same calculation as in the proof of Proposition 4.1, leading to the result:

$$\overline{\overline{V}} \in G_0^{-(\frac{1}{l^2}-\frac{4}{l}+4)}.$$

We use the same procedure to demonstrate that:

$$Q_2 \in G_0^{-(\frac{1}{l^2}-\frac{4}{l}+4)}.$$

□

We return to the proof of proposition 5.1. Given that $V \in G_0^0$, $\overline{V} \in G_0^{-\frac{1}{2l^2}}$, $Q_1, Q \in G_0^{-(\frac{1}{2l^2}+2-\frac{2}{l})}$ and $\overline{\overline{V}}, Q_2 \in G_0^{-(\frac{1}{l^2}-\frac{4}{l}+4)}$, we apply

Proposition 2.1 in equation (58), we get:

$$\begin{cases} \left\| \overline{\overline{V}}.H^{\frac{1}{2l^2}+2-\frac{2}{l}} \right\| & \leq C, \\ \left\| \left[Q_2, H^{\frac{1}{l}-1}V\right] H^{\frac{1}{2l^2}+3-\frac{3}{l}} \right\| & \leq C, \\ \left\| \left[Q, H^{\frac{1}{l}-1}\overline{V}\right] H^{\frac{1}{2l^2}+2-\frac{2}{l}} \right\| & \leq C, \\ \left\| \left[Q, \left[Q_1, H^{\frac{1}{l}-1}V\right]\right] H^{\frac{1}{2l^2}+3-\frac{3}{l}} \right\| & \leq C, \\ \left\| \left[Q, \left[Q_2, \frac{1}{l}H^{\frac{1}{l}-1}V\right]\right] H^{\frac{3}{4l^2}+4-\frac{4}{l}} \right\| & \leq C, \\ \left\| \left[Q, \overline{\overline{V}}\right] H^{\frac{3}{4l^2}+4-\frac{4}{l}} \right\| & \leq C, \\ \left\| \frac{(AdQ)^n}{(n+3)!} \left[Q, \left[Q, \left[Q, L_l\right]\right]\right] H^{\frac{3}{4l^2}+3-\frac{4}{l}} \right\| & \leq C \left\|Q\right\|^n . \end{cases} \tag{64}$$

For the last inequality, we used equation (54) along with the identity:

$$(AdQ)^n .W = \sum_{p=0}^{n} (-1)^{n-p} C_n^p Q^p W Q^{n-p},$$

from the above, we can conclude that

$$(e^Q L_l e^{-Q} - \overline{L}_l) H^{\frac{1}{2l^2}+2-\frac{2}{l}},$$

is bounded. Returning to the proof of Proposition 1.3, we deduce from Proposition 5.1 that there exists a constant $c > 0$ such that:

$$-cH^{-(\frac{1}{2l^2}+2-\frac{2}{l})} \leq e^Q L_l e^{-Q} - \overline{L}_l \leq cH^{-(\frac{1}{2l^2}+2-\frac{2}{l})},$$

according to the Min-Max theorem, we have:

$$v_k = \overline{v}_k + O(\lambda^{-(\frac{1}{2l^2}+2-\frac{2}{l})}).$$

Now, to prove Theorem 1.1, we simply combine Propositions 1.1 and 1.2. For $l \geq 2$, this yields:

$$\mu_k = \frac{l}{B(\frac{1}{2l}, \frac{1}{2l})} \int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2l}})}{(1-y^{2l})^{1-\frac{1}{2l}}} dy + O\left(\lambda_k^{-(\frac{1}{l}-\frac{1}{4l^2})}\right).$$

□

6. **Appendix: Proof of Proposition 2.1**

i) We proceed as in ([14], Theorem (II 30)), let a and b denote the Weyl symbols of operators A and B respectively. The Weyl symbol c of the operator AB is given by ([14], p. 79, with the constant h replaced by 1):

$$c(x,\xi) = \frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}a(x+\omega,\rho+\xi)b(x+r,\tau+\xi)\,d\rho\,d\omega\,d\tau\,dr, \tag{65}$$

for every $(x,\xi)\in\mathbb{R}\times\mathbb{R}$. We split the oscillator integral c into two parts $c^{(1)}$ and $c^{(2)}$, then we use the cuttof function:

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho) = \chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^{2l}+\xi^{2l})^{\frac{\eta}{2}}}\right], \tag{66}$$

and

$$\omega_{2,\varepsilon} = 1-\omega_{1,\varepsilon},$$

where $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ in $[-1,1]$, $\chi\equiv 0$ in $\mathbb{R}\backslash]-2,2[$ and $\eta>0$.

Let us consider for $j\in\{1,2\}$

$$d_j(x,\xi,\omega,\tau,r,\rho) = \omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho)a(x+\omega,\rho+\xi)b(x+r,\tau+\xi), \tag{67}$$

$c^{(1)}$ (resp $c^{(2)}$) the integral obtained in (65) by replacing the amplitude by d_1 (resp d_2).

Study of $c^{(2)}$:

On the support of d_2, we have $\omega^2+\rho^2+r^2+\tau^2 \geq 2\varepsilon(1+x^{2l}+\xi^{2l})^{\frac{\eta}{2}}$, we perform integration by parts using the operator:

$$M = \frac{1}{2i}(\omega^2+\rho^2+r^2+\tau^2)^{-1}(-\rho\partial_r - r\partial_\rho + \tau\partial_\omega + \omega\partial_\tau),$$

for all $k\in\mathbb{N}$, we have:

$$c^{(2)} = \frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}({}^tM)^k d_2\,d\rho\,d\omega\,d\tau\,dr,$$

leading to

$$c^{(2)} \in \Gamma_0^{m_1+m_2-\frac{\eta}{2}k}. \tag{68}$$

Study of $c^{(1)}$:

The function $(\omega,\tau,r,\rho) \longrightarrow d_1(x,\xi,\omega,\tau,r,\rho)$ has a compact support. From ([14], Proposition II $-$ 26), we deduce that for every $N\in\mathbb{N}$,

$$c^{(1)} = \sum_{j=0}^{N} c_j(x,\xi) + R_{N+1}(x,\xi), \tag{69}$$

where

$$c_j = \frac{1}{2^j}\sum_{\alpha+\beta=j}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(\partial_x^\alpha\partial_\xi^\beta b), \tag{70}$$

and

$$|R_{N+1}(x,\xi)| \leq c_N \left\|(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1} d_1\right\|_{H^3(\mathbb{R}^4)}, \tag{71}$$

where $H^3(\mathbb{R}^4)$ is the Sobolev space. Since $a \in \Gamma_\rho^{m_1}$ and $b \in \Gamma_0^{m_2}$, we have for every $j \leq N$:

$$c = \sum_{j=0}^{+\infty} c_j, \quad c_j \in \Gamma_0^{m_1+m_2-\rho j}. \tag{72}$$

Let us study the rest term $R_{N+1}(x,\xi)$ in equation (69). From (71) we have:

$$\begin{aligned} & |R_{N+1}(x,\xi)| \\ \leq\ & c_N \sum_{\substack{|\gamma|\leq 3\\ \gamma\in\mathbb{N}^4}} \left\|(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1} d_1 \partial^\gamma_{\omega,,\tau,r,\rho} d_1\right\|_{\mathbb{L}^2(\mathbb{R}^4)} \\ \leq\ & c_N(1+x^{2l}+\xi^{2l})^\eta \\ \times & \sup_{\substack{[\omega^2+\rho^2+r^2+\tau^2\leq 2\varepsilon(1+x^{2l}+\xi^{2l})^{\frac{\eta}{2}}]\\ |\gamma|\leq 3}} |(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,,\tau,r,\rho} d_1|, \end{aligned} \tag{73}$$

for $|\gamma| \leq 3$, we have :

$$\begin{aligned} & |(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,,\tau,r,\rho} d_1| \\ =\ & (N+1)! \sum_{\alpha+\beta=N+1} \frac{(-1)^\beta}{\alpha!\beta!} \partial^\beta_\omega \partial^\beta_\tau \partial^\alpha_r \partial^\alpha_\rho \partial^\gamma_{\omega,\tau,r,\rho} d_1. \end{aligned} \tag{74}$$

Since a (resp b) are independent of (τ, r) (resp (ω,ρ)), for $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ we have:

$$\begin{aligned} & |\partial^\beta_\omega \partial^\beta_\tau \partial^\alpha_r \partial^\alpha_\rho \partial^\gamma_{\omega,\tau,r,\rho} d_1| \\ \leq\ & C \sum_{\substack{i_1+i_2=\beta+\gamma_1\\ i_p\leq\beta+\gamma_1\\ j_1+j_2=\beta+\gamma_2\\ j_p\leq\beta+\gamma_2\\ k_1+k_2=\alpha+\gamma_3\\ k_p\leq\alpha+\gamma_3\\ r_1+r_2=\alpha+\gamma_4\\ r_p\leq\alpha+\gamma_4}} |\partial^{i_1}_\omega \partial^{r_1}_\rho a||\partial^{j_1}_\tau \partial^{k_1}_r b||\partial^{i_2}_\omega \partial^{j_2}_\tau \partial^{k_2}_r \partial^{r_2}_\rho \omega_{1,\varepsilon}|, \end{aligned} \tag{75}$$

on the support of $\omega_{1,\varepsilon}$ and for sufficiently small ε, we have:

$$|\partial^{i_2}_\omega \partial^{j_2}_\tau \partial^{k_2}_\tau \partial^{r_2}_\rho \omega_{1,\varepsilon}| \leq c(1+x^{2l}+\xi^{2l})^{\frac{-\eta}{4}(i_2+j_2+k_2+r_2)}. \tag{76}$$

We now introduce the following lemma.

Lemma 6.1. *For sufficiently small $\varepsilon > 0$ and $0 < \eta \leq \frac{2}{l}$, there exist positive constants c,c',C, C' such that:*

$$c(1 + x^{2l} + \xi^{2l})^{\frac{1}{2}} \leq (1 + (x + u)^{2l} + (\xi + v)^{2l})^{\frac{1}{2}} \leq C(1 + x^{2l} + \xi^{2l})^{\frac{1}{2}},$$

for all x, ξ, u and v in $\mathbb{R}$.

P r o o f. The convexity of the function $x \longrightarrow x^{2l}$, along with the condition $0 < \eta \leq \frac{2}{l}$ allows us to demonstrate that:

$$1 + (x + u)^{2l} + (\xi + v)^{2l} \leq C(1 + x^{2l} + \xi^{2l}).$$

Similarly, we have:

$$x^{2l} \leq 2^{2l-1}((x + u)^{2l} + u^{2l}) \quad ; \quad \xi^{2l} \leq 2^{2l-1}((\xi + v)^{2l} + v^{2l}),$$

hence

$$(1 + x^{2l} + \xi^{2l})(1 - 2^{3l}\varepsilon^{l}) \leq C(1 + (x + u)^{2l} + (\xi + v)^{2l}),$$

for sufficiently small ε, we obtain:

$$c(1 + x^{2l} + \xi^{2l}) \leq (1 + (x + u)^{2l} + (\xi + v)^{2l}),$$

which implies

$$c(1 + x^{2l} + \xi^{2l})^{\frac{1}{2}} \leq (1 + (x + u)^{2l} + (\xi + v)^{2l})^{\frac{1}{2}} \leq C(1 + x^{2l} + \xi^{2l})^{\frac{1}{2}}.$$

□

From equations (75),(76), Lemma 6.1, and the facts that $a \in \Gamma_{\rho}^{m_1}$, $b \in \Gamma_{0}^{m_2}$, we have:

$$\begin{aligned}|\partial_{\omega}^{\beta}\partial_{\tau}^{\beta}\partial_{r}^{\alpha}\partial_{\rho}^{\alpha}\partial_{\omega,\tau,r,\rho}^{\gamma}d_1| &\leq C(1 + x^{2l} + \xi^{2l})^{\frac{m_1+m_2}{2}} \\ &\times \sum (1 + x^{2l} + \xi^{2l})^{-\frac{\rho}{2}(i_1+r_1)-\frac{\eta}{4}(i_2+j_2+k_2+r_2)},\end{aligned} \tag{77}$$

assuming that $\frac{\eta}{2} \geq \rho$, we have:

$$\begin{aligned}|\partial_{\omega}^{\beta}\partial_{\tau}^{\beta}\partial_{r}^{\alpha}\partial_{\rho}^{\alpha}\partial_{\omega,\tau,r,\rho}^{\gamma}d_1| \;&\leq\; C(e + x^{2l} + \xi^{2l})^{\frac{m_1+m_2}{2}} \\ &\times\; \sum (1 + x^{2l} + \xi^{2l})^{-\frac{\rho}{2}(i_1+i_2+r_1+r_2)-\frac{\rho}{2}(j_2+k_2)},\end{aligned}$$

since $i_1 + i_2 + r_1 + r_2 = N + 1 + \gamma_1 + \gamma_4$, it follows that:

$$|\partial_{\omega}^{\beta}\partial_{\tau}^{\beta}\partial_{r}^{\alpha}\partial_{\rho}^{\alpha}\partial_{\omega,\tau,r,\rho}^{\gamma}d_1| \;\leq\; C(e + \sigma_H)^{\frac{m_1+m_2-(N+1)\rho}{2}}, \tag{78}$$

using (74) and (78), we get:

$$|(\partial_{\omega}\partial_{x} - \partial_{r}\partial_{\rho})^{N+1}\partial_{\omega,,\tau,r,\rho}^{\gamma}d_1| \;\leq\; C_N(e + \sigma_H)^{\frac{m_1+m_2-(N+1)\rho}{2}}. \tag{79}$$

Finally, combining (73) and (79), we obtain the following estimate for R_{N+1}:

$$|R_{N+1}(x,\xi)| \ \leq \ C_N(e+\sigma_H)^{\frac{m_1+m_2-(N+1)\rho+2\eta}{2}}, \tag{80}$$

this implies that

$$R_{N+1} \in \Gamma_0^{m_1+m_2-(N+1)\rho+2\eta},$$

the rest of the symbol c is expressed as:

$$\delta_{N+1}(x,\xi) = R_{N+1}(x,\xi) + c^{(2)}(x,\xi). \tag{81}$$

To estimate δ_{N+1}, we use (68),(80) and (81), by further expanding the development, i.e., writing:

$$\delta_{N+1}(x,\xi) = c_{N+1} + \cdots + c_{N+k} + \delta_{N+1+k},$$

and choosing $k \geq 4$, we find that

$$\delta_{N+1} \in \Gamma_0^{m_1+m_2-(N+1)\rho}.$$

ii) It is sufficient to do the same for $n = 2$. Let us first note that $H \in G^2_{\frac{1}{l}}$, we have:

$$B_1B_2H^{-\frac{m_1+m_2}{2}} = B_1H^{-\frac{m_1}{2}}H^{\frac{m_1}{2}}B_2H^{-\frac{m_1+m_2}{2}},$$

according to i) the operator $B_1H^{-\frac{m_1}{2}} \in G_0^0$, and according to Calderon Vaillancourt's Theorem 2.1, $B_1H^{-\frac{m_1}{2}}$ is bounded. Similarly, the operator $B_2H^{-\frac{m_1+m_2}{2}} \in G_0^{-\frac{m_1}{2}}$, thus $H^{\frac{m_1}{2}}B_2H^{-\frac{m_1+m_2}{2}} \in G_0^0$.

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