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Double Sumudu Integral Transform on Time Scales with Applications

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Abstract

In this paper, we introduce an extension of the double Sumudu transform on time scales. The fundamental properties of the double Sumudu transform are established, including its existence, shifting property, transform of the derivative, and convolution theorem.The double Sumudu transform is demonstrated to be an effective and efficient technique for solving partial and partial-integro dynamic equations on time scales.

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Key Words and Phrases: Sumudu transform, Double Sumudu transform, Time scales calculus.

1 Introduction

Time scale calculus is a relatively new theory that unites two approaches of dynamic modeling difference equation and differential equation. S. Hilger introduced this theory in his Ph. D thesis. Time scale is any non-empty closed subset of real numbers. The double Sumudu transform helps to solve partial dynamic equations and integro-dynamic equations. Motivating by this theory, integral transforms such as Laplace, Fourier, Sumudu, and Shehu transforms have been generalized on time scales [2–9, 11, 12, 17]. Hassan A. Agwa [1] introduced the Sumudu transform on time scales for the rd-continuous function $f : \mathbb{T}_0 \to \mathbb{R}$ defined as,

$$S\left\{f\right\}(u) = \frac{1}{u}\int_{t_0}^{\infty} f\left(t\right) e^{\sigma}_{\ominus\frac{1}{u}}\left(t, t_0\right)\ \Delta t.$$

for $u \in D\left\{f\right\}$, where $D\left\{f\right\}$ consists of all complex numbers $u \in \mathcal{R}$ for which the improper integral exists.

M. J. Tchueche and N. S. Mbare [18] introduced the double Sumudu transform of the function $\varphi(t, x)$ given by

$$S_2\left[\varphi\left(t, x\right); (u, v)\right] = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{t}{u}+\frac{x}{v}\right)}\varphi\left(t, x\right)\ dt\ dx$$

where $t, x \in \mathbb{R}^+$. This transform extends the classical Sumudu transform to two dimensions and is particularly useful in solving certain partial differential equations. In this

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paper, we have generalized the double Sumudu transform on time scales, which is helpful for solving partial and integral dynamic equations.

2 Preliminaries

In this section, all the results and basic terminologies follow from [1–5, 13–18]. Any non-empty closed subset of $\mathbb{R}$ is called a time scale. Forward jump Operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, it is called right scattered. If $\rho(t) < t$, then t is called left scattered. If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right dense point. If $t > \inf\mathbb{T}$ and $\rho(t) = t$, the t is called the left dense point. Points that are right-scattered and left-scattered at the same time are called isolated. It means that $\rho(t) < t < \sigma(t)$. The function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ is called graininess function.

Definition 2.1. If a function $\varphi : \mathbb{T} \to \mathbb{R}$ is continuous at a right-dense point in $\mathbb{T}$ and has a left-sided limit at left-dense points in $\mathbb{T}$, then the function is known as rd-continuous.

Definition 2.2. A function $\varphi : \mathbb{T} \to \mathbb{R}$ is called regressive provided, $1 + \mu(t)\varphi(t) \neq 0$ for all $t \in \mathbb{T}$. We denote the set of all regressive functions by $\mathscr{R}$. Further, for $h > 0$, the set of Hilger complex numbers is, $\mathbb{C}_h := \left\{z \in \mathbb{C} : z \neq \frac{-1}{h}\right\}$ and for $z \in \mathbb{C}$ the Hilger real part of z is $Re_h(z) := \frac{|zh+1|-1}{h}$.

Definition 2.3 (Regulated Function)**.** A function $\varphi : \mathbb{T} \to \mathbb{R}$ is said to be regulated if

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its right-sided limits exist(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left-dense points in $\mathbb{T}$.

Theorem 2.4. *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}$, then*

i) $\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t;$

ii) $\int_a^b (\beta f(t))\,\Delta t = \beta \int_a^b f(t)\,\Delta t;$

iii) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t;$

iv) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t;$

v) $\int_a^b f(\sigma(t))\, g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(t)\ \Delta t;$

vi) $\int_a^b f(t)\, g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(\sigma(t))\ \Delta t;$

vii) $\int_a^a f(t)\,\Delta t = 0;$

viii) *if* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*

$$\left|\int_a^b f(t)\,\Delta t\right| \leq \int_a^b g(t)\,\Delta t;$$

ix) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\ \Delta t \geq 0$.

Theorem 2.5. *[2, 4] If $\lambda_1, \lambda_2 : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous, then the following properties hold:*

1) $\lambda_1 \oplus \lambda_2 = \lambda_1 + \lambda_2 + \mu\, \lambda_1 \lambda_2,$

2) $\lambda_1 \ominus \lambda_2 = \lambda_1 \oplus (\ominus \lambda_2) = \frac{\lambda_1 - \lambda_2}{1 + \mu \lambda_2},$

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3) $\ominus\lambda_1 = \frac{-\lambda_1}{1+\mu\lambda_1}$,

4) $e_{\lambda_1}(\sigma(t), t_0) = e_{\lambda_1}(t, t_0)(1 + \mu(t)\lambda_1(t))$,

5) $e_{\ominus\lambda_1}(t, t_0) = \frac{1}{e_{\lambda_1}(t,t_0)}$,

6) $e_{\lambda_1}(t, t_0).e_{\lambda_2}(t, t_0) = e_{\lambda_1\oplus\lambda_2}(t, t_0)$,

7) $frac e_{\lambda_1}(t, t_0) e_{\lambda_2}(t, t_0) = e_{\lambda_1\ominus\lambda_2}(t, t_0)$,

8) $e^{\sigma}_{\ominus\lambda_1}(t, t_0) = \frac{e_{\ominus\lambda_1(t,t_0)}}{1+\mu\lambda_1}$.

In this article, we use the following notations

$e_{\lambda_1\oplus\lambda_2}\left(t_1, t_2, t_0, t_0^{'}\right) = e_{\lambda_1}(t_1, t_0)\, e_{\lambda_2}\left(t_2, t_0^{'}\right)$ *and* $e_{\ominus\lambda_1\ominus\lambda_2}\left(t_1, t_2, t_0, t_0^{'}\right) = e_{\ominus\lambda_1}(t_1, t_0)\, e_{\ominus\lambda_2}\left(t_2, t_0^{'}\right)$.

Definition 2.6. The function $\varphi(t_1, t_2) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is said to be of exponential type I, if there exist constants $\mathcal{M}, d_1, d_2 > 0$ such that $|\varphi(t_1, t_2)| \leq \mathcal{M}e^{d_1t_1+d_2t_2}$ and φ is said to be of exponential type II if there exist constants $\mathcal{M}, d_1, d_2 > 0$ such that $|\varphi(t_1, t_2)| \leq \mathcal{M}e_{d_1\oplus d_2}\left(t_1, t_2, t_0, t_0^{'}\right)$.

3 Double Sumudu Transform on Time Scales

In this section, we extend the double Sumudu transform for time scales and present some essential properties of the double Sumudu transform.

Definition 3.1. Suppose that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales such that $\sup\{\mathbb{T}_1, \mathbb{T}_2\} = \infty$ and $t_0 \in \mathbb{T}_1, t_0^{'} \in \mathbb{T}_2$ are fixed. Let $\varphi : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ be rd-continuous functions. Then the

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generalized double Sumudu transform of $\varphi(t_1,t_2)$ is

$$\Phi(\lambda_1,\lambda_2)=\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)]=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi(t_1,t_2)\ \Delta t_1\ \Delta t_2$$

Provided the integral exists with $1+\mu_1\frac{1}{\lambda_1}\neq 0$ and $1+\mu_2\frac{1}{\lambda_2}\neq 0$ for all $(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2$.

Theorem 3.2 (Existence Theorem). *If $\varphi(t_1,t_2)$ is rd-continuous function on $\mathbb{T}_1\bigcap[t_0,\xi_1]\times\mathbb{T}_2\bigcap\left[t_0',\xi_2\right]$ and $|\varphi(t_1,t_2)|\leq Me_{d_1\oplus d_2}\left(t_1,t_2,t_0,t_0'\right)$, then the double Sumudu transform of $\varphi(t_1,t_2)$ exists for all positively regressive $\ominus\frac{1}{\lambda_1}$ and $\ominus\frac{1}{\lambda_2}$ provided, $\lim\limits_{t_1\to\infty}e_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)\to 0$ and $\lim\limits_{t_2\to\infty}e_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\to 0$.*

Proof. We have, $|\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)]|$

$$=\left|\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi(t_1,t_2)\ \Delta t_1\ \Delta t_2\right|$$
$$\leq\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}\left|e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi(t_1,t_2)\ \Delta t_1\ \Delta t_2\right|$$
$$\leq\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)|\varphi(t_1,t_2)|\,\Delta t_1\Delta t_2$$
$$\leq\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)Me_{d_1\oplus d_2}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\Delta t_2$$
$$=\frac{M}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}\frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)}{\left(1+\mu_1(t)\frac{1}{\lambda_1}\right)\left(1+\mu_2(t)\frac{1}{\lambda_2}\right)}e_{d_1\oplus d_2}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\Delta t_2$$
$$=\frac{M}{\lambda_1}\int_{t_0}^{\infty}\frac{e_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1(t)\frac{1}{\lambda_1}\right)}\left[\int_{t_0'}^{\infty}\frac{e_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)}{\lambda_2\left(1+\mu_2(t)\frac{1}{\lambda_2}\right)}\Delta t_2\right]\Delta t_1$$
$$=\frac{M}{\lambda_1\left(d_1-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}\frac{\left(d_1-\frac{1}{\lambda_1}\right)e_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1(t)\frac{1}{\lambda_1}\right)}\left[\frac{1}{\lambda_2\left(d_2-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}\frac{\lambda_2\left(d_2-\frac{1}{\lambda_2}\right)e_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)}{\left(1+\mu_2(t)\frac{1}{\lambda_2}\right)}\Delta t_2\right]\Delta t_1$$
$$=\frac{M}{\lambda_1\left(d_1-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}\left(d_1\ominus\frac{1}{\lambda_1}\right)e_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)\left[\frac{1}{\lambda_2\left(d_2-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}\left(d_2\ominus\frac{1}{\lambda_2}\right)e_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\Delta t_2\right]\Delta t_1$$
$$=\frac{M}{\lambda_1\left(d_1-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}e^{\Delta_1}_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)\left[\frac{1}{\lambda_2\left(d_2-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}e^{\Delta_2}_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\Delta t_2\right]\Delta t_1$$
$$=\frac{M}{\lambda_1\left(d_1-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}e^{\Delta_1}_{d_1\ominus\frac{1}{\lambda_1}}(t_1,t_0)\left[\frac{1}{\lambda_2\left(d_2-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}e^{\Delta_2}_{d_2\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\Delta t_2\right]\Delta t_1$$

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$= \frac{M}{\lambda_1\lambda_2\left(\frac{1}{\lambda_1}-d_1\right)\left(\frac{1}{\lambda_2}-d_2\right)}$
$= \frac{M}{(1-d_1\lambda_1)(1-d_2\lambda_2)}$ □

3.1 Double Sumudu transform of some elementary functions

To find the double Sumudu transform of some elementary functions, we apply the Definition 3.1.

1) $\mathcal{S}_1\mathcal{S}_2\left[1\right] = 1$

Proof. $\mathcal{S}_1\mathcal{S}_2\left[1\right]$

$$
\begin{aligned}
&= \tfrac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\,\Delta t_2\\
&= \left[\tfrac{1}{\lambda_1}\int_{t_0}^{\infty}\frac{e_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)}\Delta t_1\right]\left[\tfrac{1}{\lambda_2}\int_{t_0'}^{\infty}\frac{e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)}{\left(1+\mu_2\frac{1}{\lambda_2}\right)}\Delta t_2\right]\\
&= \left[\frac{1}{\lambda_1\left(\frac{-1}{\lambda_1}\right)}\int_{t_0}^{\infty}\frac{\left(\frac{-1}{\lambda_1}\right)e_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)}\Delta t_1\right]\left[\frac{1}{\lambda_2\left(\frac{-1}{\lambda_2}\right)}\int_{t_0'}^{\infty}\frac{\left(\frac{-1}{\lambda_2}\right)e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)}{\left(1+\mu_2\frac{1}{\lambda_2}\right)}\Delta t_2\right]\\
&= \left[-\int_{t_0}^{\infty}\left(\ominus\frac{1}{\lambda_1}\right)e_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\Delta t_1\right]\left[-\int_{t_0'}^{\infty}\left(\ominus\frac{1}{\lambda_2}\right)e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\Delta t_2\right]\\
&= \left[-\int_{t_0}^{\infty}e^{\Delta_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\Delta t_1\right]\left[-\int_{t_0'}^{\infty}e^{\Delta_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\Delta t_2\right]
\end{aligned}
$$

$= 1$

Provided $\lim_{t_1\to\infty} e_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\to 0$ and $\lim_{t_2\to\infty} e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\to 0.$ □

2) $\mathcal{S}_1\mathcal{S}_2\left[e_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\right] = \frac{1}{(1-\gamma\lambda_1)(1-\tau\lambda_2)}$

Proof. $\mathcal{S}_1\mathcal{S}_2\left[e_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\right]$

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$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right) e_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\,\Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)} e_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\,\Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)e_{\tau\ominus\frac{1}{\lambda_2}}(t_1,t_0)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)}\Delta t_1\,\Delta t_2$$
$$= \left[\frac{1}{\lambda_1}\int_{t_0}^{\infty}\frac{e_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)}\Delta t_1\right]\left[\frac{1}{\lambda_2}\int_{t_0'}^{\infty}\frac{e_{\tau\ominus\frac{1}{\lambda_2}}(t_1,t_0)}{\left(1+\mu_2\frac{1}{\lambda_2}\right)}\Delta t_2\right]$$
$$= \left[\frac{1}{\lambda_1\left(\gamma-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}\frac{\left(\gamma-\frac{1}{\lambda_1}\right)e_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)}\Delta t_1\right]\left[\frac{1}{\lambda_2\left(\tau-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}\frac{\left(\tau-\frac{1}{\lambda_2}\right)e_{\tau\ominus\frac{1}{\lambda_2}}(t_1,t_0)}{\left(1+\mu_2\frac{1}{\lambda_2}\right)}\Delta t_2\right]$$
$$= \left[\frac{1}{\lambda_1\left(\gamma-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}\left(\gamma\ominus\frac{1}{\lambda_1}\right)e_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\Delta t_1\right]\left[\frac{1}{\lambda_2\left(\tau-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}\left(\tau\ominus\frac{1}{\lambda_2}\right)e_{\tau\ominus\frac{1}{\lambda_2}}(t_1,t_0)\,\Delta t_2\right]$$
$$= \left[\frac{1}{\lambda_1\left(\gamma-\frac{1}{\lambda_1}\right)}\int_{t_0}^{\infty}e^{\Delta_1}_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\Delta t_1\right]\left[\frac{1}{\lambda_2\left(\tau-\frac{1}{\lambda_2}\right)}\int_{t_0'}^{\infty}e^{\Delta_2}_{\tau\ominus\frac{1}{\lambda_2}}(t_1,t_0)\,\Delta t_2\right]$$
$$= \left[\frac{1}{\lambda_1\left(\frac{1}{\lambda_1}-\gamma\right)}\right]\left[\frac{1}{\lambda_2\left(\frac{1}{\lambda_2}-\tau\right)}\right]$$
$$= \frac{1}{(1-\gamma\lambda_1)(1-\tau\lambda_2)}$$

Provided $\lim_{t_1\to\infty} e_{\gamma\ominus\frac{1}{\lambda_1}}(t_1,t_0)\to 0$ and $\lim_{t_2\to\infty} e_{\tau\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\to 0$. □

3) $\mathcal{S}_1\mathcal{S}_2\left[\sin_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\right] = \frac{\tau\lambda_2+\gamma\lambda_1}{\left[1+\gamma^2\lambda_1^2\right]\left[1+\tau^2\lambda_2^2\right]}$

Proof. $\mathcal{S}_1\mathcal{S}_2\left[\sin_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\right]$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\sin_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\Delta t_1\,\Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\left[\frac{e_{i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0'\right)-e_{\ominus i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0'\right)}{2i}\right]\Delta t_1\,\Delta t_2$$
$$= \mathcal{S}_1\mathcal{S}_2\left[\frac{e_{i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0'\right)-e_{\ominus i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0'\right)}{2i}\right]$$
$$= \frac{1}{2i}\left[\frac{1}{(1-i\gamma\lambda_1)(1-i\tau\lambda_2)}-\frac{1}{(1+i\gamma\lambda_1)(1+i\tau\lambda_2)}\right]$$
$$= \frac{\tau\lambda_2+\gamma\lambda_1}{\left[1+\gamma^2\lambda_1^2\right]\left[1+\tau^2\lambda_2^2\right]}$$ □

4) $\mathcal{S}_1\mathcal{S}_2\left[\cos_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0'\right)\right] = \frac{1-\gamma\tau\lambda_1\lambda_2}{\left[1+\gamma^2\lambda_1^2\right]\left[1+\tau^2\lambda_2^2\right]}$

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Proof. $\mathcal{S}_1\mathcal{S}_2\left[\cos_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\right]$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\cos_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\Delta t_1\ \Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\left[\frac{e_{i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)+e_{\ominus i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]\Delta t_1\ \Delta t_2$$
$$= \mathcal{S}_1\mathcal{S}_2\left[\frac{e_{i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)+e_{\ominus i(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{(1-i\gamma\lambda_1)(1-i\tau\lambda_2)}+\frac{1}{(1+i\gamma\lambda_1)(1+i\tau\lambda_2)}\right]$$
$$= \frac{1-\gamma\tau\lambda_1\lambda_2}{\left[1+\gamma^2\lambda_1^2\right]\left[1+\tau^2\lambda_2^2\right]}$$ □

5) $\mathcal{S}_1\mathcal{S}_2\left[\sinh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\right] = \frac{\tau\lambda_2+\gamma\lambda_1}{\left[1-\gamma^2\lambda_1^2\right]\left[1-\tau^2\lambda_2^2\right]}$

Proof. $\mathcal{S}_1\mathcal{S}_2\left[\sinh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\right]$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\sinh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\Delta t_1\ \Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\left[\frac{e_{(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)-e_{\ominus(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]\Delta t_1\ \Delta t_2$$
$$= \mathcal{S}_1\mathcal{S}_2\left[\frac{e_{(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)-e_{\ominus(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{(1-\gamma\lambda_1)(1-\tau\lambda_2)}-\frac{1}{(1+\gamma\lambda_1)(1+\tau\lambda_2)}\right]$$
$$= \frac{\tau\lambda_2+\gamma\lambda_1}{\left[1-\gamma^2\lambda_1^2\right]\left[1-\tau^2\lambda_2^2\right]}$$ □

6) $\mathcal{S}_1\mathcal{S}_2\left[\cosh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\right] == \frac{1+\gamma\tau\lambda_1\lambda_2}{\left[1-\gamma^2\lambda_1^2\right]\left[1-\tau^2\lambda_2^2\right]}$

Proof. $\mathcal{S}_1\mathcal{S}_2\left[\cosh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\right]$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\cosh_{\gamma\oplus\tau}\left(t_1,t_2,t_0,t_0^{'}\right)\Delta t_1\ \Delta t_2$$
$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0^{'}}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0^{'}\right)\left[\frac{e_{(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)+e_{\ominus(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]\Delta t_1\ \Delta t_2$$
$$= \mathcal{S}_1\mathcal{S}_2\left[\frac{e_{(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)+e_{\ominus(\gamma\oplus\tau)}\left(t_1,t_2,t_0,t_0^{'}\right)}{2}\right]$$

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$$= \frac{1}{2}\left[\frac{1}{(1-\gamma\lambda_1)(1-\tau\lambda_2)} + \frac{1}{(1+\gamma\lambda_1)(1+\tau\lambda_2)}\right]$$

$$= \frac{1+\gamma\tau\lambda_1\lambda_2}{\left[1-\gamma^2\lambda_1^2\right]\left[1-\tau^2\lambda_2^2\right]} \qquad \square$$

4 Main Results

Theorem 4.1 (Linearity Property). *If $\varphi_1(t_1, t_2)$ and $\varphi_2(t_1, t_2)$ are functions with the double Sumudu transform $\mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)]$ and $\mathcal{S}_1\mathcal{S}_2[\varphi_2(t_1, t_2)]$ respectively. Then*

$\mathcal{S}_1\mathcal{S}_2[b_1\varphi_1(t_1, t_2) + b_2\varphi_2(t_1, t_2)] = b_1\mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)] + b_2\mathcal{S}_1\mathcal{S}_2[\varphi_2(t_1, t_2)]$.

Proof. The proof follows from the definition 3.1 of the double Sumudu transform. $\square$

Theorem 4.2 (Shifting Theorem). *For $\eta_1 \in \mathbb{T}_1$ and $\eta_2 \in \mathbb{T}_2$ with $\eta_1, \eta_2 > 0$, we have*

$$G_{\eta_1,\eta_2}(t_1, t_2) = \begin{cases} 0 & : \ t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \ \&\ t_1 < \eta_1, t_2 < \eta_2 \\ 1 & : \ t_1 \in \mathbb{T}_1, t_2 \in \mathbb{T}_2 \ \&\ t_1 \geq \eta_1, t_2 \geq \eta_2. \end{cases}$$

Then $\mathcal{S}_1\mathcal{S}_2[G_{\eta_1,\eta_2}(t_1, t_2)\varphi(t_1, t_2)] = e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\eta_1, \eta_2, t_0, t_0'\right)\mathcal{S}_1\mathcal{S}_2[\varphi(t_1, t_2)]$

Proof. On applying the definition of the double Sumudu transform $\mathcal{S}_1\mathcal{S}_2[G_{\eta_1,\eta_2}(t_1, t_2)\varphi(t_1, t_2)]$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1, t_2, t_0, t_0'\right)[G_{\eta_1,\eta_2}(t_1, t_2)\varphi(t_1, t_2)]\,\Delta t_1\,\Delta t_2$$

$$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1, t_2, t_0, t_0'\right)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)}[G_{\eta_1,\eta_2}(t_1, t_2)\varphi(t_1, t_2)]\,\Delta t_1\,\Delta t_2$$

$$= \frac{1}{\lambda_1\lambda_2}\int_{\eta_1}^{\infty}\int_{\eta_2}^{\infty} \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\eta_1, \eta_2, t_0, t_0'\right)e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}(t_1, t_2, \eta_1, \eta_2)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)}\varphi(t_1, t_2)\,\Delta t_1\,\Delta t_2$$

$$= \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\eta_1, \eta_2, t_0, t_0'\right)}{\lambda_1\lambda_2}\int_{\eta_1}^{\infty}\int_{\eta_2}^{\infty} \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}(t_1, t_2, \eta_1, \eta_2)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)}\varphi(t_1, t_2)\,\Delta t_1\,\Delta t_2$$

$$= e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\eta_1, \eta_2, t_0, t_0'\right)\mathcal{S}_1\mathcal{S}_2[\varphi(t_1, t_2)] \qquad \square$$

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Theorem 4.3 (Derivative property of the double Sumudu transform). *Let* $\varphi(t_1,t_2) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ *be rd-continuous function such that*

$$\varphi^{\Delta_1}(t_1,t_2) = \frac{\partial\varphi(t_1,t_2)}{\Delta_1 t_1},\ \varphi^{\Delta_1^2}(t_1,t_2) = \frac{\partial^2\varphi(t_1,t_2)}{\Delta_1 t_1^2},$$
$$\varphi^{\Delta_2}(t_1,t_2) = \frac{\partial\varphi(t_1,t_2)}{\Delta_2 t_2},\ \varphi^{\Delta_2^2}(t_1,t_2) = \frac{\partial^2\varphi(t_1,t_2)}{\Delta_2 t_2^2},$$

are also rd-continuous, then

1) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_1}(t_1,t_2)\right] = \frac{1}{\lambda_1}\left\{\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right] - \mathcal{S}_2\left[\varphi(t_0,t_2)\right]\right\}$,

2) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_2}(t_1,t_2)\right] = \frac{1}{\lambda_2}\left\{\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right] - \mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right]\right\}$,

3) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_1^2}(t_1,t_2)\right] = \frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right] - \frac{1}{\lambda_1^2}\mathcal{S}_2\left[\varphi(t_0,t_2)\right] - \frac{1}{\lambda_1}\mathcal{S}_2\left[\varphi^{\Delta_1}(t_0,t_2)\right]$,

4) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_2^2}(t_1,t_2)\right] = \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right] - \frac{1}{\lambda_2^2}\mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right] - \frac{1}{\lambda_2}\mathcal{S}_1\left[\varphi^{\Delta_2}\left(t_1,t_0'\right)\right]$

Provided $\lim\limits_{t_1\to\infty} e_{\ominus\frac{1}{\lambda_1}}(t_1,t_0) \to 0$ *and* $\lim\limits_{t_2\to\infty} e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right) \to 0$.

Proof.

1) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_1}(t_1,t_2)\right] = \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi^{\Delta_1}(t_1,t_2)\,\Delta t_1\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi^{\Delta_1}(t_1,t_2)\,\Delta t_1\right]\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[\int_{t_0}^{\infty}\left[\left(e_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi(t_1,t_2)\right)^{\Delta_1} - \left(e^{\Delta_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi(t_1,t_2)\right)\right]\Delta t_1\right]\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[-\varphi(t_0,t_2) - \int_{t_0}^{\infty}\left(\ominus\frac{1}{\lambda_1}\right)e_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi(t_1,t_2)\,\Delta t_1\right]\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[-\varphi(t_0,t_2) + \frac{1}{\lambda_1}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi(t_1,t_2)\,\Delta t_1\right]\,\Delta t_2$

$= \frac{1}{\lambda_1}\left[\frac{-1}{\lambda_2}\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi(t_0,t_2)\,\Delta t_2\right] + \frac{1}{\lambda_1}\left[\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi(t_1,t_2)\,\Delta t_1\Delta t_2\right]$

$= \frac{-1}{\lambda_1}\mathcal{S}_2\left[\varphi(t_0,t_2)\right] + \frac{1}{\lambda_1}\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right]$

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$= \frac{1}{\lambda_1}\left\{\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1,t_2\right)\right]-\mathcal{S}_2\left[\varphi\left(t_0,t_2\right)\right]\right\}$

2) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_2}\left(t_1,t_2\right)\right]=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi^{\Delta_2}\left(t_1,t_2\right)\Delta t_1\ \Delta t_2$

$=\frac{1}{\lambda_1\lambda_1}\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\left[\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi^{\Delta_2}\left(t_1,t_2\right)\Delta t_2\right]\ \Delta t_1$

$=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\left[\int_{t_0'}^{\infty}\left[\left(e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi\left(t_1,t_2\right)\right)^{\Delta_2}-\left(e^{\Delta_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi\left(t_1,t_2\right)\right)\right]\Delta t_2\right]\ \Delta t_1$

$=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\left[-\varphi\left(t_1,t_0'\right)-\int_{t_0'}^{\infty}\left(\ominus\frac{1}{\lambda_2}\right)e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi\left(t_1,t_2\right)\Delta t_2\right]\ \Delta t_1$

$=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\left[-\varphi\left(t_1,t_0'\right)+\frac{1}{\lambda_2}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi\left(t_1,t_2\right)\Delta t_2\right]\ \Delta t_1$

$=\frac{1}{\lambda_2}\left[\frac{-1}{\lambda_1}\int_{t_0'}^{\infty}e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\varphi\left(t_1,t_0'\right)\Delta t_1\right]+\frac{1}{\lambda_2}\left[\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi\left(t_1,t_2\right)\Delta t_1\Delta t_2\right]$

$=\frac{-1}{\lambda_2}\mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right]+\frac{1}{\lambda_2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1,t_2\right)\right]$

$=\frac{1}{\lambda_2}\left\{\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1,t_2\right)\right]-\mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right]\right\}$

3) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_1^2}\left(t_1,t_2\right)\right]=\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi^{\Delta_1^2}\left(t_1,t_2\right)\Delta t_1\ \Delta t_2$

$=\frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[\int_{t_0}^{\infty}\left[\left(e_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\varphi^{\Delta_1}\left(t_1,t_2\right)\right)^{\Delta_1}-\left(e^{\Delta_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\varphi^{\Delta_1}\left(t_1,t_2\right)\right)\right]\Delta t_1\right]\ \Delta t_2$

$=\frac{1}{\lambda_1\lambda_2}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\left[-\varphi^{\Delta_1}\left(t_0,t_2\right)+\frac{1}{\lambda_1}\int_{t_0}^{\infty}\left(e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}\left(t_1,t_0\right)\varphi^{\Delta_1}\left(t_1,t_2\right)\right)\Delta t_1\right]\ \Delta t_2$

$=\frac{-1}{\lambda_1}\left[\frac{1}{\lambda_2}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi^{\Delta_1}\left(t_0,t_2\right)\Delta t_2\right]+\frac{-1}{\lambda_1^2}\left[\frac{1}{\lambda_2}\int_{t_0'}^{\infty}e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi\left(t_0,t_2\right)\Delta t_2\right]$

$+\frac{1}{\lambda_1^2}\left[\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi\left(t_1,t_2\right)\Delta t_1\ \Delta t_2\right]$

$=-\frac{1}{\lambda_1}\mathcal{S}_2\left[\varphi^{\Delta_1}\left(t_0,t_2\right)\right]-\frac{1}{\lambda_1^2}\mathcal{S}_2\left[\varphi\left(t_0,t_2\right)\right]+\frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1,t_2\right)\right]$

$=\frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1,t_2\right)\right]-\frac{1}{\lambda_1^2}\mathcal{S}_2\left[\varphi\left(t_0,t_2\right)\right]-\frac{1}{\lambda_1}\mathcal{S}_2\left[\varphi^{\Delta_1}\left(t_0,t_2\right)\right]$

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4) $\mathcal{S}_1\mathcal{S}_2\left[\varphi^{\Delta_2^2}(t_1,t_2)\right] = \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi^{\Delta_2^2}(t_1,t_2)\,\Delta t_1\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\left[\int_{t_0'}^{\infty}\left[\left(e_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi^{\Delta_2}(t_1,t_2)\right)^{\Delta_2} - \left(e^{\Delta_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi^{\Delta_2}(t_1,t_2)\right)\right]\Delta t_2\right]\,\Delta t_1$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\left[-\varphi^{\Delta_2}\left(t_1,t_0'\right) + \frac{1}{\lambda_2}\int_{t_0'}^{\infty}\left(e^{\sigma_2}_{\ominus\frac{1}{\lambda_2}}\left(t_2,t_0'\right)\varphi^{\Delta_2}(t_1,t_2)\right)\Delta t_2\right]\,\Delta t_1$

$= \frac{-1}{\lambda_2}\left[\frac{1}{\lambda_1}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi^{\Delta_2}\left(t_1,t_0'\right)\Delta t_1\right] + \frac{-1}{\lambda_2^2}\left[\frac{1}{\lambda_1}\int_{t_0}^{\infty} e^{\sigma_1}_{\ominus\frac{1}{\lambda_1}}(t_1,t_0)\,\varphi\left(t_1,t_0'\right)\Delta t_2\right]$

$+ \frac{1}{\lambda_2^2}\left[\frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)\varphi(t_1,t_2)\,\Delta t_1\,\Delta t_2\right]$

$= -\frac{1}{\lambda_2}\mathcal{S}_1\left[\varphi^{\Delta_2}\left(t_1,t_0'\right)\right] - \frac{1}{\lambda_2^2}\mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right] + \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right]$

$= \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi(t_1,t_2)\right] - \frac{1}{\lambda_2^2}\mathcal{S}_1\left[\varphi\left(t_1,t_0'\right)\right] - \frac{1}{\lambda_2}\mathcal{S}_1\left[\varphi^{\Delta_2}\left(t_1,t_0'\right)\right]$ □

Theorem 4.4 (Double Sumudu transform of integrals)**.** *If* $\varphi(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ *is regulated, then* $\mathcal{S}_1\mathcal{S}_2\left[\int_{t_0}^{t_1}\int_{t_0'}^{t_2} q(\eta_1,\eta_2)\,\Delta\eta_1\,\Delta\eta_2\right] = \lambda_1\lambda_2\mathcal{S}_1\mathcal{S}_2\left[q(t_1,t_2)\right]$.

Proof. Let $Q(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0'}^{t_2} q(\eta_1,\eta_2)\,\Delta\eta_1\,\Delta\eta_2$

$\mathcal{S}_1\mathcal{S}_2\left[Q(t_1,t_2)\right]$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)Q(t_1,t_2)\,\Delta t_1\,\Delta t_2$

$= \frac{1}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)}{\left(1+\mu_1\frac{1}{\lambda_1}\right)\left(1+\mu_2\frac{1}{\lambda_2}\right)}Q(t_1,t_2)\,\Delta t_1\,\Delta t_2$

$= \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2}\int_{t_0}^{\infty}\int_{t_0'}^{\infty}\left(\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}\right)e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)Q(t_1,t_2)\,\Delta t_1\,\Delta t_2$

By using integration by parts rule and Fundamental theorem of calculus and $Q\left(t_0,t_0'\right)=0$, we get

$= \lambda_1\lambda_2\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1,t_2,t_0,t_0'\right)q(t_1,t_2)\,\Delta t_1\,\Delta t_2$

$= \lambda_1\lambda_2\mathcal{S}_1\mathcal{S}_2\left[q(t_1,t_2)\right]$. □

Definition 4.5 (Convolution Property)**.** [5, 8, 13–17] If $\varphi_1:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ is rd-continuous

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and

$\varphi_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is piecewise rd-continuous function of exponential type II, then the double convolution of φ_1 and φ_2 denoted by $\varphi_1 * *\varphi_2$ given by

$$[\varphi_1 * \varphi_2](t_1, t_2) = \int_{t_0}^{t_1} \int_{t_0'}^{t_2} \varphi_1(\xi_1, \xi_2) \varphi_2(t_1, t_2, \sigma_1(\xi_1), \sigma_2(\xi_2)) \ \Delta\xi_1 \Delta\xi_2$$

Where $\varphi_2(t_1, t_2, \sigma_1(\xi_1), \sigma_2(\xi_2))$ is delay of $\varphi_2(t_1, t_2)$ by $\sigma_1(\xi_1) \in \mathbb{T}_1$ and $\sigma_2(\xi_2) \in \mathbb{T}_2$.

Theorem 4.6 (Convolution theorem for the double Sumudu transform). *Let $\varphi_1 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ and $\varphi_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ be rd-continuous functions of exponential type II, having the double Sumudu transform $\mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)]$ and $\mathcal{S}_1\mathcal{S}_2[\varphi_2(t_1, t_2)]$ respectively. Then $\mathcal{S}_1\mathcal{S}_2\{[\varphi_1 * \varphi_2](t_1, t_2)\} = \lambda_1\lambda_2\mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)] \cdot \mathcal{S}_1\mathcal{S}_2[\varphi_2(t_1, t_2)]$.*

***Proof*.** By the definition of the double Sumudu transform on time scales, we obtain

$$\mathcal{S}_1\mathcal{S}_2\{[\varphi_1 * \varphi_2](t_1, t_2)\} = \tfrac{1}{\lambda_1\lambda_2} \int_{t_0}^{\infty} \int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1, t_2, t_0, t_0'\right) \{[\varphi_1 * \varphi_2](t_1, t_2)\} \Delta t_1 \ \Delta t_2$$
$$= \tfrac{1}{\lambda_1\lambda_2} \int_{t_0}^{\infty} \int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1, t_2, t_0, t_0'\right) \left\{\int_{t_0}^{t_1} \int_{t_0'}^{t_2} \varphi_1(t_1, t_2, \sigma_1(\xi_1), \sigma_2(\xi_2)) \varphi_2(\xi_1, \xi_2) \ \Delta\xi_1\Delta\xi_2\right\} \Delta t_1 \ \Delta t_2$$
$$= \int_{t_0}^{\infty} \int_{t_0'}^{\infty} \varphi_2(\xi_1, \xi_2)$$
$$\times \left\{\tfrac{1}{\lambda_1\lambda_2} \int_{t_0}^{\infty} \int_{t_0'}^{\infty} \varphi_1(t_1, t_2, \sigma_1(\xi_1), \sigma_2(\xi_2)) G_{\sigma_1(\xi_1), \sigma_2(\xi_2)}(t_1, t_2) e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(t_1, t_2, t_0, t_0'\right) \ \Delta t_1 \ \Delta t_2\right\} \Delta\xi_1\Delta\xi_2$$
$$= \int_{t_0}^{\infty} \int_{t_0'}^{\infty} \varphi_2(\xi_1, \xi_2) \mathcal{S}_1\mathcal{S}_2\left[G_{\sigma_1(\xi_1), \sigma_2(\xi_2)}(t_1, t_2) \varphi_1(t_1, t_2, \sigma_1(\xi_1), \sigma_2(\xi_2))\right] \Delta\xi_1\Delta\xi_2$$
$$= \int_{t_0}^{\infty} \int_{t_0'}^{\infty} \varphi_2(\xi_1, \xi_2) e_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\sigma_1(\xi_1), \sigma_2(\xi_2), t_0, t_0'\right) \mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)] \Delta\xi_1\Delta\xi_2$$
$$= \mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)] \left[\tfrac{\lambda_1\lambda_2}{\lambda_1\lambda_2} \int_{t_0}^{\infty} \int_{t_0'}^{\infty} \varphi_2(\xi_1, \xi_2) e^{\sigma_1\sigma_2}_{\ominus\frac{1}{\lambda_1}\ominus\frac{1}{\lambda_2}}\left(\xi_1, \xi_2, t_0, t_0'\right) \Delta\xi_1\Delta\xi_2\right]$$
$$= \lambda_1\lambda_2\mathcal{S}_1\mathcal{S}_2[\varphi_1(t_1, t_2)] \cdot \mathcal{S}_1\mathcal{S}_2[\varphi_2(t_1, t_2)]. \qquad \square$$

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5 Applications

Example 5.1. [14, 18] Consider the following partial-integro dynamic equation for $\mathbb{T}_1 \times \mathbb{T}_2$ such that $t_0 \in \mathbb{T}_1, t_0^{'} \in \mathbb{T}_2$.

$$\varphi^{\Delta_1}(t_1,t_2)+\varphi^{\Delta_2}(t_1,t_2) = -1+e_1(t_1,0)+e_1(t_2,0)+e_{1\oplus 1}(t_1,t_2,0,0)+\int_{t_0}^{t_1}\int_{t_0^{'}}^{t_2}\varphi(\tau_1,\tau_2)\ \Delta\tau_1\Delta\tau_2$$

with initial conditions,

$$\varphi\left(t_1,t_0^{'}\right) = e_1\left(t_1,t_0^{'}\right), \ \ \varphi(t_0,t_2) = e_1(t_2,t_0).$$

Solution: Applying the double Sumudu transform on both sides, we get

$$\frac{1}{\lambda_1}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)] - \frac{1}{\lambda_1}\mathcal{S}_2[\varphi(t_0,t_2)] + \frac{1}{\lambda_2}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)] - \frac{1}{\lambda_2}\mathcal{S}_1\left[\varphi\left(t_1,t_0^{'}\right)\right] = \mathcal{S}_1\mathcal{S}_2[-1] + \mathcal{S}_1\mathcal{S}_2[e_1(t_1,0)]+\mathcal{S}_1\mathcal{S}_2[e_1(t_2,0)]+\mathcal{S}_1\mathcal{S}_2[e_{1\oplus 1}(t_1,t_2,0,0)]+\mathcal{S}_1\mathcal{S}_2\left[\int_{t_0}^{t_1}\int_{t_0^{'}}^{t_2}\varphi(\tau_1,\tau_2)\ \Delta\tau_1\Delta\tau_2\right].$$

Which gives

$$\frac{1}{\lambda_1}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)] - \frac{1}{\lambda_1}\mathcal{S}_2[e_1(t_2,t_0)] + \frac{1}{\lambda_2}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)] - \frac{1}{\lambda_2}\mathcal{S}_1\left[e_1\left(t_1,t_0^{'}\right)\right] = \mathcal{S}_1\mathcal{S}_2[-1] + \mathcal{S}_1\mathcal{S}_2\left[e_1\left(t_1,t_0^{'}\right)\right]+\mathcal{S}_1\mathcal{S}_2[e_1(t_2,t_0)]+\mathcal{S}_1\mathcal{S}_2\left[e_{1\oplus 1}\left(t_1,t_2,t_0,t_0^{'}\right)\right]+\mathcal{S}_1\mathcal{S}_2\left[\int_{t_0}^{t_1}\int_{t_0^{'}}^{t_2}\varphi(\tau_1,\tau_2)\ \Delta\tau_1\Delta\tau_2\right].$$

By substituting the initial conditions, we have

$$\frac{1}{\lambda_1}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)]-\frac{1}{\lambda_1}\left[\frac{1}{1-\lambda_2}\right]+\frac{1}{\lambda_2}\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)]-\frac{1}{\lambda_2}\left[\frac{1}{1-\lambda_1}\right] = -1+\frac{1}{1-\lambda_1}+\frac{1}{1-\lambda_2}+\frac{1}{(1-\lambda_1)(1-\lambda_2)}+ \lambda_1\lambda_2\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)].$$

On simplification, we obtain

$$\mathcal{S}_1\mathcal{S}_2[\varphi(t_1,t_2)] = \frac{1}{(1-\lambda_1)(1-\lambda_2)}.$$

Taking the inverse Sumudu transform, we get

$$\varphi(t_1,t_2) = e_{1\oplus 1}(t_1,t_2,0,0).$$

presents a necessary solution. □

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Example 5.2. Consider the following partial dynamic equation for $\mathbb{T}_1 \times \mathbb{T}_2$ such that $t_0 \in \mathbb{T}_1, t_0^{'} \in \mathbb{T}_2$.

$$\varphi^{\Delta_1^2}\left(t_1, t_2\right) + \varphi^{\Delta_2^2}\left(t_1, t_2\right) = 0$$

with the conditions,

$$\varphi\left(t_1, t_0^{'}\right) = \sinh_1\left(t_1, t_0^{'}\right),\ \varphi^{\Delta_2}\left(t_1, t_0^{'}\right) = 0$$

$$\varphi\left(t_0, t_2\right) = 0,\ \varphi^{\Delta_1}\left(t_0, t_2\right) = \cos_1\left(t_2, t_0\right).$$

Solution: Applying the double Sumudu transform on both sides, we get

$$\frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - \frac{1}{\lambda_1^2}\mathcal{S}_2\left[\varphi\left(t_0, t_2\right)\right] - \frac{1}{\lambda_1}\mathcal{S}_2\left[\varphi^{\Delta_1}\left(t_0, t_2\right)\right] + \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - \frac{1}{\lambda_2^2}\mathcal{S}_1\left[\varphi\left(t_1, t_0^{'}\right)\right] - \frac{1}{\lambda_2}\mathcal{S}_1\left[\varphi^{\Delta_2}\left(t_1, t_0^{'}\right)\right]$$

$$= 0.$$

By substituting the initial conditions, we obtain

$$\frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - 0 - \frac{1}{\lambda_1}\mathcal{S}_2\left[\cos_1\left(t_2, t_0\right)\right] + \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - \frac{1}{\lambda_2^2}\mathcal{S}_1\left[\sinh_1\left(t_1, t_0^{'}\right)\right] - 0 = 0.$$

Which gives

$$\frac{1}{\lambda_1^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - \frac{1}{\lambda_1}\left[\frac{1}{\left(1+\lambda_2^2\right)}\right] + \frac{1}{\lambda_2^2}\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] - \frac{1}{\lambda_2^2}\left[\frac{\lambda_1}{\left(1-\lambda_1^2\right)}\right] = 0.$$

On simplification

$$\mathcal{S}_1\mathcal{S}_2\left[\varphi\left(t_1, t_2\right)\right] = \frac{\lambda_1}{\left(1-\lambda_1^2\right)\left(1+\lambda_2^2\right)}.$$

Taking the inverse Sumudu transform, we get

$$\varphi\left(t_1, t_2\right) = \sinh_1\left(t_1, t_0^{'}\right)\cos_1\left(t_2, t_0\right).$$

presents a necessary solution. □

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Conclusion

In this article, we generalized the double Sumudu transform on time scales. We proved the existence theorem, along with the transform of integral, transform of derivative, shifting theorem, and convolution theorem. The double Sumudu transform can solve partial dynamic and integro-dynamic equations.

Conflict of interests

The authors declare that there is no conflict of interests.

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References

[1] H. A. Agwa, F. M. Ali and Adem KIlicman *A new integral transform on time scales and its applications*, Advances in Difference Equations, *2012*(60) (2012), 1–14.

[2] R. A. Agrwal, M. Bohner, D. O'Regan and A. Peterson*Dynamic equations on time*

Received: August 03, 2025

International Journal of Applied Mathematics
Volume 38 No. 1 2025
ISSN: 1311-1728 (printed); ISSN: 1314-8060 (online)

scales: a survey, Journal of Computational and Applied Mathematics, *141*, (2002), 1–26.

[3] M. Bohner and A. Peterson, *The Laplace transform and Z-transform:unification and extension*, Methods and Applications of Analysis, *9*(1) (2002), 155-158.

[4] M. Bohner and A. Peterson, *Dynamic Equation on time scales: An introduction with applications*, Birkhäuser, Boston, (2002), Mass USA.

[5] M. Bohner and G. SH. Guseinov, *The convolution on time scales*, Abstr. Appl. Anal., *2007* (2007), 1–24.

[6] T. Cuchta and S. Georgiev, *Analysis of bilateral Laplace transform on time scales with applications*, Int. J. Dyn. Syst. Differ. Equ., *11*, 3-4, (2001), 255–274.

[7] H. Eltayeb and A. Kilicman, *On double Sumudu transform and double Laplace transform*, Malays. J. Math. Sci., *4*(1), (2010), 17–30.

[8] H. Eltayeb, A. Kilicman, and B. Fisher, *On multiple convolutions and time scales*, Math. Prob. Engrg., *2013* (2013), 1–7.

[9] S. Hilger, *Special functions, Laplace and Fourier transform on measure chains*, Dynam. Systems. Appl., *8*(1999), 471–488.

[10] Kratuloek, K., Kumam, P., Nikam, V., Gopal, D. and Seangwattana, T., 2025. Examination of fractional order model for the population of diabetes and the effects of

Received: August 03, 2025

changes in lifestyle on remission. Mathematical and Computer Modelling of Dynamical Systems, 31(1), p.2488182.

[11] R. Khandelwal, P. Choudhary, and Y. Khandelwal, *Solution of fractional ordinary differential equation by Kamal transform*, International Journal of Statistics and Applied Mathematics,*3*(2) (2018), 279–284.

[12] A. Kamal and H. Sedeeg, *The new integral transform "Kamal Transform"*, Advances in Theoretical and Applied Mathematics, *11*(4) (2016), 451–458.

[13] T. G. Thange and S. M. Chhatraband, *Laplace-Sumudu Integral Transform On Time Scales*, South East Asian J. of Mathematics and Mathematical Sciences, *19* (1), (2023), 91-102.

[14] T. G. Thange and S. M. Chhatraband, *A new α- Laplace transform on time scales*, *Jñānābha*, *53*(2), (2023), 1–10.

[15] T. G. Thange and S. M. Chhatraband, *On Nabla Shehu Transform and Its Applications*, Journal of Fractional Calculus and Applications, *15*(1), (2024), 1–13.

[16] T. G. Thange and S. M. Chhatraband, *New general integral transform on time scales*, J. Math. Model., *12*(4), (2024), 655–669.

[17] T. G. Thange and S. M. Chhatraband, *Double Shehu transform for time scales with applications*, Journal of Classical Analysis, *25*(1), (2025), 75–94.

International Journal of Applied Mathematics
Volume 38 No. 1 2025
ISSN: 1311-1728 (printed); ISSN: 1314-8060 (online)

[18] M. J. Tchueche and N. S. Mbare, *An application of double Sumudu transform*, Applied Mathematical Science, *1*(1), (2007), 31–39.

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