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GENERALIZED NESTED FUNCTIONS AND SOME GENERALIZATIONS OF WILKER AND HUYGEN'S INEQUALITIES

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Abstract

In this work, we are using the special case of Mittag-Leffler functions, namely, H and T Nested Functions T_{pj} and H_{pj}. We study some properties and identities of these functions. Then, we derive new results about the generalizations of Wilker and Huygen's type inequalities based on T_{pj} and H_{pj}, with improving some recent inequalities.

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1. **Introduction**

Many studies have been conducted in the field of Mittag-Leffler functions due to their applications in the study of integral equations, fractional calculus and recently, in the field of inequalities. Especially, many authors have developed new results about the trigonometric inequalities based on Mittag-Leffler functions (see e.g. [14]). In recent years, trigonometric inequalities and hyperbolic inequalities have attracted the attention of many researchers and interesting results have been obtained (see [3]-[22]). Here, we can mention some well-known trigonometric inequalities, which are known as Wilker's inequality, Huygen's inequality and Cusa's inequality (see [3], [5] and [7]) pointed out in this context and are shown here, respectively:

$$\left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} > 2 \qquad (0 < |x| < \frac{\pi}{2}), \tag{1}$$

$$2\left(\frac{\sin(x)}{x}\right) + \frac{\tan(x)}{x} > 3 \qquad (0 < |x| < \frac{\pi}{2}), \tag{2}$$

$$\frac{\sin(x)}{x} < \frac{2}{3} + \frac{1}{3}\cos(x) \qquad (0 < |x| < \frac{\pi}{2}). \tag{3}$$

A lot of research have been done to develop these trigonometric inequalities into other functions, such as lemniscate functions, Bessel functions, hyperbolic functions, exponential functions and weighted functions.

We can refer to the following inequalities based on hyperbolic functions from the article [21], which are respectively known as Wilker's first type inequality, Wilker's second type inequality, Cusa type inequality and Huygen's type inequality:

$$\left(\frac{\sinh(x)}{x}\right)^2 + \frac{\tanh(x)}{x} > 2 \qquad (x \neq 0), \tag{4}$$

$$\left(\frac{x}{\sinh(x)}\right)^2 + \frac{x}{\tanh(x)} > 2 \qquad (x \neq 0), \tag{5}$$

$$\left(\frac{\sinh(x)}{x}\right) - \frac{1}{3}\ \cosh(x) < \frac{2}{3} \qquad (x \neq 0), \tag{6}$$

$$2\left(\frac{\sinh(x)}{x}\right) + \frac{\tanh(x)}{x} > 3 \qquad (x \neq 0). \tag{7}$$

This paper consists of three main sections. In Section 2, we introduce some important inequalities, including Wilker's and Huygens' inequalities. In

Section 3, we examine the functions of H_{pj} and T_{pj} based on Mittage-Leffler functions. We will study them and prove some of their properties. In Section 4, we introduce the generalized nested function H_{apj} and T_{apj}. We give some properties of these functions. Then, we present new generalizations of the well-known Wilker and Huygen's type inequalities and prove them.

We give a generalization of Wilker's inequality for hyperbolic functions [16].

We continue this section by stating the following lemmas and theorem that are important to prove some upcoming results in the next sections.

LEMMA 1.1. *For each $x \neq 0$ the following inequalities hold:*

$$\frac{\tanh(x)}{x} < 1,$$

$$\frac{T_{p1}(x)}{x} > 1.$$

THEOREM 1.1. [16] *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{\sinh(x)}{x}\right)^n + \frac{n}{2}\frac{\tanh(x)}{x} > \frac{n+2}{2} .$$

LEMMA 1.2. *Let x and y be positive real numbers. Then,*

i. *(Mitrinovic et al.* [11]*) For $\mu \in [0,1]$,*

$$\mu x + (1-\mu) \geq x^{\mu} y^{1-\mu}.$$

ii. *(Issa and Ibrahimov* [8]*) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, 1\right]$,*

$$\mu x + (1-\mu) \geq x^{1-\mu} y^{\mu} + (2\mu - 1)(x-y) \geq x^{\mu} y^{1-\mu}.$$

iii. *(Issa and Ibrahimov* [8]*) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, \frac{3}{4}\right]$,*

$$\mu x + (1-\mu) \geq x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + \frac{(x-y)}{2} \geq x^{\mu} y^{1-\mu}.$$

2. On some properties of H and T nested functions

In this section, as the special case of Mittag-Leffler functions, we consider H and T Nested Functions T_{pj} and H_{pj} in order to prove some results about the generalizations of Wilker and Huygens type inequalities.

We begin this section by definition of T_{pj} and H_{pj} and give some properties of these special functions, that are in fact particular cases of the Mittag-Leffler functions and of the higher order trigonometric (cosine and sine) functions (see for example, [9]).

Definition 2.1. [1] H and T Nested Functions $T_{pj}, H_{pj} : R \to R,\ j = 0, 1, 2, \cdots, p-1,\ p \in N$, are defined as follows:

$$T_{pj}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Theorem 2.1. [1] *For each $t \in R$, we have*

$$\begin{aligned} T'_{p0}(t) &= -T_{pp-1}(t), & H'_{p0}(t) &= H_{pp-1}(t),\\ T'_{p1}(t) &= T_{p0}(t), & H'_{p1}(t) &= H_{p0}(t),\\ &\vdots & &\vdots\\ T'_{pp-1}(t) &= T_{pp-2}(t), & H'_{pp-1}(t) &= H_{pp-2}(t). \end{aligned}$$

Theorem 2.2. [2] *Let $\lambda^p = 1, \lambda \neq 1, w^p = -1$ and $w \neq -1$, then, we have*

$$\begin{aligned} H_{p0}(t) &= \frac{\sum_{j=0}^{p-1} e^{\lambda^j t}}{p},\ T_{p0}(t) = \frac{\sum_{j=0}^{p-1} e^{w^j t}}{p}, j = 0, 1, ..., p-1,\\ H_{p1}(t) &= \frac{\sum_{j=0}^{p-1} \lambda^{p-j} e^{\lambda^j t}}{p},\ T_{p1}(t) = \frac{\sum_{j=0}^{p-1} w^{p-j} e^{w^j t}}{p}, j = 0, 1, ..., p-1,\\ &\vdots\\ H_{pp-1}(t) &= \frac{\sum_{j=0}^{p-1} \lambda^j e^{\lambda^j t}}{p},\ T_{pp-1}(t) = \frac{\sum_{j=0}^{p-1} w^j e^{w^j t}}{p}, j = 0, 1, ..., p-1. \end{aligned}$$

Example 2.1. [2] For $p = 3$, $\lambda^3 = 1$ and $\lambda \neq 1$,we have

$$\begin{aligned} H_{30}(t) &= \frac{e^t + e^{\lambda t} + e^{\lambda^2 t}}{3},\\ H_{31}(t) &= \frac{e^t + \lambda^2 e^{\lambda t} + \lambda e^{\lambda^2 t}}{3},\\ H_{32}(t) &= \frac{e^t + \lambda e^{\lambda t} + \lambda^2 e^{\lambda^2 t}}{3}. \end{aligned}$$

EXAMPLE 2.2. [2] For $p=3$, $w^3=-1$ and $w\neq -1$, we have

$$\begin{aligned} T_{30}(t) &= \frac{e^{-t}+e^{wt}+e^{-w^2t}}{3}, \\ T_{31}(t) &= \frac{-e^{-t}-w^2e^{wt}+we^{-w^2t}}{3}, \\ T_{32}(t) &= \frac{e^{-t}-we^{wt}+w^2e^{-w^2t}}{3}. \end{aligned}$$

THEOREM 2.3. [2] *For each $x\in R$, we have the following identities:*

$$\begin{aligned} T_{30}^3(x)-T_{31}^3(x)+T_{32}^3(x)+3T_{30}(x)T_{31}(x)T_{32}(x) &= 1, \\ H_{30}^3(x)+H_{31}^3(x)+H_{32}^3(x)-3H_{30}(x)H_{31}(x)H_{32}(x) &= 1. \end{aligned}$$

DEFINITION 2.2. [16] The functions ${}_p\tan_{ij}$, ${}_p\tanh_{ij} : R\to R$, $i,j=0,1,2,\cdots,p-1$, $p\in N, i\neq j$ are defined as follows:

$$\begin{aligned} {}_p\tan_{ij}(t) &= \frac{T_{pi}(t)}{T_{pj}(t)}, \quad {}_p\tanh_{ij}(t)=\frac{H_{pi}(t)}{H_{pj}(t)}, \\ &\Longrightarrow \\ {}_p\tan_{10}(t) &= \frac{T_{p1}(t)}{T_{p0}(t)}, \quad {}_p\tanh_{10}(t)=\frac{H_{p1}(t)}{H_{p0}(t)}. \end{aligned}$$

REMARK 2.1. By Theorem 2.2 in [16], for $p=3$, we have

$$\begin{aligned} {}_3\tan_{10}(t) &= \frac{H_{31}(t)}{H_{30}(t)}=\frac{e^t+\lambda^2e^{\lambda t}+\lambda e^{\lambda^2t}}{e^t+e^{\lambda t}+e^{\lambda^2t}}, \\ {}_3\tanh_{21}(t) &= \frac{H_{32}(t)}{H_{31}(t)}=\frac{e^t+\lambda e^{\lambda t}+\lambda^2e^{\lambda^2t}}{e^t+\lambda^2e^{\lambda t}+\lambda e^{\lambda^2t}}, \\ {}_3\tanh_{02}(t) &= \frac{H_{30}(t)}{H_{32}(t)}=\frac{e^t+e^{\lambda t}+e^{\lambda^2t}}{e^t+\lambda e^{\lambda t}+\lambda^2e^{\lambda^2t}}. \end{aligned}$$

LEMMA 2.1. [16] *For each $x\neq 0$, the following inequalities hold:*

$$\frac{{}_p\tan_{10}(x)}{x}<1 \qquad \text{and (here } x \text{ are limited)} \qquad \frac{T_{p1}(x)}{x}>1,$$

$$\frac{{}_p\tanh_{10}(x)}{x}<1 \quad \text{and} \quad \frac{H_{p1}(x)}{x}>1.$$

LEMMA 2.2. [16] *For each $x\neq 0$, the following inequalities hold:*

$$\left(\frac{H_{p1}(x)}{x}\right)^{p+1}>H_{p0}(x), \ \text{for} \ \ p\geq 2.$$

Lemma 2.3. [16] *For each $x \neq 0$, the following inequalities hold:*

$$p\left(\frac{H_{p1}(x)}{x}\right)+\frac{{}_p\tanh_{10}(x)}{x}>p+1 \qquad (x\neq 0),$$

$$p\left(\frac{T_{p1}(x)}{x}\right)+\frac{{}_p\tan_{10}(x)}{x}>p+1 \qquad (x\neq 0). \quad \textit{(Open problem)}$$

Theorem 2.4. [16] *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{H_{p1}(x)}{x}\right)^n+\frac{n}{p}\frac{{}_p\tanh_{10}(x)}{x}>\frac{n+p}{p}\ .$$

Theorem 2.5. [16] *Let $x>0, a>0, b>0, p\geq 2,\ p\in N$ and $m\geq pnb/a$. Then, for $n>0$, the following inequality holds:*

$$\frac{a}{a+b}\left(\frac{H_{p1}(x)}{x}\right)^m+\frac{b}{a+b}\left(\frac{{}_p\tanh_{10}(x)}{x}\right)^n>1\ .$$

Lemma 2.4. [16] *For each $x \neq 0$, the following inequalities hold:*

$$\frac{H_{pi}(x)}{x^i}>1,\ \ i=0,1,2,\cdots,p-1.$$

and

$$\frac{{}_p\tanh_{i0}(x)}{x^i}<1,\ \ i=0,1,2,\cdots,p-1.$$

Lemma 2.5. [16] *For each $x \neq 0$, the following inequalities hold:*

$$\left(\frac{H_{pi}(x)}{x^i}\right)^{p+i}>H_{p0}(x),\ \text{for}\ \ p\geq 2,\ \ i=0,1,2,\cdots,p-1.$$

Lemma 2.6. [16] *For each $x \neq 0$, the following inequalities hold:*

$$p\left(\frac{H_{pi}(x)}{x^i}\right)+i\frac{{}_p\tanh_{i0}(x)}{x^i}>p+i \qquad (x\neq 0),$$

$$p\left(\frac{T_{pi}(x)}{x^i}\right)+i\frac{{}_p\tan_{i0}(x)}{x^i}>p+i. \qquad (x\neq 0) \quad \textit{(Open problem)}$$

Theorem 2.6. [16] *For each $x \neq 0, i=1,2,...,p-1$ and $n\geq 1$, the following inequality holds:*

$$\left(\frac{H_{pi}(x)}{x^i}\right)^n+\frac{in}{p}\frac{{}_p\tanh_{i0}(x)}{x^i}>\frac{in+p}{p}\ .$$

THEOREM 2.7. [16] *Let* $x > 0, a > 0, b > 0, p \geq 2, p \in N, i = 1, 2, ..., p-1$ *and* $m \geq (p+i-1)nb/a$. *Then, for* $n > 0$, *the following inequality holds:*

$$\frac{a}{a+b}(\frac{H_{pi}(x)}{x^i})^m + \frac{b}{a+b}(\frac{p\tanh_{i0}(x)}{x^i})^n > 1 \ .$$

LEMMA 2.7. [16] *Let* $x > 0$, *the following inequality holds:*

$$\left(\frac{H_{p1}(x)}{x}\right)^p - \frac{p}{p+1}H_{p0}(x) > \frac{1}{p+1} \ .$$

LEMMA 2.8. [16] *Let* $x > 0$, *the following inequality holds:*

$$\frac{p}{p+1}\left[1 - H_{p0}(x)\right]\left[1 - \frac{H_{p1}(x)}{x}\right] + \left[\frac{H_{p1}(x)}{x} - \frac{H_{p0}(x)}{p+1} - \frac{p}{p+1}\right] > 0 \ .$$

3. Generalized nested functions and generalizations of Wilker-Huygen's inequalities

In this section, inspired by the work of Kwara [12], using generalized H and T nested functions, we prove new results about the generalization of Wilker's and Huygen's type inequalities.

DEFINITION 3.1. The generalized H and T nested functions T_{apj}, H_{apj} : $R \to R, a > 0$, $a \neq 1, j = 0, 1, 2, \cdots, p-1$, $p \in N$, are defined as follows:

$$\begin{aligned} T_{apj}(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n (t\ln a)^{pn+j}}{(pn+j)!} = T_{pj}(t\ln a), \\ H_{apj}(t) &= \sum_{n=0}^{\infty} \frac{(t\ln a)^{pn+j}}{(pn+j)!} = H_{pj}(t\ln a). \end{aligned}$$

THEOREM 3.1. *For each* $t \in R$, *we have*

$$\begin{aligned} T'_{ap0}(t) &= -\ln a T_{app-1}(t) & H'_{ap0}(t) &= \ln a H_{app-1}(t), \\ T'_{ap1}(t) &= \ln a T_{ap0}(t) & H'_{ap1}(t) &= \ln a H_{ap0}(t), \\ &\vdots & &\vdots \\ T'_{app-1}(t) &= \ln a T_{app-2}(t) & H'_{app-1}(t) &= \ln a H_{app-2}(t). \end{aligned}$$

By Theorem 2.2 and Definition 3.1, we have the following theorem.

THEOREM 3.2. *Let $\lambda^p = 1, \lambda \neq 1, w^p = -1, w \neq -1, a > 0$ and $a \neq 1$. Then, we have*

$$H_{ap0}(t) = \frac{\sum_{j=0}^{p-1} a^{\lambda^j t}}{p},\ T_{ap0}(t) = \frac{\sum_{j=0}^{p-1} a^{w^j t}}{p}, j = 0, 1, ..., p-1,$$
$$H_{ap1}(t) = \frac{\sum_{j=0}^{p-1} \lambda^{p-j} a^{\lambda^j t}}{p},\ T_{ap1}(t) = \frac{\sum_{j=0}^{p-1} w^{p-j} a^{w^j t}}{p}, j = 0, 1, ..., p-1,$$
$$\vdots\vdots$$
$$H_{ap-1}(t) = \frac{\sum_{j=0}^{p-1} \lambda^j a^{\lambda^j t}}{p},\ T_{app-1}(t) = \frac{\sum_{j=0}^{p-1} w^j a^{w^j t}}{p}, j = 0, 1, ..., p-1.$$

EXAMPLE 3.1. For $p = 3$, we have

$$\begin{aligned} H_{a30}(t) &= \frac{a^t + a^{\lambda t} + a^{\lambda^2 t}}{3}, \\ H_{a31}(t) &= \frac{a^t + \lambda^2 a^{\lambda t} + \lambda a^{\lambda^2 t}}{3}, \\ H_{a32}(t) &= \frac{a^t + \lambda a^{\lambda t} + \lambda^2 a^{\lambda^2 t}}{3}. \end{aligned}$$

And

$$\begin{aligned} T_{a30}(t) &= \frac{a^{-t} + a^{wt} + a^{-w^2 t}}{3}, \\ T_{a31}(t) &= \frac{-a^{-t} - w^2 a^{wt} + w a^{-w^2 t}}{3}, \\ T_{a32}(t) &= \frac{a^{-t} - w a^{wt} + w^2 a^{-w^2 t}}{3}. \end{aligned}$$

By using Theorem 2.3 and Definition 3.1, we have the following theorem.

THEOREM 3.3. *For each $x \in R$, we have the following identities.*

$$\begin{aligned} H_{a30}^3(x) + H_{a31}^3(x) + H_{a32}^3(x) - 3H_{a30}(x)H_{a31}(x)H_{a32}(x) &= 1, \\ T_{a30}^3(x) - T_{a31}^3(x) + T_{a32}^3(x) + 3T_{a30}(x)T_{a31}(x)T_{a32}(x) &= 1. \end{aligned}$$

DEFINITION 3.2. The functions ${}_p\tan_{ij}, {}_p\tanh_{ij} : R \to R,\ i,j = 0,1,2,\cdots, p-1,\ p \in N, i \neq j$ are defined as follows:

$$\begin{aligned} {}_p\tan_{ij}(t) &= \frac{T_{api}(t)}{T_{apj}(t)}, \quad {}_p\tanh_{ij}(t) = \frac{H_{api}(t)}{H_{apj}(t)}, \\ &\Longrightarrow \\ {}_p\tan_{10}(t) &= \frac{T_{ap1}(t)}{T_{ap0}(t)}, \quad {}_p\tanh_{10}(t) = \frac{H_{ap1}(t)}{H_{ap0}(t)}. \end{aligned}$$

LEMMA 3.1. *For each $x \neq 0$, the following inequalities hold:*

$$(\frac{H_{ap1}(x)}{x})^{p+1} > \ln^{p+1} a H_{p0}(x), \ \textit{for} \ \ a > 1, \ p \geq 2.$$

P r o o f. By Definition 3.1 and Lemma 2.2, we have

$$\begin{aligned} (\frac{H_{ap1}(x)}{x})^{p+1} &= (\frac{H_{p1}(x\ln a)}{x})^{p+1} = \ln^{p+1} a(\frac{H_{p1}(x\ln a)}{x\ln a})^{p+1} \\ &= \ln^{p+1} a(\frac{H_{p1}(t)}{t})^{p+1} > \ln^{p+1} a H_{p0}(x). \end{aligned}$$

□

COROLLARY 3.1. *For each $x \neq 0$, the following inequalities hold:*

$$(\frac{H_{ap1}(x)}{x})^{p+1} > H_{ap0}(x), \ \textit{for} \ \ a > e, \ p \geq 2.$$

REMARK 3.1. For each $a > 1$, the inequalities of Theorem 3.11 in [13] are not true. For example, if we choice $a = \frac{11}{10}, z = 5,$ we have:

$$\begin{aligned} (\frac{(\frac{11}{10})^5 - (\frac{11}{10})^{-5}}{10})^3 &\cong 0.00096, \\ \frac{(\frac{11}{10})^5 + (\frac{11}{10})^{-5}}{10} &\cong 1.11571, \\ &\Longrightarrow \\ (\frac{(\frac{11}{10})^5 - (\frac{11}{10})^{-5}}{10})^3 &< \frac{(\frac{11}{10})^5 + (\frac{11}{10})^{-5}}{10}. \end{aligned}$$

THEOREM 3.4. [6] (***Monotone form of L'Hôpital's rule***). *Let f, g be continuous functions defined in $[a,b]$, differentiable in (a,b). Suppose that $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$, and assume that $g'(x) \neq 0$ for all $x \in (a,b)$. If f'/g' is increasing (decreasing) on (a,b), then so is f/g.*

REMARK 3.2. By considering Theorem 3.4 and Remark 3.1, condition $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$ is essential to apply Theorem 3.4.

LEMMA 3.2. *For each $x \neq 0$, the following inequalities hold:*

$$p\left(\frac{H_{ap1}(x)}{x}\right) + \frac{{}_p\tanh_{a10}(x)}{x} > (p+1)\ln a \qquad (a>1, x\neq 0),$$

$$p\left(\frac{T_{ap1}(x)}{x}\right) + \frac{{}_p\tan_{a10}(x)}{x} > (p+1)\ln a \qquad (a>1, x\neq 0)$$

(*Open problem*).

P r o o f. By Definition 3.1 and Lemma 2.3, we have

$$p\left(\frac{H_{ap1}(x)}{x}\right) + \frac{{}_p\tanh_{a10}(x)}{x} = p\left(\frac{H_{p1}(x\ln a)}{x}\right) + \frac{{}_p\tanh(x\ln a)}{x}$$

$$= \ln a[p\left(\frac{H_{p1}(x\ln a)}{x\ln a}\right) + \frac{{}_p\tanh(x\ln a)}{x\ln a}]$$

$$= \ln a[p\left(\frac{H_{p1}(t)}{t}\right) + \frac{{}_p\tanh(t)}{t}] > (p+1)\ln a.$$

□

COROLLARY 3.2. *For each $x \neq 0$, the following inequalities hold:*

$$p\left(\frac{H_{ap1}(x)}{x}\right) + \frac{{}_p\tanh_{a10}(x)}{x} > p+1 \qquad (a>e, x\neq 0),$$

$$p\left(\frac{T_{ap1}(x)}{x}\right) + \frac{{}_p\tan_{a10}(x)}{x} > p+1 \qquad (a>e, x\neq 0) \quad (\textit{Open problem}).$$

THEOREM 3.5. *For each $x \neq 0$ and $n \geq 1$, the following inequality holds:*

$$\left(\frac{H_{ap1}(x)}{x}\right)^n + \frac{n}{p}\frac{{}_p\tanh_{a10}(x)}{x}\ln^{n-1} a > \frac{n+p}{p}\ln^n a \quad (a>e, x\neq 0) \qquad .$$

P r o o f. By Definition 3.1 and Lemma 2.4, we have

$$\left(\frac{H_{ap1}(x)}{x}\right)^n + \frac{n}{p}\frac{{}_p\tanh_{a10}(x)}{x}\ln^{n-1} a = \ln^n a\left(\frac{H_{p1}(x\ln a)}{x\ln^n a}\right)^n +$$

$$\frac{n}{p}\left[\frac{{}_p\tanh_{10}(x\ln a)}{x\ln a}\right]\ln^n a = \ln^n a[\left(\frac{H_{p1}(x)}{x}\right)^n + \frac{n}{p}\frac{{}_p\tanh_{10}(x)}{x}]$$

$$> \frac{n+p}{p}\ln a.$$

□

LEMMA 3.3. *For each $x \neq 0$, the following inequalities hold:*

$$p\left(\frac{H_{api}(x)}{x^i}\right) + i\frac{{}_p\tanh_{ai0}(x)}{x^i} > (p+i)\ln^i a \qquad (x \neq 0),$$

$$p\left(\frac{T_{api}(x)}{x^i}\right) + i\frac{{}_p\tan_{ai0}(x)}{x^i} > (p+i)\ln^i a \qquad (x \neq 0) \quad (\textit{Open problem}).$$

P r o o f. By Definition 3.1 and Lemma 2.6, we have

$$\left(\frac{H_{ap1}(x)}{x^i}\right) + i\frac{{}_p\tanh_{a10}(x)}{x^i} = \ln^i a\left(\frac{H_{p1}(x\ln a)}{x^i\ln^i a}\right)$$

$$+i\left[\frac{{}_p\tanh_{10}(x\ln a)}{(x\ln a)^i}\right]\ln^i a \ = \ln^i a[\left(\frac{H_{p1}(t)}{t^i}\right) + i\frac{{}_p\tanh_{10}(t)}{t^i}]$$

$$> (p+i)\ln^i a.$$

□

THEOREM 3.6. *For each $x \neq 0$, $i = 1, 2, ..., p-1$ and $n \geq 1$, following inequality holds:*

$$\left(\frac{H_{api}(x)}{x^i}\right)^n + \frac{in}{p}\frac{{}_p\tanh_{ai0}(x)}{x^i}\ln^{in-i} a > (\frac{in+p}{p})\ln^{in} a \ .$$

P r o o f. By Definition 3.1 and Lemma 2.6, we have

$$\left(\frac{H_{api}(x)}{x^i}\right)^n + \frac{in}{p}\frac{{}_p\tanh_{ai0}(x)}{x^i}\ln^{in-i} a$$

$$\ln^{in} a[\left(\frac{H_{p1}(x\ln a)}{x^i\ln^i a}\right)^n + \frac{in}{p}\frac{{}_p\tanh_{10}(x\ln a)}{x^i\ln^i a}]$$

$$= \ln^{ni} a[\left(\frac{H_{p1}(t)}{t^i}\right)^n + \frac{in}{p}\frac{{}_p\tanh_{10}(t)}{t^i}] > (\frac{in+p}{p})\ln^{ni} a.$$

□

We are inspired by [10] for the following theorem, and we will improve the inequality in Theorem 5.1 in [10].

THEOREM 3.7. *Let $x > 0, \alpha > 0, \beta > 0, p \geq 2, p \in N$, and $m \geq pn\beta/\alpha$. Then, for $n > 0$, the following inequality holds:*

$$\frac{\alpha}{\alpha+\beta}(\frac{H_{ap1}(x)}{x})^m + \frac{\beta}{\alpha+\beta}(\frac{{}_p\tanh_{a10}(x)}{x})^n > (\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}} \ .$$

P r o o f. By Definition 3.1, Lemma 1.2 and Lemma 2.2, we have

$$\begin{aligned}
& \frac{\alpha}{\alpha+\beta}(\frac{H_{ap1}(x)}{x})^m + \frac{\beta}{\alpha+\beta}(\frac{{}_p\tanh_{a10}(x)}{x})^n \\
\geq\ & (\frac{H_{ap1}(x)}{x})^{\frac{m\alpha}{\alpha+\beta}}(\frac{{}_p\tanh_{a10}(x)}{x})^{\frac{n\beta}{\alpha+\beta}} \\
=\ & (\frac{H_{p1}(x\ln a)}{x\ln a})^{\frac{m\alpha}{\alpha+\beta}}(\frac{H_{p1}(x\ln a)}{x\ln a})^{\frac{n\beta}{\alpha+\beta}}(\frac{1}{H_{p0}(x\ln a)})^{\frac{n\beta}{\alpha+\beta}}(\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}} \\
>\ & (\frac{H_{ap1}(t)}{t})^{\frac{m\alpha}{\alpha+\beta}}(\frac{H_{ap1}(t)}{t})^{\frac{n\beta}{\alpha+\beta}}(\frac{H_{ap1}(t)}{t})^{\frac{-n(p+1)\beta}{\alpha+\beta}}(\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}} \\
=\ & (\frac{H_{ap1}(x)}{x})^{\frac{m\alpha-np\beta}{\alpha+\beta}}(\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}} > (\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}}.
\end{aligned}$$

□

With choice $p = 2$, we have the following corollary.

Corollary 3.3. *Let $x > 0, \alpha > 0, \beta > 0$, and $m \geq 2n\beta/\alpha$. Then, for $n > 0$, the following inequality holds:*

$$\frac{\alpha}{\alpha+\beta}(\frac{H_{a21}(x)}{x})^m + \frac{\beta}{\alpha+\beta}(\frac{{}_2\tanh_{a10}(x)}{x})^n > (\ln a)^{\frac{m\alpha+n\beta}{\alpha+\beta}}.$$

Remark 3.3. For proof of Theorem 5.1 in [10], we see that

$$\begin{aligned}
{}_sF_kh(x) &= \frac{\sigma_k^x - \sigma_k^{-x}}{\sqrt{k^2+4}} = \frac{2}{\sqrt{k^2+4}}\sinh(x\ln\sigma_k), \\
{}_cF_kh(x) &= \frac{\sigma_k^x + \sigma_k^{-x}}{\sqrt{k^2+4}} = \frac{2}{\sqrt{k^2+4}}\cosh(x\ln\sigma_k), \\
{}_tF_kh(x) &= \frac{{}_sF_kh(x)}{{}_cF_kh(x)} = \tanh(x\ln\sigma_k).
\end{aligned}$$

Therefore, by Lemma 1.2 and Lemma 2.5, with $p = 2$, we have

$$\frac{\alpha}{\alpha+\beta}(\frac{{}_sF_kh(x)}{x})^m + \frac{\beta}{\alpha+\beta}(\frac{{}_tF_kh(x)}{x})^n$$

$$= \frac{\alpha}{\alpha+\beta}(\frac{2\ln\sigma_k}{\sqrt{k^2+4}}\frac{\sinh(x\ln\sigma_k)}{x\ln\sigma_k})^m$$
$$+\frac{\beta}{\alpha+\beta}(\ln\sigma_k\frac{\tanh(x\ln\sigma_k)}{x\ln\sigma_k})^n$$
$$\geq (\frac{2\ln\sigma_k}{\sqrt{k^2+4}}\frac{\sinh(x\ln\sigma_k)}{x\ln\sigma_k})^{\frac{m\alpha}{\alpha+\beta}}(\ln\sigma_k\frac{\tanh(x\ln\sigma_k)}{x\ln\sigma_k})^{\frac{n\beta}{\alpha+\beta}}$$
$$= (\frac{\sinh(t)}{t})^{\frac{m\alpha}{\alpha+\beta}}(\frac{\sinh(t)}{t})^{\frac{n\beta}{\alpha+\beta}}(\frac{1}{\cosh(t)})^{\frac{n\beta}{\alpha+\beta}}(\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}$$
$$\times(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}}$$
$$> (\frac{\sinh(t)}{t})^{\frac{m\alpha}{\alpha+\beta}}(\frac{\sinh(t)}{t})^{\frac{n\beta}{\alpha+\beta}}(\frac{\sinh(t)}{t})^{\frac{-3n\beta}{\alpha+\beta}}(\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}$$
$$\times(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}}$$
$$= (\frac{\sinh(t)}{t})^{\frac{m\alpha-2n\beta}{\alpha+\beta}}(\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}}$$
$$> (\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}}.$$

So,

$$\frac{\alpha}{\alpha+\beta}(\frac{{}_sF_kh(x)}{x})^m+\frac{\beta}{\alpha+\beta}(\frac{{}_tF_kh(x)}{x})^n > (\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}}. \quad (8)$$

Thus, according to $\frac{m\alpha}{\alpha+\beta}<\frac{m\alpha+n\beta}{\alpha+\beta}$ and $1>\frac{2}{\sqrt{k^2+4}}$, we obtain

$$(\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha}{\alpha+\beta}} > (\ln\sigma_k)^{\frac{m\alpha+n\beta}{\alpha+\beta}}(\frac{2}{\sqrt{k^2+4}})^{\frac{m\alpha+n\beta}{\alpha+\beta}}.$$

Hence, the proof is completed.

Note that, inequality in (8) is stronger than the inequality of Theorem 5.1 in [10].

THEOREM 3.8. [10] *For nonzero real number x and any positive real number k, the following inequality holds:*

$$(\frac{{}_sF_kh(x)}{x})^2+(\frac{{}_tF_kh(x)}{x}) > \frac{8\ln^2\sigma_k}{(k^2+4)}+\frac{32\ln^5\sigma_k}{45(k^2+4)}x_t^3F_kh(x)\ .$$

REMARK 3.4. In proof of Theorem 4.1 in [10], the author did not care about the condition $f(a)=g(a)=0$ or $f(b)=g(b)=0$.

THEOREM 3.9. *For nonzero real number x and any positive real number k, the following inequality holds:*

$$
\begin{aligned}
&(\frac{{}_sF_kh(x)}{x})^2+(\frac{{}_tF_kh(x)}{x})\\
>\;& \frac{4\ln^2\sigma_k}{(k^2+4)}+\frac{4\ln^4\sigma_k}{3(k^2+4)}x^2\;+\frac{8\ln^6\sigma_k}{45(k^2+4)}x^4+\frac{\ln^8\sigma_k}{336(k^2+4)}x^6.
\end{aligned}
$$

P r o o f. Note that $(\,0<\frac{\tanh(t)}{t}<1)$,

$$
\begin{aligned}
&(\frac{{}_sF_kh(x)}{x})^2+\frac{{}_tF_kh(x)}{x}=(\frac{2\ln\sigma_k}{\sqrt{k^2+4}}\frac{\sinh(x\ln\sigma_k)}{x\ln\sigma_k})^2\\
&\qquad\qquad+\ln\sigma_k\frac{\tanh(x\ln\sigma_k)}{x\ln\sigma_k}\\
&(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2(\frac{\sinh(t)}{t})^2+\ln\sigma_k\frac{\tanh(t)}{t}\\
&>(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2(1+\frac{t^2}{6}+\frac{t^4}{120}+\frac{t^6}{5040})^2+\ln\sigma_k\frac{\tanh(t)}{t}\\
&>(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2(1+\frac{t^2}{3}+\frac{2t^4}{45}+\frac{t^6}{336})^2\\
&>(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2+(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2\frac{t^2}{3}+(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2\frac{2t^4}{45}\\
&\qquad\qquad+(\frac{2\ln\sigma_k}{\sqrt{k^2+4}})^2\frac{t^6}{336}\\
&=m\frac{4\ln^2\sigma_k}{(k^2+4)}+\frac{4\ln^4\sigma_k}{3(k^2+4)}x^2\;+\frac{8\ln^6\sigma_k}{45(k^2+4)}x^4+\frac{\ln^8\sigma_k}{336(k^2+4)}x^6.
\end{aligned}
$$

□

LEMMA 3.4. *For each $x\neq 0$, the following inequality holds:*

$$
(\frac{H_{api}(x)}{x^i})^{p+i}>(\ln a)^{i(p+i)}H_{ap0}(x),\;\textit{for } p\geq 2,\; i=0,1,2,\cdots,p-1.
$$

P r o o f.

$$
\begin{aligned}
(\frac{H_{api}(x)}{(x)^i})^{p+i}&=(\ln a)^{i(p+i)}\;(\frac{H_{pi}(x\ln a)}{(x\ln a)^i})^{p+i}\\
&=(\ln a)^{i(p+i)}\;(\frac{H_{pi}(t)}{(t)^i})^{p+i}>(\ln a)^{i(p+i)}H_{p0}(t)\\
&=(\ln a)^{i(p+i)}H_{p0}(x\ln a)=(\ln a)^{i(p+i)}H_{ap0}(x).
\end{aligned}
$$

□

THEOREM 3.10. [16] *Let $x > 0, \alpha > 0, \beta > 0, p \geq 2, p \in N, i = 1, 2, ..., p-1$ and $m \geq (p+i-1)n\beta/\alpha$. Then, for $n > 0$, the following inequality holds:*

$$\frac{\alpha}{\alpha+\beta}(\frac{H_{pi}(x)}{x^i})^m + \frac{\beta}{\alpha+\beta}(\frac{p\tanh_{i0}(x)}{x^i})^n > (\ln a)^{i\frac{m\alpha+n\beta}{\alpha+\beta}} .$$

P r o o f. By Definition 3.1, Lemma 1.2 and Lemma 2.5, we have

$$\begin{aligned}
&\frac{\alpha}{\alpha+\beta}(\frac{H_{pi}(x)}{x^i})^m + \frac{\beta}{\alpha+\beta}(\frac{p\tanh_{i0}(x)}{x^i})^n \\
&\geq (\frac{H_{pi}(x)}{x^i})^{\frac{m\alpha}{\alpha+\beta}}(\frac{p\tanh_{i0}(x)}{x^i})^{\frac{n\beta}{\alpha+\beta}} \\
&= (\frac{H_{pi}(x\ln a)}{(x\ln a)^i})^{\frac{m\alpha}{\alpha+\beta}}(\frac{H_{pi}(x\ln a)}{(x\ln a)^i})^{\frac{n\beta}{\alpha+\beta}}(\frac{1}{H_{p0}(x\ln a)})^{\frac{n\beta}{\alpha+\beta}}(\ln a)^{i\frac{m\alpha+n\beta}{\alpha+\beta}} \\
&> (\frac{H_{ap1}(t)}{t^i})^{\frac{m\alpha}{\alpha+\beta}}(\frac{H_{ap1}(t)}{t^i})^{\frac{n\beta}{\alpha+\beta}}(\frac{H_{api}(t)}{t^i})^{\frac{-n(p+i)\beta}{\alpha+\beta}}(\ln a)^{i\frac{m\alpha+n\beta}{\alpha+\beta}} \\
&= (\frac{H_{api}(t)}{t^i})^{\frac{m\alpha-(p+i-1)n\beta}{\alpha+\beta}}(\ln a)^{i\frac{m\alpha+n\beta}{\alpha+\beta}} > (\ln a)^{i\frac{m\alpha+n\beta}{\alpha+\beta}}.
\end{aligned}$$

□

4. Conclusion

In this paper, we introduced some important inequalities, including Wilker's and Huygen's inequalities. Then, we studied some properties of H_{pj} and T_{pj} based on Mittag-Leffler functions. Finally, we introduced the generalized nested function H_{apj} and T_{apj}. Some properties of these functions are shown. Then, we presented new generalizations of the well-known Wilker and Huygen's type inequalities and proved them. Also, we improved and corrected very recently inequality.

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