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ON UNIFORM STRUCTURES IN THE SPACE OF $n-$PERMUTATION DEGREE

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Abstract

In this paper, we consider uniform structures in the space of n-permutation degree SP^n_GX. Also, we obtain some results related to the uniform spaces and of the space of n-permutation degree. It is proved that weight of uniform space to be equal to weight of the space of n-permutation degree of this uniform spaces.

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1. Introduction and preliminaries

Recently, the uniform structures in topology have been studied by many authors (see [1], [2], [3]). They considered several uniform properties and studied the relation between a space X satisfying such a property and its hyperspaces with the Vietoris topology.

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In [5, 6] V. V. Fedorchuk and V. V. Filippov investigated a functor of n-permutation degree and it was proved that a functor of n-permutation degree is a normal functor in the category of compact spaces and their continuous mappings.

In recent researches an interest in the theory of cardinal invariants, homotopy properties, some classes of topological space and their behavior under the influence of various covariant functors is increasing fast. In [7, 8, 9, 10, 11, 12, 13] the authors investigated several cardinal invariants, homotopy properties, some classes of topological space under the influence of some covariant functors and the space of G-permutation degree.

In [7, 8], studied index boundedness, uniform connectedness and homotopy properties of space of the G-permutation degree. In [8], shown that the functor SP^n_G preserves the homotopy and the retraction of topological spaces. In addition, proved that if the spaces X and Y are homotopically equivalent, then the spaces $SP^n_G X$ and $SP^n_G Y$ are also homotopically equivalent. As a result, it has been proved that the functor SP^n_G is a covariant homotopy functor.

The spaces of the $G-$permutation degree was investigated in [9, 10, 11] and it was studied that some tightness-type properties, network-type properties and Lindelŏf-type properties of the space of G-permutation degree. In [11] has been proved that the functor SP^n_G preserves cs-network, cs^*-network, cn-network and ck-network of topological spaces. In [12, 13] has been devoted to the investigation some classes of topological space such as the developable space, Moore space, M_1-space, M_2-space and some other topological properties of the space of permutation degree, as well as, Lašnev property and Nagata property are studied.

To consider uniformly maps below we recall the following equivalent definition of uniformity.

Definition 1.1. Let X be a non-empty set. A family $\mathcal{U}$ of coverings of the set X is said to be a uniformity (in term of coverings) on X if the following conditions are satisfied:

(C1) if $\alpha \in \mathcal{U}$ and α refines a covering γ of X then $\gamma \in \mathcal{U}$;

(C2) for any pair $\alpha, \beta \in \mathcal{U}$ there exists $\gamma \in \mathcal{U}$ that is refines both α and β;

(C3) for any pair $\alpha \in \mathcal{U}$ there exists $\beta \in \mathcal{U}$ such that β strongly star-refines α;

(C4) for any pair x, y of distinct elements of X there exists $\alpha \in \mathcal{U}$ such that no element of α contains both x and y [**?**].

A pair $(X,\mathcal{U})$ consisting of a set X and a uniformity $\mathcal{U}$ on it is said to be a *uniform space*.

Definition 1.2. A continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ of a uniform space $(X,\mathcal{U})$ to a uniform space $(Y,\mathcal{V})$ is said to be *uniformly continuous* if for each open uniform covering $\beta \in \mathcal{V}$ inverse $f^{-1}(\beta) \in \mathcal{U}$ [3].

It is known that a permutation group is the group of all permutations, that is one-to-one mappings $X \to X$. A permutation group of a set X is usually denoted by $S(X)$. Especially, if $X = \{1, 2, \dots, n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the n-th power of a compact space X. The permutation group S_n of all permutations acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n.

Two points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ are considered to be *equivalent* if there exists a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space SP^nX is called the *n-permutation degree* of the space X.

Equivalent relation by which we obtain the space SP^nX is called the *symmetric equivalence relation.* The n-th permutation degree is a quotient of X^n. Therefore, the quotient map, denoted by $\pi_n^s : X^n \to SP^nX$, is defined as

$$\pi_n^s((x_1, x_2, \dots, x_n)) = [(x_1, x_2, \dots, x_n)],$$

for every $(x_1, x_2, \dots, x_n) \in X^n$.

In this paper, we study the relation between a uniform space $(X,\mathcal{U})$ satisfying topological and cardinal properties and the space of the n-permutation degree $(SP^nX, SP^n\mathcal{U})$ satisfying the same properties.

Throughout this paper, all spaces are assumed to be T_1 and regular, N denotes the set of all positive integers.

For some undefined or related concepts, we refer the reader to [4].

2. Main results

Theorem 2.1. *If $\mathcal{B}$ is the base of a uniform space $(X,\mathcal{U})$, then $\mathcal{S}(\mathcal{B}) = \{\mathcal{S}(\alpha) : \alpha \in \mathcal{B}\}$ forms the base of some uniformity $SP^n\mathcal{U}$ on SP^nX where $\mathcal{S}(\alpha) = \{[U_1 \times U_2 \times ... \times U_n] : U_i \in \alpha\}$.*

P r o o f. Let α, β be arbitrary coverings in $\mathcal{B}$. Then there is a uniform covering γ of $\mathcal{B}$ such that $\gamma \succ \alpha$ and $\gamma \succ \beta$. Let us show that $\mathcal{S}(\gamma) \succ \mathcal{S}(\alpha)$ and $\mathcal{S}(\gamma) \succ \mathcal{S}(\beta)$. It is enough to show one of them, for example, $\mathcal{S}(\gamma) \succ \mathcal{S}(\alpha)$. Choose $[U_1 \times U_2 \times ... \times U_n] \in \mathcal{S}(\gamma)$. It is clear that $U_i \in \gamma$ for each $i = 1, 2, ..., n$. Since γ is inscribed in α there is $V_i \in \alpha$ such that $U_i \subset V_i$ for each $i = 1, 2, ..., n$. In this case, we have $[U_1 \times U_2 \times ... \times U_n] \subset [V_1 \times V_2 \times ... \times V_n]$.

If β are arbitrary coverings in $\mathcal{B}$, then there is a uniform covering γ of $\mathcal{B}$ such that $\gamma^* \succ \beta$. Let us show that $\mathcal{S}(\gamma)$ strongly stellar inscribed in $S(\beta)$. Let us put

$$(S(\gamma))^* = \{\mathcal{S}(\gamma)\left([U_1 \times U_2 \times ... \times U_n]\right) : \ \ U_i \in \gamma, \quad i = 1, 2, ..., n\}.$$

Take an arbitrary U_i from γ for each $i = 1, 2, ..., n$. Then there exists $V_i \in \beta$ such that $\gamma(U_i) \subset V_i$ for all $i = 1, 2, ..., n$. By definition of a star set, we have

$$\mathcal{S}(\gamma)\left([U_1 \times U_2 \times ... \times U_n]\right) = \{[W_1 \times W_2 \times ... \times W_n] \in \mathcal{S}(\gamma) :$$

$$[U_1 \times U_2 \times ... \times U_n] \bigcap [W_1 \times W_2 \times ... \times W_n] \neq \emptyset\}.$$

Take an arbitrary element $[W_1 \times W_2 \times ... \times W_n]$ from $\mathcal{S}(\gamma)$, and

$$[U_1 \times U_2 \times ... \times U_n] \bigcap [W_1 \times W_2 \times ... \times W_n] \neq \emptyset.$$

In this case, there is a permutation $\sigma \in S_n$ such that $U_i \bigcap W_{\sigma(i)} \neq \emptyset$ for each $i = 1, 2, ..., n$. So $W_{\sigma(i)} \subset \gamma(U_i) \subset V_i$ for all $i = 1, 2, ..., n$. Therefore,

$$[W_1 \times W_2 \times ... \times W_n] = [W_{\sigma(1)} \times W_{\sigma(2)} \times ... \times W_{\sigma(n)}] \subset [V_1 \times V_2 \times ... \times V_n].$$

From the last inclusion, we obtain $(S(\gamma))^* \succ S(\beta)$.

Let us show the equality $\{[x]\} = \cap\{\mathcal{S}(\alpha)([x]) : \ \ \alpha \in \mathcal{B}\}$ for any $[x] = [(x_1, x_2, ..., x_n)] \in SP^n X$. Assume, that $[y] \neq [x]$ and $[y] \in \cap\{\mathcal{S}(\alpha)([x]) : \ \ \alpha \in \mathcal{B}\}$. In in this case, we have $[y] \in \mathcal{S}(\alpha)([x])$, i.e.

$$[y] \in \bigcup \{[U_1 \times U_2 \times ... \times U_n] : \ \ [x] \in [U_1 \times U_2 \times ... \times U_n], \ U_i \in \alpha, \ i = 1, 2, ..., n\}$$

for all $\alpha \in \mathcal{B}$. Therefore, $[x], \ [y] \in [V_1 \times V_2 \times ... \times V_n]$ for some $V_1, V_2, ..., V_n \in \alpha$. At the same time, there are permutations $\delta_1, \ \delta_2 \in S_n$ such that $x_j \in V_{\delta_1(j)}$ and $y_j \in V_{\delta_2(j)}$ for each $j = 1, 2, ..., n$. Let us carry out the transformations $j = \delta_2^{-1}\delta_1(i)$. Thus, $y_{\delta_2^{-1}\delta_1(i)} \in \alpha(x_i)$ for all $i = 1, 2, ..., n$ and $\alpha \in \mathcal{B}$. So $y_{\delta_2^{-1}\delta_1(i)} = x_i$ for each $i = 1, 2, ..., n$, i.e. $[x] = [y]$. □

Corollary 2.1. *Let $(X, \mathcal{U})$ be a uniform space. Then,*

$$w(X, \mathcal{U}) = w(SP^n X, SP^n \mathcal{U}).$$

Theorem 2.2. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous map of the uniform space $(X, \mathcal{U})$ onto the uniform space $(Y, \mathcal{V})$. Then the map $SP^n f : (SP^n X, \ SP^n \mathcal{U}) \to (SP^n Y, \ SP^n \mathcal{V})$ is uniformly continuous.*

P r o o f. Let $\beta \in \mathcal{V}$ be an arbitrary covering. Due to the uniform continuity of the mapping f, there is $\alpha \in \mathcal{U}$ such that $\alpha \succ f^{-1}(\beta)$. Let us show that $\mathcal{S}(\alpha) \succ (SP^n f)^{-1}(\mathcal{S}(\beta))$. Let $[U_1 \times U_2 \times ... \times U_n] \in \mathcal{S}(\alpha)$ arbitrary element. For every $U_i \in \alpha$ $(i = 1, 2, ..., n)$ there is $V_i \in \beta$ such that $U_i \subset f^{-1}(V_i)$. It is easy to see that $[U_1 \times U_2 \times ... \times U_n] \subset (SP^n f)^{-1}([V_1 \times V_2 \times ... \times V_n])$. □

Corollary 2.2. *Let* $(X,\mathcal{U})$ *be a uniform space. Then,*

$$d(X,\mathcal{U}) \leq d(SP^nX, SP^n\mathcal{U}).$$

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