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IRREDUCIBLE REPRESENTATIONS OF THE GROUP OF CONJUGATING AUTOMORPHISMS OF A FREE GROUP

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Abstract

As E. Formanek has characterized low dimensional representations of the braid group B_n, we extend these representations to the group of conjugating automorphisms C_n, when $n \geq 5$. We then give a classification for irreducible representations of C_n in dimensions of at most $n-3$. Next, we determine representations of C_n in dimension $n-1$ when each of the restrictions to the symmetric group S_n and the braid group B_n are irreducible.

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1. Introduction

The classical braid group B_n on n strings is the abstract group with generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and a presentation as follows:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2.$$

B_n has a normal subgroup called the pure braid group and is denoted by P_n. The pure braid group is the kernel of the surjective map $\nu : B_n \to S_n$ defined by $\sigma_i \mapsto \alpha_i$, where S_n is the symmetric group on n elements, with generators: $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and a presentation as follows:

$$\alpha_i^2 = 1, \quad i = 1, 2, \ldots, n-2,$$
$$\alpha_i\alpha_{i+1}\alpha_i = \alpha_{i+1}\alpha_i\alpha_{i+1}, \quad i = 1, 2, \ldots, n-2,$$
$$\alpha_i\alpha_j = \alpha_j\alpha_i, \quad |i-j| \geq 2.$$

One of the famous representations of the group B_n is the Burau representation [3]. Let $z \in \mathbb{C}^*$, the reduced Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is defined by specializing $t \mapsto z$ in $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm}])$, where t is an indeterminate [4]. Other linear representations of the braid group were constructed, where the question of irreducibility has been the focus of many studies [1].

A generalization of the braid group is the group of conjugating automorphisms C_n, a subgroup of $Aut(F_n)$, where $F_n =$
$< x_1,\ x_2, \ldots,\ x_n >$ is the free group of rank n. For all $\beta \in C_n$, $\beta(x_i) = f_i^{-1}x_{\pi(i)}f_i$, where $\pi \in S_n$ and $f_i \in F_n$. In addition, if $\beta(x_1x_2\ldots x_n) = x_1x_2\ldots x_n$ then $\beta \in B_n$. By theorem of Artin [9], we define a faithful representation of the braid group B_n in $Aut(F_n)$.

Furthermore, C_n has a normal subgroup called the group of basis conjugating automorphisms and denoted by Cb_n. Note that Cb_n satisfies for all $\beta \in Cb_n$, $\beta(x_i) = f_i^{-1}x_if_i$. According to [2] , the structure of Cb_n is similar to the structure of P_n. Also, the quotient groups B_n/P_n and C_n/Cb_n are isomorphic to S_n. Moreover, the generators of C_n are those of the braid group with those of the symmetric group (see Definition 2.5).

Let $F_n' = [F_n,\ F_n]$ be the commutator subgroup of F_n and $A_n = F_n/F_n'$. The extension of Burau representation on C_n is obtained by restricting

$$\mu : IA(F_n) \to GL_n(\mathbb{Z}[{t_1}^{\pm1}, \ldots, {t_n}^{\pm1}])$$

to Cb_n and by putting $t_1 = \ldots = t_n = t$, where $IA(F_n)$ is the kernel of the epimorphism $Aut(F_n) \to Aut(A_n)$ and $t_1, \ldots, t_n$ are indeterminate variables (see [3] and [2]).

In this paper, we prove that every irreducible representation of S_n of non trivial dimension, having the image of the generator α_1 under this representation a pseudoreflection, is equivalent to the restrictions of the extension of the reduced Burau representation to S_n namely $\hat{\phi}_{B/S_n}(z)$, for $n \geq 5$ (Proposition 3.1). We also get a similar proposition, for $n \geq 7$, without requiring the image of a generator of S_n to be a pseudoreflection (Proposition 3.2). Our first result is Theorem 3.7, which classifies all irreducible complex representations of C_n of dimensions r, when $2 \leq r \leq n-3$ and $n \geq 5$. Our second result is Theorem 4.1, which classifies representations $\rho : C_n \mapsto GL_{n-1}(\mathbb{C})$, where $n \geq 7$ and both of the restrictions of ρ to S_n and B_n are irreducible.

2. **Preliminaries and notations**

Let $\mathbb{C}^r$ denote the $r \times 1$ vectors, and $\bar{\mathbb{C}}^r$ denote the $1 \times r$ vectors. We say that a matrix $Z \in M_r(\mathbb{C})$ is a pseudoreflection if the rank of $Z - I$ is 1. If Z is a pseudoreflection then there exist $X \in \mathbb{C}^r$ and $Y \in \bar{\mathbb{C}}^r$ such that $Z = I - XY$. The eigenvalues of Z are 1 (with multiplicity $r-1$) and $1 - YX$ (with multiplicity 1). Note that Z is invertible if and only if $YX \neq 1$.

Definition 2.1. [4] Let $z \in \mathbb{C}^*$. The complex specialization of the reduced Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is defined on the generators σ_i of B_n, $1 \leq i \leq n-1$, by pseudoreflections $\beta_n(z)(\sigma_i) = I - P_i Q_i$, where

$$P_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ P_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ P_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\0\end{matrix}}\right\} i-1 \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\0\end{matrix}}\right\} n-i-2 \end{matrix} \quad \text{for } 1 \leq i \leq n-2$$

and

$$Q_1 = \begin{pmatrix} z^{-1} & -1 & 0 & \ldots & 0 \end{pmatrix}, \ Q_{n-1} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 + z^{-1} \end{pmatrix},$$

$$Q_i = (\overbrace{0 \ \ \ldots \ \ 0}^{i-1} \quad z^{-1} \quad -1 \quad \overbrace{0 \ \ \ldots \ \ 0}^{n-i-2}) \ \text{ for } 1 \leq i \leq n-2.$$

The associated matrix (Q_iP_j) is

$$(Q_iP_j)=\begin{pmatrix} z^{-1}+1 & -z^{-1} & 0 & \cdots & 0 & 0\\ -1 & z^{-1}+1 & -z^{-1} & \ddots & 0 & 0\\ 0 & -1 & z^{-1}+1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & z^{-1}+1 & -z^{-1}\\ 0 & 0 & 0 & \cdots & -1 & z^{-1}+1 \end{pmatrix}.$$

DEFINITION 2.2. ([4]) Let z be a root of $f_{n+1}(t)=t^n+t^{n-1}+\ldots+t+1$, where $n\geq 3$. Then the extension of $\beta_n(z)$ to B_{n+1}, namely $\hat{\beta}_{n+1}(z):B_{n+1}\to GL_{n-1}$ is the irreducible representation defined by $\hat{\beta}_{n+1}(z)(\sigma_i)=\beta_n(z)(\sigma_i)$ for $1\leq i\leq n-1$, and $\hat{\beta}_{n+1}(z)(\sigma_n)=I-PQ$, where $P=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$ and $Q=(-1)^{n-2}z(1,\ -(1+z),\ (1+z+z^2),\ldots,(-1)^{n-2}(1+z+\ldots+z^{n-2}))$.

LEMMA 2.1. ([4]) *The specialization of the reduced Burau representation* $\beta_n(z):B_n\to GL_{n-1}(\mathbb{C})$ *is either irreducible if* $z\in\mathbb{C}^*$ *is not a root of* $f_n(t)=t^{n-1}+t^n+\ldots+t+1$*, or has an irreducible composition factor* $\hat{\beta}_n(z)$ *of degree* $n-2$ *if* $z\in\mathbb{C}^*$ *is a root of* $f_n(t)$ *and* $n\geq 4$*.*

The standard representation was first discovered in 1996 by D. Tong and others [12, Eq.(19)]. I. Sysoeva used the complex specialization of the standard representation to classify the irreducible representations of the braid group of degree n.

DEFINITION 2.3. [10] Let $u\in\mathbb{C}^*$. A specialization of the standard representation is the representation

$$\tau_n(u):B_n\to GL_n(\mathbb{C})$$

defined by

$$\tau_n(u)(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & u & \\ & 1 & 0 & \\ & & & I_{n-1-i} \end{pmatrix}$$

for $1 \le i \le n-1$, where I_k is the $k \times k$ identity matrix. This representation is irreducible if and only if $u \neq 1$.

Lemma 2.2. ([10, Theorem 6.1]) *Suppose that $\rho : B_n \to GL_n(\mathbb{C})$ is an irreducible representation of B_n of degree $n \ge 9$. Then it is equivalent to the tensor product of a one-dimensional representation and a specialization of the standard representation.*

Definition 2.4. [4] A representation of the braid group $B_n \to GL_r(\mathbb{C})$ is of Burau type if $r \ge 2$ and it is equivalent to an irreducible representation which is the tensor product of a one dimensional representation and $\beta_n(z)$ or $\hat{\beta}_n(z)$.

We now introduce the group of conjugating automorphisms C_n as an abstract group with generators and relations.

Definition 2.5. [8] The group of conjugating automorphisms, denoted by C_n, is generated by $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$, where $\sigma_i, 1 \le i \le n-1$, generate the braid group B_n and $\alpha_i, 1 \le i \le n-1$, generate the symmetric group S_n, with the defining relations:

$$\begin{aligned} \sigma_i\alpha_j &= \alpha_j\sigma_i, \quad if\ |i-j| \ge 2, \\ \sigma_i\alpha_{i+1}\alpha_i &= \alpha_{i+1}\alpha_i\sigma_{i+1}, \\ \sigma_i\sigma_{i+1}\alpha_i &= \alpha_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \quad if\ |i-j| \ge 2, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \alpha_i^2 &= 1, \\ \alpha_i\alpha_j &= \alpha_j\alpha_i, \quad if\ |i-j| \ge 2, \\ \alpha_i\alpha_{i+1}\alpha_i &= \alpha_{i+1}\alpha_i\alpha_{i+1}. \end{aligned}$$

Theorem 2.1. ([6, Theorem 6]) *The extension of Burau representation on C_n of degree n is reducible.*

Definition 2.6. ([6]) Let $z \in \mathbb{C}^*$ and let $\hat{\phi}_B(z) : C_n \to GL_{n-1}(\mathbb{C})$ be the specialization of the extension of the reduced Burau representation on C_n, defined as follows:

$$\hat{\phi}_B(z)(\sigma_i) = \beta_n(z)(\sigma_i) \quad \text{and} \quad \hat{\phi}_B(z)(\alpha_i) = I - R_i S_i, \quad 1 \le i \le n-1,$$

where

$$R_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; R_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \; R_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} i-1 \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n-i-2 \end{matrix} \qquad \text{for} \quad 1 \le i \le n-2,$$

and

$$S_1 = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \end{pmatrix}, \; S_{n-1} = \begin{pmatrix} 1 & 1 & \dots & 1 & 2 \end{pmatrix}, \quad \text{and}$$

$$S_i = (\overbrace{0 \quad \dots \quad 0}^{i-1} \quad -1 \quad 1 \quad \overbrace{0 \quad \dots \quad 0}^{n-i-2}), \quad \text{for} \quad 1 \le i \le n-2.$$

The associated matrix $(S_i R_j)$ is

$$(S_i R_j) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Theorem 2.2. ([6]) *For $z \in \mathbb{C}^*$, the representation $\hat{\phi}_B(z) : C_n \to GL_{n-1}(\mathbb{C})$ is irreducible.*

3. Classification of irreducible representations of the group of conjugating automorphisms C_n of degree at most $n-3$

E. Formanek has classified all irreducible representations of B_n of which the generators are given by pseudoreflections. It is well known that C_n is generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ and $\{\alpha_1, \ldots, \alpha_{n-1}\}$, the generators of B_n and S_n respectively. It is then natural to classify all those irreducible representations of the group of conjugating automorphisms $C_n \mapsto GL_r(\mathbb{C})$, where $2 \leq r \leq n+1$. In our work, we consider r such that $2 \leq r \leq n-3$ and classify all irreducible representations of C_n of degree r. First, we state some previous results of E. Formanek and R. Rasala.

Theorem 3.1. ([4, Theorem 10]) *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $n \geq 5$ and $r \geq 2$. Suppose $\rho(\sigma_1)$ is a pseudoreflection. Then one of the following is satisfied.*
(i) ρ is equivalent to $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$, where $z \in \mathbb{C}^*$ is not a root of $f_n(t) = t^{n-1} + t^{n-2} + \ldots + 1$,
(ii) ρ is equivalent to $\hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$, where $z \in \mathbb{C}^*$ is a root of $f_n(t)$.

The next theorem is a classification of all irreducible complex representations of B_n of degree r, where $2 \leq r \leq n-1$ for $n \geq 7$.

Theorem 3.2. ([4]) *Let $n \geq 7$, all irreducible complex representations of B_n of dimension r, where $2 \leq r \leq n-1$, are of Burau type.*

Theorem 3.3. ([4, Theorem 15]) *Let $\rho : B_n \to GL_r$ be an irreducible representation, where $n \geq 5$. Then the following are equivalent:*
(a) *the representation ρ is of Burau type;*
(b) *for some $y \in \mathbb{C}^*$, $y^{-1}\rho(\sigma_1)$ is pseudoreflection;*
(c) *for some $y \in \mathbb{C}^*$, $\rho(\sigma) - yI$ is a matrix of rank 1.*

Theorem 3.4. ([7, p.144-145]0 *For $n \geq 5$, the symmetric group S_n has no irreducible representation of dimension r, such that $1 < r < n-1$.*

Theorem 3.5. ([7, Result 2]) *For $n \geq 9$, the first four minimal degrees of S_n are:*
(a) 1,
(b) $n-1$,
(c) $\frac{1}{2}n(n-3)$,
(d) $\frac{1}{2}(n-1)(n-2)$.

Theorem 3.6. ([5]) *Every representation of a finite group G over a field F with characteristic not dividing the order of G is semi-simple; that is a direct sum of irreducible representations.*

We use this theorem to give a classification of irreducible complex representations of the symmetric group and hence of the group of conjugating automorphisms C_n, when $n \geq 5$.

We now prove a result about symmetric groups, which is similar to Theorem 3.1 about braid groups.

PROPOSITION 3.1. *Let $\psi : S_n \to GL_r(\mathbb{C})$ be an irreducible representation such that $n \geq 5$, $r \geq 2$ and $\psi(\alpha_1)$ is pseudoreflection. Then ψ is equivalent to the restriction of $\hat{\phi}_B(z) : C_n \to GL_{n-1}(\mathbb{C})$ to the symmetric group S_n.*

P r o o f. Let $\nu : B_n \to S_n$ be a surjective map and $\psi : S_n \to GL_r(\mathbb{C})$ be an irreducible representation, such that $\psi(\alpha_1)$ is a pseudoreflection, then $\psi \circ \nu$ is an irreducible representation of B_n. So, by Theorem 3.1 and Theorem 3.4, $\psi \circ \nu$ is equivalent to $\beta_n(z)$, when z is not a root of $f_n(t) = t^{n-1} + t^{n-2} + \ldots + t + 1$ for $n \geq 5$. This implies that $\psi(\alpha_i)$ is equivalent to $\beta_n(z)(\alpha_i)$. It's easy to see that $z = \pm 1$. Direct calculations show that $\beta_n(\pm 1)$ is equivalent to $\hat{\phi}_{B/S_n}$. □

We now improve this result by not requiring that the image of a generator is a pseudoreflection. We get the following proposition.

PROPOSITION 3.2. *Let $\rho : S_n \to GL_r(\mathbb{C})$ be an irreducible representation, such that $n \geq 7$ and $2 \leq r \leq n-1$, then ρ is equivalent to $y\hat{\phi}_{B(z)/S_n}$ where $y, z \in \mathbb{C}^*$ and y is unique.*

P r o o f. We have $\nu : B_n \to S_n$ is a surjective map and $\rho : S_n \to GL_r(\mathbb{C})$ is an irreducible representation, then $\rho \circ \nu : B_n \to GL_r(\mathbb{C})$ is irreducible. Since $n \geq 7$, by Theorem 3.2, $\rho \circ \nu$ is of Burau type. Then, by Theorem 3.3, there exists $y \in \mathbb{C}^*$ such that $y^{-1}\rho \circ \nu(\sigma_1)$ is a pseudoreflection. This implies that $y^{-1}\rho(\alpha_1)$ is a pseudoreflention. Using Proposition 3.1, we obtain $y^{-1}\rho$ is equivalent to $\hat{\phi}_{B/S_n}$. Hence ρ is equivalent to $y\hat{\phi}_{B/S_n}$. □

For $n \geq 9$, Theorem 3.5 provides a stronger result.

LEMMA 3.1. ([11]) *For $n \geq 10$, there are no irreducible complex representations of the braid group B_n on n strings of dimension $n+1$.*

Using Lemma 3.1, we deduce that there are no irreducible representations of S_n of degree $n+1$, where $n \geq 10$. This result was initially found by R. Rasala (Theorem 3.5).

COROLLARY 3.1. *There are no irreducible complex representations of S_n of dimension $n+1$, for all $n \geq 10$.*

P r o o f. We have $\nu : B_n \to S_n$ is surjective map, such that $\nu(\sigma_i) = \alpha_i$. Suppose that for all $n \geq 10$, there exists $\gamma : S_n \to GL_{n+1}(\mathbb{C})$ an irreducible representation of S_n of dimension n+1. The representation $\gamma \circ \nu : B_n \to GL_{n+1}(\mathbb{C})$ is irreducible because $\gamma \circ \nu(\sigma_i)V = \gamma(\alpha_i)V$ for all V subspaces of $GL_{n+1}(\mathbb{C})$. This is because γ is irreducible. This gives an irreducible representation of B_n of degree $n+1$, which gives a contradiction with Lemma 3.1. Hence, we deduce that there are no irreducible complex representations of S_n of dimenion $n+1$, for all $n \geq 10$. □

Using previous results, we describe the restrictions of the representations of C_n onto B_n and S_n of dimension r when $2 \leq r \leq n-3$, and $n \geq 5$. Then we attempt to give a classification of irreducible complex representations of C_n of degree at most $n-3$.

Using results of E. Formanek [4], we easily see that for $n \geq 5$, there are no irreducible complex representations of the braid group B_n of degree $\leq n-3$ except the one-dimensional representation.

LEMMA 3.2. *Let K be an $r \times r$ matrix defined by $K = \begin{pmatrix} I_s & 0 \\ 0 & -I_{r-s} \end{pmatrix}$, where $1 \leq s < r$. Here I_s is the $s \times s$ identity matrix and I_{r-s} is the $(r-s) \times (r-s)$ identity matrix. If M is an $r \times r$ matrix that commutes with K, then $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix}$, where M_1 is an $s \times s$ matrix and M_4 is an $(r-s) \times (r-s)$ matrix.*

P r o o f. Let M be an $r \times r$ matrix that commutes with K. Set $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where M_1 is an $s \times s$ matrix and M_4 is an $(r-s) \times (r-s)$ matrix. Since $KM = MK$, it follows that $\begin{pmatrix} M_1 & M_2 \\ -M_3 & -M_4 \end{pmatrix} = \begin{pmatrix} M_1 & -M_2 \\ M_3 & -M_4 \end{pmatrix}$. Thus M_2 and M_3 are zero matrices, and so $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix}$ as required. □

THEOREM 3.7. *Let $\rho : C_n \mapsto GL_r(\mathbb{C})$ be a representation such that $n \geq 5$ and $2 \leq r \leq n-3$. Then ρ is reducible.*

P r o o f. We have that $\rho_{/S_n}$ and $\rho_{/B_n}$ are both reducible by Theorem 3.4 and results in [4]. For some basis, the representation of S_n is semi-simple, that is equivalent to a direct sum of one-dimensional representations. Let ρ' be an equivalent representation of ρ such that ρ' is semi-simple. Since $\rho'(\alpha_i)$ is direct sum of blocks and $\alpha_i^2 = 1$ for all $1 \leq i \leq n-1$, it follows that $\rho'(\alpha_i)$

is a diagonal matrix with diagonal entries either 1 or -1. We consider two cases:

(i) Assume first that $\rho'(\alpha_i)$ is constant for all $1 \leq i \leq n-1$, that is $\rho'(\alpha_i) = c_i I_r$, where $c_i \in \{-1, 1\}$ for all $1 \leq i \leq n-1$. In this case, we easily see that ρ' is reducible as $\rho'_{/B_n}$ is reducible. Hence, ρ is reducible.

(ii) Assume now, without loss of generality, that $\rho'(\alpha_1)$ is not constant. Let P be the permutation matrix such that $P^{-1}\rho'(\alpha_1)P = \begin{pmatrix} I_s & 0 \\ 0 & -I_{r-s} \end{pmatrix}$, where $1 \leq s < r$. Let $\rho'' = P^{-1}\rho' P$. Since $\rho''(\alpha_i)$ is still a diagonal matrix as P is a permutation matrix, it follows that we may write $\rho''(\alpha_i) = \begin{pmatrix} A_s^{(i)} & 0 \\ 0 & A_{r-s}^{(i)} \end{pmatrix}$, where $A_s^{(i)}$ is an $s \times s$ diagonal matrix and $A_{r-s}^{(i)}$ is an $(r-s) \times (r-s)$ diagonal matrix for all $1 \leq i \leq n-1$. Using the relations $\alpha_1\sigma_j = \sigma_j\alpha_1$ for all $3 \leq j \leq n-1$, and using Lemma 3.2, we see that $\rho''(\sigma_i) = \begin{pmatrix} B_s^{(i)} & 0 \\ 0 & B_{r-s}^{(i)} \end{pmatrix}$, where $B_s^{(i)}$ is an $s \times s$ matrix and $B_{r-s}^{(i)}$ is an $(r-s) \times (r-s)$ matrix for all $3 \leq i \leq n-1$. Now, using $\sigma_2\alpha_3\alpha_2 = \alpha_3\alpha_2\sigma_3$, we get that $\sigma_2 = \alpha_3\alpha_2\sigma_3\alpha_2\alpha_3$ and so $\rho''(\sigma_2) = \rho''(\alpha_3)\rho''(\alpha_2)\rho''(\sigma_3)\rho''(\alpha_2)\rho''(\alpha_3)$. This implies that $\rho''(\sigma_2) = \begin{pmatrix} B_s^{(2)} & 0 \\ 0 & B_{r-s}^{(2)} \end{pmatrix}$, where $B_s^{(2)}$ is an $s \times s$ matrix and $B_{r-s}^{(2)}$ is an $(r-s) \times (r-s)$ matrix. In the same way, and using the relation $\sigma_1\alpha_2\alpha_1 = \alpha_2\alpha_1\sigma_2$, we get that $\rho''(\sigma_1) = \begin{pmatrix} B_s^{(1)} & 0 \\ 0 & B_{r-s}^{(1)} \end{pmatrix}$, where $B_s^{(1)}$ is an $s \times s$ matrix and $B_{r-s}^{(1)}$ is an $(r-s) \times (r-s)$ matrix. Thus, $\rho''(\alpha_i) = \begin{pmatrix} A_s^{(i)} & 0 \\ 0 & A_{r-s}^{(i)} \end{pmatrix}$ and $\rho''(\sigma_i) = \begin{pmatrix} B_s^{(i)} & 0 \\ 0 & B_{r-s}^{(i)} \end{pmatrix}$, where $A_s^{(i)}$ are diagonal matrices for all $1 \leq i \leq n-1$. Therefore ρ'' is reducible and so ρ is reducible. □

4. **Irreducible representations of C_n of degree $n-1$**

In this section, we prove a partial result which determines complex representations $\rho : C_n \mapsto GL_{n-1}(\mathbb{C})$, where both restrictions to S_n and B_n are irreducible.

LEMMA 4.1. *For $n \geq 4$, consider a representation $\rho : C_n \mapsto GL_{n-1}(\mathbb{C})$, where $\rho(\sigma_i) = \hat{\phi_B}(\sigma_i)$ and $\rho(\alpha_i) = K^{-1}\hat{\phi_B}(\alpha_i)K$ for $i = 1, \ldots, n-1$. Here K is an $(n-1) \times (n-1)$ invertible matrix. Using one type of the relations in C_n, $\hat{\phi_B}(\sigma_i)K^{-1}\hat{\phi_B}(\alpha_j)K = K^{-1}\hat{\phi_B}(\alpha_j)K\hat{\phi_B}(\sigma_i)$ ($|j-i| \geq 2$), we get that K*

is a block lower triangular matrix $\begin{pmatrix} K' & 0 \\ * & c \end{pmatrix}$, *where* K' *is an* $(n-2)\times(n-2)$ *invertible matrix and* $c \in \mathbb{C}^*$.

Proof. We take into account the general form of α_i and σ_i under ρ. We make computations for the 3×3 main blocks. For simplicity, we prove the lemma for $n=4$; that is $\rho : C_4 \mapsto GL_3(\mathbb{C})$. We determine the form of the matrix M that commutes with the image of the generator σ_3 of C_4. We have $\rho(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -t^{-1} \end{pmatrix}$. Consider a 3×3 invertible matrix $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Now, $\rho(\sigma_3)M = M\rho(\sigma_3)$ implies that

$$\begin{pmatrix} a & b & c \\ d & e & f \\ -a-d-t^{-1}g & -b-e-t^{-1}h & -c-f-t^{-1}i \end{pmatrix}$$

$$= \begin{pmatrix} a-c & b-c & -ct^{-1} \\ d-f & e-f & -ft^{-1} \\ g-i & h-i & -it^{-1} \end{pmatrix}.$$

Direct computations show that $c=f=0$, and so $M = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & i \end{pmatrix}$. Let us put $i=1$. It is then easy to see that the vector $v=(0,\ldots,0,1)^T$ is an eigenvector of the matrix M with eigenvalue equals 1. Now let $K = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Using the relation $\hat{\phi_B}(\sigma_3)K^{-1}\hat{\phi_B}(\alpha_1)K = K^{-1}\hat{\phi_B}(\alpha_1)K\hat{\phi_B}(\sigma_3)$, we see that $K^{-1}\hat{\phi_B}(\alpha_1)K$ commutes with $\hat{\phi_B}(\sigma_3)$. So we take $M = K^{-1}\hat{\phi_B}(\alpha_1)K$ and assume that $(3,3)$ entry of this matrix is one. Since the vector $v=(0,\ldots,0,1)^T$ is fixed by M, it follows that $\hat{\phi_B}(\alpha_1)(Kv) = Kv$, and so Kv is an eigenvector of $\hat{\phi_B}(\alpha_1)$ with eigenvalue equals one. Note that $Kv = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$. On the other hand, the eigenvectors of $\hat{\phi_B}(\alpha_1)$ that correspond to the eigenvalue 1 are $u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence, $Kv = c_1u_1 + c_2u_2$, where c_1 and c_2 are scalars. Thus, $Kv = \begin{pmatrix} c_2 \\ c_2 \\ c_1 \end{pmatrix}$ and so $K = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Using the other relation $\hat{\phi_B}(\sigma_1)K^{-1}\hat{\phi_B}(\alpha_3)K = K^{-1}\hat{\phi_B}(\alpha_3)K\hat{\phi_B}(\sigma_1)$, we get $a_{13} = 0$, and so K is a block lower triangular matrix. □

THEOREM 4.1. *Let $\rho : C_n \to GL_{n-1}(\mathbb{C})$ be a representation such that $n \geq 7$. Assume that $\rho_{/S_n}$ and $\rho_{/B_n}$ are irreducible, then ρ is equivalent to a representation ρ_1 given by*

$$\rho_1(\alpha_i) = y_1\hat{\phi}_B(z_1)(\alpha_i)$$

and

$$\rho_1(\sigma_i) = y_2\hat{\phi}_B(z_2)(\sigma_i)$$

for all $1 \leq i \leq n-1$, where y_1, y_2, z_1 and z_2 are non zero complex numbers. Here, $\hat{\phi}_B(z_i)$ $(i = 1, 2)$ are the extensions of the reduced Burau representation.

P r o o f. For $n \geq 7$, consider the representation $\rho : C_n \to GL_{n-1}(\mathbb{C})$ such that $\rho_{/S_n}$ and $\rho_{/B_n}$ are both irreducible. Then both representations are equivalent to the extension of the reduced Burau representation $\hat{\phi}_B(z)$ restricted to S_n and B_n respectively, up to tensoring by a one-dimensional representation (See Theorem 3.2, Proposition 3.2). Without loss of generality, we might consider an equivalent representation of ρ and we still denote the obtained representation by ρ. More precisely, we write the images under the representation ρ of the generators α_i and σ_i as follows: $\rho(\alpha_i) = y_1K^{-1}\hat{\phi}_B(z_1)(\alpha_i)K$ and $\rho(\sigma_i) = y_2\hat{\phi}_B(z_2)(\sigma_i)$, where y_1, y_2, z_1 and z_2 are non zero complex numbers, $1 \leq i \leq n-1$ and K is an $(n-1)\times(n-1)$ invertible matrix. For $n \geq 7$, we take into account that general form of α_i and σ_i under ρ. We make computations for the 3×3 main blocks in them and we use mathematical induction on n. We consider the generators of S_4 and B_4 as a reference, where the matrices of the generators α_i and σ_i under ρ are equivalent to $\hat{\phi}_B(z)$ restricted to S_n and B_n respectively.

By Lemma 4.1, the matrix K is block lower triangular, and so consider $K = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & i \end{pmatrix}$, where $a, b, d, e, g, h, i \in \mathbb{C}$, $i \neq 0$, $ae - bd \neq 0$. The images of the generators α_i and σ_i under $\hat{\phi_B}(z)$ are

$$\alpha_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \alpha_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \alpha_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix},$$

and

$$\sigma_1 \mapsto \begin{pmatrix} 1-z^{-1} & 1 & 0 \\ z^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-z^{-1} & 1 \\ 0 & z^{-1} & 0 \end{pmatrix},$$

$$\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -z^{-1} \end{pmatrix}.$$

Applying the relations between the generators $\rho(\alpha_i)$ and $\rho(\sigma_i)$ (Definition 2.5), we obtain equations in seven unknowns. Using Mathematica software to simplify computations, we get that the matrix K is a constant matrix. This implies that ρ is equivalent to $\hat{\phi}_B(z)$. We now use mathematical induction on $n \geq 7$.

We assume that representations $\hat{\phi}_{B/B_{n-1}}$ and $\hat{\phi}_{B/S_{n-1}}$ satisfy the relations in C_{n-1} implies that the $(n-2)\times(n-2)$ matrix K involved in conjugating $\hat{\phi}_{B/S_{n-1}}$ is the constant matrix. We now prove that this result remains true for our representation $\rho : C_n \to GL_{n-1}(\mathbb{C})$, where the restriction to S_n and B_n are $\hat{\phi}_{B/B_n}$ and $\hat{\phi}_{B/S_n}$. Under the hypothesis, both representations ρ_{S_n} and ρ_{B_n} are equivalent to $\hat{\phi}_B(z)$ restricted to S_n and B_n respectively. We consider the restrictions of the representation ρ on S_{n-1} and B_{n-1}. It is easy to see that the representation $\rho_{/B_{n-1}}$ of degree $n-1$ is reducible because the row vector $(1,1,\ldots,1)$ is fixed under the representation. In this case, we write the generator σ_i of $\rho_{/B_n}$, $i = 1,2,\ldots,n-2$, as $\begin{pmatrix} \sigma'_i & 0 \\ * & 1 \end{pmatrix}$ for a choice of a basis, where $\sigma'_i = \hat{\phi}_{B/B_{n-1}}(\sigma_i)$ of degree $n-2$. It is worth saying that this representation of σ_i, given by block lower triangular matrices, is obtained with respect to some basis; That is, all our representations are defined up to equivalence. Similarly, the representation $\rho_{/S_{n-1}}$ is reducible. By the result of Maschke (Theorem 3.6), the complex representation of the finite group S_n is semi-simple, that is equivalent to a direct sum of a representation of degree $n-2$ and a one-dimensional representation. So, with respect to a basis, each α_i under $\rho_{/S_{n-1}}$, $i = 1,2,\ldots,n-2$, is written as $\begin{pmatrix} \alpha'_i & 0 \\ 0 & 1 \end{pmatrix}$, where α'_i is similar to $\hat{\phi}_{B/S_{n-1}}(\alpha_i)$ of degree $n-2$.

Upon conjugation of the matrices α_i and σ_i, we still call the matrix involved in conjugating the matrices α_i by K, which is assumed to be block lower triangular by Lemma 4.1. Writing the relations between the generators of C_n, α_i and σ_i, for $1 \leq i \leq n-1$, we perform multiplication of block lower triangular matrices to obtain the same relations as in C_{n-1}, involving the generators $\alpha_1,\ldots,\alpha_{n-2}$ and the generators $\sigma_1,\ldots,\sigma_{n-2}$. For instance, considering the left hand side of the relation in C_n, $\rho(\alpha_i)\rho(\sigma_j) = \rho(\sigma_j)\rho(\alpha_i)$, $|i-j| \geq 2$, we obtain

$$\begin{pmatrix} K'^{-1} & 0 \\ * & c^{-1} \end{pmatrix} \begin{pmatrix} \alpha'_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K' & 0 \\ * & c \end{pmatrix} \begin{pmatrix} \sigma'_j & 0 \\ * & 1 \end{pmatrix} = \begin{pmatrix} K'^{-1}\alpha'_i K'\sigma'_j & 0 \\ * & 1 \end{pmatrix}.$$ Similarly, the right hand side of the relation is $\begin{pmatrix} \sigma'_j K'^{-1}\alpha'_i K' & 0 \\ * & 1 \end{pmatrix}$.

Likewise, all the other relations in C_{n-1} are obtained in this way. We remark that $\sigma'_j = \hat{\phi}_{B/B_{n-1}}(\sigma_j)$ and $\alpha'_i = \hat{\phi}_{B/S_{n-1}}(\alpha_i)$. Having the relations satisfied in C_{n-1}, this implies that K' is a block lower triangular matrix by Lemma 4.1. By induction, since the result is true for C_{n-1}, we conclude that K' is the constant matrix. This means that $K' = dI_{n-2}$, where $d \in \mathbb{C}^*$. This means that the $(n-1)\times(n-1)$ matrix K is a block lower triangular matrix of the form $\begin{pmatrix} dI_{n-2} & 0 \\ * & c \end{pmatrix}$. On the other hand, we use the remaining relations in C_n involving the generators, not used so far, namely α_{n-1} and σ_{n-1}. Direct calculations show that $d = c$ and the entries off the main diagonal are zeros, and so $K = cI_{n-1}$. □

For $n = 3$, we prove a result similar to that of Theorem 4.1 when $r = n-1$.

THEOREM 4.2. *Let $\rho : C_3 \mapsto GL_2(\mathbb{C})$ be an irreducible representation. Assume that $\rho_{/S_3}$ and $\rho_{/B_3}$ are both irreducible. Then we have non trivial representations, among them is the extension of the reduced Burau representation $\hat{\phi}_B(z)$, where $z \in \mathbb{C}^*$.*

P r o o f. By a result of Formanek [4], Theorem 3.2 remains true for $n = 3$. This means that $\rho_{/B_3}$ is equivalent to $\hat{\phi}_{B/B_3}$, up to tensoring by 1-dimensional representation. On the other hand, $\rho_{/S_3}$ is equivalent to $\hat{\phi}_{B/S_3}$. Consider a matrix K, which is used to conjugate the representation $\rho_{/S_3}$. Thus, we let $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. In this case, we have:

$$\sigma_1 \mapsto \begin{pmatrix} 1-t^{-1} & 1 \\ t^{-1} & 0 \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & -t^{-1} \end{pmatrix},$$

$$\alpha_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \alpha_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Using the only two relations: $\sigma_1\sigma_2 K\alpha_1 K^{-1} = K\alpha_2 K^{-1}\sigma_1\sigma_2$ and $\sigma_1 K\alpha_2\alpha_1 K^{-1} = K\alpha_2\alpha_1 K^{-1}\sigma_2$, we get that $K = \begin{pmatrix} -c+d & -c \\ c & d \end{pmatrix}$. Thus, we see that we have infinite solutions if $d^2 - cd + c^2 \neq 0$. If we let $c = 0$, then $K = dI_2$, where $d \neq 0$. This gives us the extension of the reduced Burau representation $\hat{\phi}_B(z)$, for $z \in \mathbb{C}^*$. □

5. **Conclusion**

As E. Formanek gave classification for irreducible complex representations $\rho : B_n \mapsto GL_r(\mathbb{C}), n \geq 7$ and $2 \leq r \leq n-1$, we extend this work to the group of conjugating automorphisms of a free group C_n, namely $C_n \mapsto GL_r(\mathbb{C})$, for specific values of r; more precisely, when $2 \leq r \leq n-3$. We also show that if $r = n-1$ and the restrictions of ρ to S_n and B_n are both irreducible then the representation ρ is equivalent to the extension of the Burau representation, up to tensoring by one-dimensional representation.

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