International Journal of Applied Mathematics

Volume 37 No. 5 2024, 501–510
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v37i5.2

GENERALIZED POISSON SEMIGROUP ASSOCIATED TO THE BESSEL OPERATOR AND APPLICATIONS

Adam Zakria [1,§], Mohamed Vall Ould Moustapha [2], Ibrahim-Elkhalil Ahmed [3] and Husam Elfadil Mohammed [4]

[1,3,4] Department of Mathematics, College of Science
Jouf University, Sakaka 2014, SAUDI ARABIA
e-mails: [1] azsidig@ju.edu.sa, [3] ielkhail@ju.edu.sa,
[4] husamelfadil@ju.edu.sa [4]
[2] Faculté des Sciences et Techniques
Université de Nouakchott Al-Aasriya, Nouakchott-Mauritanie
e-mail: [2] mohamedvallouldmoustapha230@gmail.com
[1] Faculty of Science, Department of Mathematics
University of Kordofan, El-Obeid, SUDAN
[3,4] Shendi University, Faculty of Sciences and Technology
Departement of Mathematics, Shendi SUDAN

Abstract

In this manuscript we give explicit formulas for the generalized Poisson semigroups associated to the operator of Bessel type

$$L^a = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - a^2x^2.$$

Received: January 7, 2024

As applications of our results we give explicit formulas for the generalized Poisson semigroup with Morse potential and on the real hyperbolic space $\mathbb{H}^n$.

MSC 2020: 35C05, 35C15

Key Words and Phrases: Poisson, heat, semigroup, Bessel operator, Laplace-Beltrami operator, hyperbolic space

1. Introduction

The aim of this paper is to solve explicitly the generalized Poisson equation associated to the Bessel operator. It is known that the differential operator of the Bessel type is very important in analysis and its applications, and there are many interesting research papers published in this area of research (see for example Abdelhaye et al. [1], Betancor et al. [2], Adam et al. [4]) and the references therein. Betancor et al. in [3] investigated the heat and Poisson semigroups of Fourier-Neumann expansions. Isolda-Cardoso [6], explain the importance of the pointwise convergence to initial data of heat and Poisson problems for the Bessel operator.

The main objective of this article is to solve explicitly the following generalized Poisson problem

$$\begin{cases} L^a u(y,x) = \left(i\frac{\partial}{\partial y} + \nu\right)^2 u(y,x), (y,x) \in I\!R^2_+ \\ \lim_{y\longrightarrow 0} y^{-\nu} u(y,x) = u_0(x), u_0 \in C_0^\infty(I\!R^+) \end{cases} \tag{1}$$

associated to the second order differential operator of Bessel type:

$$L^a = x^2 \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - a^2 x^2, \tag{2}$$

where a and ν are real parameters.

As an application of our results we give explicit formula the generalized Poisson semigroups with the Morse potential on the real line $I\!R$, and we consider also the generalized Poisson problem on the real hyperbolic space $\mathbb{H}^n$.

Note that for $\nu = 0$, the generalized Poisson problem (1) reduces to the classical Poisson problems for the Bessel operators L^a and this particular case is considered recently by author Adam et al. [4].

For the proof of our main results we need the following technical lemmas:

LEMMA 1.1. *If* $z = x^2 + x'^2 - 2xx'\cos y$, $x, x', y \in I\!R^+$ *and* $f(z) = \phi(x, x', y)$, *then the following formulas hold:*

- *i)* $\frac{\partial\phi}{\partial x} = (2x - 2x'\cos y)\frac{\partial f}{\partial z}$, $\frac{\partial^2\phi}{\partial x^2} = (2x - 2x'\cos y)^2 \frac{\partial^2 f}{\partial z^2} + 2\frac{\partial f}{\partial z}$,
- *ii)* $\frac{\partial\phi}{\partial y} = 2xx'\sin y \frac{\partial f}{\partial z}$, $\frac{\partial^2\phi}{\partial y^2} = (2xx'\sin y)^2 \frac{\partial^2 f}{\partial z^2} + 2xx'\cos y \frac{\partial f}{\partial z}$,

- *iii)* $\left(L^a + \frac{\partial^2}{\partial y^2}\right)\phi = 4x^2\left(z\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} - \frac{a^2}{4}\right)f,$
- *iv)* $x^{-\alpha}L^a x^\alpha = x^2\frac{\partial^2}{\partial x^2} + (2\alpha+1)x\frac{\partial}{\partial x} + \alpha^2 - a^2x^2.$

The proof of this lemma is simple, and is left to the reader.

Lemma 1.2. *The Lommel differential equation (see Magnus et al.* [8]*)*

$$x^2\frac{\partial^2 f}{\partial x^2} + (1-2\alpha)x\frac{\partial f}{\partial x} - (\beta\gamma x^\gamma)^2 f + (\alpha^2 - \nu^2\gamma^2)f = 0,$$

has two independent solutions $x^\alpha I_\nu(\beta x^\gamma)$ *and* $x^\alpha K_\nu(\beta x^\gamma)$*, where* I_ν *and* K_ν *are the modified Bessel functions of the first and second kind.*

Lemma 1.3. *For* $f \in L^1(\mathbb{R}^n)$ *the Fourier and inverse Fourier transforms of* f *are given by*

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} f(x)\exp(-i\xi.x)dx,$$

and

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} f(\xi)\exp(i\xi.x)d\xi.$$

For radial function $f \in L^1(I\!R)$*, according to Taylor* [11]*, p.226, we have:*

$$(\mathcal{F}^{-1}f)(x) = |x|^{1-\frac{n}{2}}\int_0^{+\infty} f(r)J_{\frac{n}{2}-1}(r\,|x|)r^{\frac{n}{2}}dr.$$

Lemma 1.4. *The following formula hold (Prudnikov et al.* [10]*, p.365):*

$$\int_0^{+\infty} x^{\alpha-1}J_\mu(bx)K_\nu(cx)dx = A^\alpha_{\mu,\nu},$$

with

$$A^\alpha_{\mu,\nu} = 2^{\alpha-2}b^\mu c^{-(\alpha+\mu)}\frac{\Gamma((\alpha+\mu+\nu)/2\Gamma((\alpha+\mu-\nu)/2)}{\Gamma(\mu+1)}$$
$$\times {}_2F_1((\alpha+\mu+\nu)/2, (\alpha+\mu-\nu)/2, \mu+1, -\frac{b^2}{c^2}),$$

where J_μ *is the Bessel functions of the first kind,* K_ν *is modified Bessel functions of the second kind and* ${}_2F_1$ *is the Gauss hypergeometric function*

$$F(a,b,c,z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad |z| < 1,$$

$(a)_n$ *is defined by* $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Lemma 1.5. *We have the following integral formulas:*

- *i)* $\int_{-\infty}^{\infty}(1+s^2)^\alpha s^\beta ds = (1+(-1)^\beta)\frac{\Gamma(\beta+1)\Gamma(-\alpha-(\beta+1)/2))}{\Gamma(-\alpha)}$: $0<\beta+1<-2\alpha$,
- *ii)* $\int_{-\infty}^{\infty}(1+s^2)^\nu s^{-\nu-1}ds = (1+(-1)^{-\nu-1})\frac{1}{2}\Gamma((-1-\nu)/2) : -\nu<0$,
- *iii)* $\int_{-\infty}^{\infty}(1+s^2)^{\nu-1}s^{-\nu}ds = (1+(-1)^{-\nu})\frac{1}{2}\Gamma((1+\nu)/2) : \nu<1$.

P r o o f. The proof of the lemma uses essentially the formula from Magnus et al. [8], p.6,

$$\int_0^\infty \frac{t^x}{(1+t^z)^{1+y}} = \frac{1}{z}\frac{\Gamma((x+1)/z)\Gamma(y-(x-z+1)/z))}{\Gamma(1+y)},$$
$$Re\, z>0, Re\, y > Re\,(x-z+1)/z, Re\, x>-1, Re\, y>-1.$$

□

2. Generalized Poisson equation associated to the the Bessel operator

In this section we present our main result of this paper:

THEOREM 2.1. *For $a\in\mathbb{R}^*$ and for $\nu : 0<\nu<1$, the problem* (1) *has the unique solution given by*

$$u(y,x) = \int_0^\infty P_a^\nu(y,x,x')u_0(x')\frac{dx'}{x'^{1+2\nu}}, \tag{3}$$

where

$$P_a^\nu(y,x,x') = c_\nu\frac{\partial}{\partial y}\left[\left(\frac{x-x'e^{-iy}}{x-x'e^{iy}}\right)^{\nu/2}K_{-\nu}\left(|a|\sqrt{x^2+x'^2-2xx'\cos y}\right)\right],$$

with $c_\nu = \frac{4|a|^{-\nu}(-1)^{\nu+1}}{i\nu\Gamma(-\nu)\Gamma(\frac{-1-\nu}{2})(1+(-1)^{-\nu-1})+\Gamma(-1-\nu)\Gamma(\frac{1+\nu}{2})(1+(-1)^{-\nu})}$, *and $K_{-\nu}$ is the modified Bessel function of the second kind.*

P r o o f. Set $A = x - x'e^{iy}$ and $D^\nu = -\left(i\frac{\partial}{\partial y}+\nu\right)^2$. We like to solve the Poisson equation in (1):

$$(L^a + D^\nu)\, p_\nu = 0. \tag{4}$$

We look for a solution of equation (4) in the form

$$p_\nu(y,x,x') = A^{-\nu}e_\nu(y,x,x'), \tag{5}$$

that is: $(L^a+D^\nu)\,A^{-\nu}e_\nu(y,x,x')=0$, we have:

$$L^aA^{-\nu}e_\nu(y,x,x') = A^{-\nu-2}\{A^2x^2\frac{\partial^2}{\partial x^2} + A(-2\nu x + A)x\frac{\partial}{\partial x}$$
$$+\nu(\nu+1)x^2 - \nu xA - a^2x^2A^2\}e_\nu(y,x,x'),$$

and $D^\nu A^{-\nu} e_\nu(y,x,x') = A^{-\nu-2}\left\{A^2 \frac{\partial^2}{\partial y^2} + 2i\nu x A \frac{\partial}{\partial y}\right\} e_\nu(y,x,x')$,
thus we obtain:

$$(L^a + D^\nu)\, A^{-\nu} e_\nu(y,x,x') = A^{-\nu-2}\{A^2 x^2 \frac{\partial^2}{\partial x^2} + A^2 \frac{\partial^2}{\partial y^2}$$
$$+Ax(-2\nu x + A)\frac{\partial}{\partial x} + (A^2 + 2i\nu x' e^{iy} A)\frac{\partial}{\partial y} - a^2 x^2 A^2\} e_\nu(y,x,x') = 0. \quad (6)$$

Using Lemma 1.1 we have

$$\begin{aligned} A^2 x^2 \frac{\partial^2}{\partial x^2} + A^2 \frac{\partial^2}{\partial y^2} &= A^2 x^2 \left((2x - 2x'\cos y)^2 \frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z}\right) \\ &\quad + A^2 \left((2xx'\sin y)^2 \frac{\partial^2}{\partial z^2} + 2xx'\sin y \frac{\partial}{\partial z}\right) \\ &= A^2 x^2 \left(4z \frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z} + 2xx'\sin y \frac{\partial}{\partial z}\right), \end{aligned} \quad (7)$$

$$\begin{aligned} Ax(-2\nu x + A)\frac{\partial}{\partial x} + (A^2 + 2i\nu x' e^{iy} A)\frac{\partial}{\partial y} - a^2 x^2 A^2 &= Ax(-2\nu x + A) \\ \times(2x - 2x'\cos y)\frac{\partial}{\partial z} + (A^2 + 2i\nu x' e^{iy} A) 2xx'\sin y \frac{\partial}{\partial z} &- a^2 x^2 A^2 \\ = A^2 x^2 \left(2\frac{\partial}{\partial z} - 4\nu \frac{\partial}{\partial z} - 2xx'\sin y \frac{\partial}{\partial z}\right) &- a^2 x^2 A^2. \end{aligned} \quad (8)$$

Substituting equation (7) and equation (8) into equation (6), we get the following

$$(L^a + D^\nu)\, A^{-\nu} e_\nu(y,x,x') = A^{-\nu-2}\{A^2 x^2 \left(4z \frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z} + 2xx'\sin y \frac{\partial}{\partial z}\right)$$
$$+A^2 x^2 \left(2\frac{\partial}{\partial z} - 4\nu \frac{\partial}{\partial z} - 2xx'\sin y \frac{\partial}{\partial z}\right) - a^2 x^2 A^2\} e_\nu(y,x,x') = 0,$$

where $z = x^2 + x'^2 - 2xx'\cos y$ as in Lemma 1.1, and we obtain

$$z \frac{\partial^2}{\partial z^2} e_\nu^a + (1-\nu)\frac{\partial}{\partial z} e_\nu^a - \frac{a^2}{4} e_\nu^a = 0. \quad (9)$$

The equation (9) is a generalized Lommel modified Bessel equation as in Lemma 1.2, with $\alpha = \nu/2$, $\nu = \nu$, $\beta = |a|$ and $\gamma = 1/2$ an appropriate solution is $e_\nu^a = c z^{\nu/2} K_{\pm\nu}(z^{1/2})$ and a solution of the equation in (1) is

$$A^{-\nu} z^{\nu/2} K_{\pm\nu}(z^{1/2}) = \left(\frac{x - x' e^{-iy}}{x - x' e^{iy}}\right)^{\nu/2} K_{-\nu}\left(|a|\sqrt{x^2 + x'^2 - 2xx'\cos y}\right).$$

It is clear that

$$P_\nu^a(y,x,x') = -\frac{\partial}{\partial y}\left[\left(\frac{x-x'e^{-iy}}{x-x'e^{iy}}\right)^{\nu/2} K_{-\nu}\left(|a|\sqrt{x^2+x'^2-2xx'\cos y}\right)\right],$$

satisfies equation (1). To prove Theorem 2.1 it remains to prove the limit conditions (1) and for this we need the following lemma.

LEMMA 2.1. *Set* $z = 4e^{X+X'}\{\sinh^2\frac{(X-X')}{2} + \sin^2(y/2)\}$ *and* $A = e^X - e^{X'+iy}$, *set also* $\sinh(X-X')/2 = s|\sin(y/2)|$, *that is*

$$X' = X + 2\arg\sinh(s|\sin y/2|),$$

then we have

$$z = 4e^{2X+2\arg\sinh(s|\sin(y/2)|)}\sin^2(y/2)(1+s^2)$$

and

$$A \sim -2s|\sin(y/2)|e^X, \quad z^{\nu/2}K_{-\nu}(az^{1/2}) \sim 2^{\nu-1}\Gamma(-\nu)|a|^\nu$$
$$\times\exp\left(2\nu X + 2\nu Arg\sinh(s|\sin(y/2)|)\right)\sin^{2\nu}y/2(s^2+1)^\nu,$$
$$A^{-\nu-1}z^{\nu/2}K_{-\nu}(az^{1/2}) \sim (-1)^{-\nu-1}2^{-2}\Gamma(-\nu)|a|^\nu e^{-(\nu+1)X}$$
$$\times\exp\left(2\nu X + 2\nu Arg\sinh(s|\sin(y/2)|)\right)\sin^{\nu-1}(y/2)(s^2+1)^\nu s^{-\nu-1},$$
$$-|a|e^{X+X'}\sin y A^{-\nu}z^{(\nu-1)/2}K_{-\nu+1}(az^{1/2}) \sim (-1)^{-\nu-1}2^{-1}\Gamma(1-\nu)|a|^\nu$$
$$\times\exp(2\nu X + 2\nu Arg\sinh(s|\sin(y/2)|)\cos(y/2)\sin^{\nu-1}(y/2)(s^2+1)^{\nu-1}s^{-\nu}.$$

P r o o f. The proof of this lemma uses essentially the formula

$$K_\nu(x) \sim \frac{2^{\nu-1}\Gamma(\nu)|}{x^\nu}$$

from Lebedev [7], p.136. □

Now we continue to prove Theorem 2.1, set

$$z = x^2 + x'^2 - 2xx'\cos y = 2xx'\left(\frac{x^2+x'^2}{2xx'} - \cos y\right),$$

$x = e^X$ and $x' = e^{X'}$, we obtain
$z = 4e^{X+X'}\{\sinh^2\frac{(X-X')}{2} + \sin^2(y/2)\}$, and we can write

$$P_a^\nu(y,X,X') = c_\nu\frac{\partial}{\partial y}\left\{\left(\frac{e^X - e^{X'-iy}}{e^X - e^{X'+iy}}\right)^{\nu/2}\right.$$
$$\left.\times K_{-\nu}\left(|a|\sqrt{\sinh^2\frac{(X-X')}{2} + \sin^2 y/2)}\right)\right\}.$$

From the formula in Magnus et al. [8], p. 67,

$$(z^\nu K_{-\nu+1}(z))' = -z^\nu K_{-\nu+1}(z),$$

we have

$$P_\nu^a(y,x,x') = c_\nu\{i\nu x' e^{i\nu y} A^{-\nu-1} z^{\nu/2} K_{-\nu}(|a|z^{1/2}) \\ -|a|xx' \sin y A^{-\nu} z^{(\nu-1)/2} K_{-\nu+1}(|a|z^{1/2})\}.$$

Using the formula (3) and setting $\sinh\frac{(X-X')}{2} = s\sin y/2$, we obtain

$$u(y,x) = \int_0^\infty P_\nu^a(y,x,x')u_0(x')\frac{dx'}{x'^{1+2\nu}} = I + J,$$

where

$$I = i\nu c_\nu \int_0^\infty x' e^{i\nu y} A^{-\nu-1} z^{\nu/2} K_{-\nu}(|a|z^{1/2}) u_0(x')\frac{dx'}{x'^{1+2\nu}},$$

and

$$J = -c_\nu|a| \int_0^\infty xx' \sin y A^{-\nu} z^{(\nu-1)/2} K_{-\nu+1}(|a|z^{1/2})\frac{dx'}{x'^{1+2\nu}}.$$

Now we use Lemmas 1.1 and 1.5 to obtain:

$$I = \frac{i\nu c_\nu(-1)^{-\nu-1}|a|^\nu}{4}\Gamma(-\nu)\Gamma(\frac{-1-\nu}{2})(1+(-1)^{-\nu-1}),$$

and

$$J = \frac{c_\nu(-1)^{-\nu-1}|a|^\nu}{4}\Gamma(-1-\nu)\Gamma(\frac{1+\nu}{2})(1+(-1)^{-\nu}),$$

and finally we get

$$\lim_{y\to 0} u(y,x) = \lim_{y\to 0}\tilde{u}(y,X) = \tilde{u}_0(X)c_\nu\frac{(-1)^{-\nu-1}|a|^{-\nu}}{4} \\ \times\left(i\nu\Gamma(-\nu)\Gamma(\frac{-1-\nu}{2})(1+(-1)^{-\nu-1}) \\ +\Gamma(-1-\nu)\Gamma(\frac{1+\nu}{2})(1+(-1)^{-\nu})\right) \\ = \tilde{u}_0(X) = u_0(x),$$

and this finishes the proof of Theorem 2.1. □

3. **Generalized Poisson equation with Morse Potential**

The main objective of this section is first to solve the following generalized Poisson problem

$$\begin{cases} M^a V(y,X) = \left(i\frac{\partial}{\partial y}+\nu\right)^2 V(y,X), (y,X) \in I\!R_+ \times I\!R \\ \lim_{y\longrightarrow 0} y^{-\nu}V(y,X) = V_0(X), v_0 \in C_0^\infty(I\!R^+) \end{cases} \tag{10}$$

associated to the second order differential operator of Morse type: $M^a = \frac{\partial^2}{\partial X^2} - a^2\exp 2X$. See Abdelhaye et al. [1], Ikeda et al. [5] and Morse [9].

THEOREM 3.1. *For a real number the problem* (10) *has the unique solution given by*

$$U(y,X)=\int_0^\infty P_a^\nu(y,X,X')U_0(X')dX' \tag{11}$$

with

$$P_a^\nu(y,X,X')=c_\nu\frac{\partial}{\partial y}\left(\frac{e^X-e^{X'-iy}}{e^X-e^{X'+iy}}\right)^{\nu/2}$$
$$\times K_\nu\left(|a|\sqrt{\sinh^2\frac{(X-X')}{2}+\sin^2 y/2}\right),$$

where $c_\nu=\frac{2^{2\nu-1}\nu i\Gamma(\nu)\Gamma(-\nu-3/2)|a|^\nu}{\Gamma(-\nu-1}$, *and* K_ν *are the modified Bessel function of the second kind.*

P r o o f. By changing the variable $x=e^X$ the problem (10)is transformed into the problem (1). □

4. Generalized Poisson equation on the hyperbolic space

In this section we consider the generalized Poisson equation on the hyperbolic space modelled by the upper half space:
$\mathbb{H}^n=\{x=(x_1,x_2,...x_n)\in\mathbb{R}^n,x_n>0\}$ endowed with the usual hyperbolic metric $\widetilde{ds}^2=x_n^{-2}[dx_1^2+dx_1^2+dx_2^2+...+dx^2]$.

The metric $\widetilde{ds}$ is invariant with respect to the group $G=SO(n,1)$.

The hyperbolic surface form $\widetilde{d\mu}(x)=\frac{1}{x_n^n}dx_1dx_2...dx_n$. And the hyperbolic distance $\rho(x,x')$ given respectively by

$$\cosh^2(\rho(x,x')/2)=\frac{(x_1-x_1')^2+...+(x_{n-1}-x_{n-1}')^2+(x_n+x_n')^2}{4x_nx_n'},$$

with the Laplace-Beltrami operator

$$\mathcal{L}_n=x_n^2\Delta_n+(2-n)\frac{\partial^2}{\partial x_n}+((n-1)/2)^2,$$

where $\Delta_n=\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}$, is the classical Laplacian on $\mathbb{R}^n$.

The main objective of this section is to solve the following generalized Poisson problem

$$\begin{cases}\mathcal{L}_nv(y,x)=\left(i\frac{\partial}{\partial y}+\nu\right)^2v(y,x),(y,x)\in I\!R_+^2\\ \lim_{y\longrightarrow 0}y^{-\nu}v(y,x)=v_0(x),v_0\in C_0^\infty(I\!R^+),\end{cases} \tag{12}$$

associated to the Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^n$.

COROLLARY 4.1. *If $a \in \mathbb{R}^*$, the problem* (12) *has the unique solution given by $v(y,x) = \int_0^\infty p_a(y,x,x',\eta,\eta')v_0(x')\frac{dx'}{x'}$, where*

$$p_n^\nu(y,x,x',\eta,\eta') = c_\nu \frac{\partial}{\partial y}\left(\frac{x-x'e^{-iy}}{x-x'e^{iy}}\right)^{\nu/2}$$

$$\times\left(x^2+x'^2-2xx'\cos y+|\eta-\eta'|^2\right)^{-(n-1)/2+\nu},$$

with $c_\nu = \frac{2^{(n-5)/2+2\nu}i\Gamma(\nu+1)\Gamma(-\nu-3/2)\Gamma((n-1)/2+\nu)}{\Gamma(-\nu-1)}$.

P r o o f. We use the following formula intertwining the Laplace Beltrami operator $\mathcal{L}_n$ on the hyperbolic space $\mathbb{H}^n$ and the Bessel operator $L^{|\xi|}$:

$$\mathcal{F}\left[x_n^{-(n-1)/2}\mathcal{L}_n x_n^{(n-1)/2}\phi\right](\xi) = L^{|\xi|}\mathcal{F}\phi(\xi).$$

The Poisson problem on the hyperbolic space (12) is transformed into the Bessel Poisson problem (1), with $v(y,x_n) = \mathcal{F}\left[x^{(1-n)/2}v(y,x,x_n)\right](\xi)$ and $v_0(x_n) = x_n^{(1-n)/2}\mathcal{F}[v_0(x,x_n)](\xi)$, that is

$$\mathcal{F}\left[x^{(1-n)/2}v(y,x,x_n)\right](\xi)$$

$$= \int_0^\infty P_{|\xi|}^\nu(y,x_n,x'_n)x_n'^{(1-n)/2}x^{(1-n)/2}\mathcal{F}\left[v_0\right](\xi,x'_n)\frac{dx'_n}{x'_n},$$

and

$$v(y,x,x_n) = \int_0^\infty \mathcal{F}^{-1}\left[P_{|\xi|}^\nu(y,x_n,x'_n)x_n'^{(1-n)/2}x^{(1-n)/2}\mathcal{F}\left[v_0\right](\xi,x'_n)\right](x)\frac{dx'_n}{x'_n},$$

$$v(y,x,x_n) = (2\pi)^{-(n-1)/2}\int_0^\infty \mathcal{F}^{-1}\left[P_{|\xi|}^\nu(y,x_n,x'_n)\right](x) * v_0(x,x'_n)$$

$$\times x_n'^{(1-n)/2}x^{(1-n)/2}\frac{dx'_n}{x'_n}.$$

Thus we can write

$$v(y,x,x_n) = (2\pi)^{-(n-1)/2}\int_0^\infty\int_{\mathbb{R}^{n-1}} \mathcal{F}^{-1}\left[P_{|\xi|}^\nu(y,x_n,x'_n)\right](x-x')$$

$$\times v_0(x',x'_n)x_n^{(n-1)/2}x_n'^{(n-1)/2}\frac{dx'dx'_n}{x_n'^n},$$

and $v(y,x,x_n) = \int_{\mathbb{H}^n} P_n^\nu\left(y,x,x',\eta,\eta'\right)v_0(w')d\mu(w')$. We compute the Fourier transforms $\mathcal{F}^{-1}\left[P_{|\xi|}^\nu(y,x,x',\eta,\eta')\right](x-x')$ by using Lemma 1.3 and Lemma 2.1 and we obtain the result. $\square$

Corollary 4.2. *If $a \in I\!R^*$ and $\nu = 0$ the Poisson equation in hyperbolic space has the unique solution given by*

$$v(y,x) = \int_0^\infty p_n(y,x,x')v_0(x')\frac{dx'}{x'},$$

where

$$p_n(y,x,x') = \frac{\Gamma(\frac{n+1}{2})}{(2\pi)^{\frac{n+1}{2}}}\frac{\sin y}{(\cosh\rho - \cos y)^{\frac{n+1}{2}}}.$$

References

[1] Abdelhaye Y., Badahi M., Ould Moustapha M.V., Wave kernel for the Schrödinger operator with the Morse potential and applications, *F. J. Math. Sci.*, **102** (2017), 1523-1532.

[2] Betancor Jorge J. and M. De Leon-Contreras, Parabolic equations involving Bessel operators and singular integrals. *Integr. Equ. Oper. Theory* (2018).

[3] Betancor Jorge J., Oscar Ciaurri, Teresa Martınez, Mario Perez, Jose L. Torrea, Juan L. Varona, Heat and Poisson semigroups for Fourier-Neumann expansions, *J. Semi. Forum.*, **73** (2006).

[4] Zakria A., Ahmed, I.-E. and Moustapha M.V.O., Poisson and heat semigroups for the Bessel operator and on the hyperbolic space. *International Journal of Applied Mathematics*, **33**, No 2 (2020), 237-252; doi:10.12732/ijam.v33i2.4.

[5] Ikeda N., Matsumoto H., Brownian motion one the hyperbolic plane and Selberg trace formula, *J. Funct. Anal.*, **163** (1999).

[6] Cardoso I., On the pointwise convergence to initial data of heat and Poisson problems for the Bessel operator, *J. Evol. Equ.* **17** (2017).

[7] Lebedev, N., *Special Functions and Their Applications*, Dover Publications Inc., New York (1972).

[8] Magnus, F., Oberhettinger and R.P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics*, 3rd enlarged ed., Springer-Verlag, Berlin-Heidelberg-New York (1966).

[9] Morse P.M., Diatomic molecules according to the wave mechanics. II. Vibrational levels, *Phys. Rev.*, **34** (1929), 57-64.

[10] Prudnikov A.P., Brychkov Yu. and O.I. Marichev, *Special Functions: Integrals and Series*, Vol. 2, Gordon and Beach Science Publ., New York-London-Paris (1986).

[11] Taylor M.E., *Partial Differential Equations I*, Springer, New York-Berlin-Heidelberg (1996).