International Journal of Applied Mathematics

Volume 37 No. 4 2024, 455–460
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v37i4.5

ANALYTIC SOLUTION OF A NONLINEAR BLACK–SCHOLES EQUATION VIA LONG AND SHORT GAMMA POSITIONS

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Abstract

We study a nonlinear Black–Scholes equation whose nonlinearity is due to feedback effects. The market involved here is illiquid as a result of transaction costs. An analytic solution to the equation via long and short gamma positions is currently unknown. After transforming the equation into a parabolic nonlinear porous medium-type equation, we find that the assumption of a traveling wave profile to the later equation reduces it to Ordinary Differential Equations (ODEs). This together with the use of long and short gamma positions facilitate a twice continuously differentiable solution. The solution can be used to price a call option. Both positive and negative gamma exposures can lead to the value of a short call.

Math. Subject Classification: 35C05, 35K55, 91G80

Key Words and Phrases: nonlinear Black–Scholes equation, illiquid markets, transaction costs, gamma positions, analytic solution

Received: December 13, 2023

1. Introduction

In formulating classical arbitrage pricing theory, two primary assumptions used are that markets are *frictionless* and *competitive*. A frictionless market has no transaction costs and restrictions on trade. A competitive market allows a trader to buy or sell any quantity of a security without changing its price. Relaxing the two assumptions leads to liquidity risk.

Although analytic solutions of the nonlinear Black–Scholes Partial Differential Equation have been obtained (see [3, 4, 5]), the solution via long and short gamma positions is currently unknown. Long (short) call prices have negative (positive) gamma. Positive and negative gamma exposure can lead to the value of a short call.

The purpose of this paper is to obtain an analytic solution of the nonlinear Black–Scholes equation arising from transaction costs via long and short gamma positions. This is done by transforming the equation into a nonlinear porous medium-type equation and then assuming a solitary wave solution.

This paper is outlined as follows. Section 2 describes the modified option pricing theory. The solution to the nonlinear Black–Scholes equation is presented in Section 3. Section 4 concludes the paper.

2. Modified Option Valuation Model

Nonlinearities in diffusion models can arise from source terms, insect dispersal, heat conduction and illiquid market effects among others.

In this work, we will consider the continuous-time (quadratic) transaction-cost model for modelling illiquid markets. Two assets are used in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time t is $B_t \equiv 1$, and a stock (*i.e.* a risky and illiquid asset). The bond's market is assumed to be liquid (or perfectly elastic) [2].

Cetin *et al.* [2] has put forward the predominant model in the transaction-cost model where a fundamental stock price process s_t^0 follows the dynamics

$$ds_t^0 = \mu s_t^0 dt + \sigma s_t^0 dW_t, \qquad t \in [0, T],$$

where μ is drift, σ is volatility, and W_t is the Wiener process. Use of a Markovian trading strategy $\Phi_t = \phi(t, s_t^0)$ for a smooth function $\phi = u_s$ resulted into the equation

$$u_t + \tfrac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0, \qquad u(s_T^0, T) = h(s_T^0), \tag{1}$$

where $\rho > 0$ is a measure of the liquidity of the market and $h(s_T^0)$ is a terminal claim whose hedge cost $u(s_t^0, t)$ is the solution to (1).

3. **Solution to a Nonlinear Black–Scholes Equation**

To solve (1), differentiate it twice with respect to the spatial variable s. Then, set $u_{ss} = w$ to get

$$w_t + \frac{\sigma^2 s^2}{2}(1+4\rho s w)w_{ss} + 2\rho\sigma^2 s^3 w_s^2 + 2\sigma^2 s(1+6\rho s w)w_s + \sigma^2(1+6\rho s w)w = 0.$$

Apply the transformations $w = \frac{\upsilon}{\rho s}$ and $x = \ln s$ to the reaction-advection-diffusion-type equation above to get

$$\upsilon_t + \frac{\sigma^2}{2}(1+4\upsilon)\upsilon_{xx} + 2\sigma^2\upsilon_x^2 + \frac{\sigma^2}{2}(1+4\upsilon)\upsilon_x = 0. \tag{2}$$

Letting $\upsilon = \frac{V-1}{4}$ gives $\upsilon_t = \frac{V_t}{4}$, $\upsilon_x = \frac{V_x}{4}$, and $\upsilon_{xx} = \frac{V_{xx}}{4}$. Substitute these expressions into (2) and simplify to get

$$V_t + \frac{\sigma^2}{2}(VV_{xx} + V_x^2 + VV_x) = 0 \quad \text{in} \quad \mathbb{R}\times(0,\infty). \tag{3}$$

We now look for a twice continuously differentiable solution of (3) on $\mathbb{R}$.

PROPOSITION 3.1. *If $\nu(\xi)$ is a twice continuously differentiable function, and x and t are the spatial and time variables, respectively, there exists a traveling wave solution to (3) of the form*

$$V(x,t) = \nu(x-ct) = \nu(\xi) \quad \text{where} \quad \xi = x - ct \tag{4}$$

for all $(x,t) \in \mathbb{R}\times(0,\infty)$ such that $V(x,t)$ is a traveling wave of permanent form which translates to the right with constant speed $c > 0$.

P r o o f. Apply the chain rule to (4) to get

$$V_t = -c\nu'(\xi), \quad V_x = \nu'(\xi), \quad \text{and} \quad V_{xx} = \nu''(\xi).$$

The prime denotes $\frac{d}{d\xi}$. Substituting these expressions into (3) leads to the conclusion that $\nu(\xi)$ must satisfy the nonlinear ODE

$$-c\nu' + \frac{\sigma^2}{2}\left(\nu\nu'' + (\nu')^2 + \nu\nu'\right) = 0. \tag{5}$$

Hence $V = V(x,t)$ solves (3).

We now assume that the traveling wave solution is *localized*. This means that

$$\lim_{x\to\pm\infty} V(x,t) = \lim_{x\to\pm\infty} V_x(x,t) = \lim_{x\to\pm\infty} V_{xx}(x,t) = 0.$$

The function V together with the form (4) in this case is referred to as a *solitary* wave (see [6]). We now impose the localizing boundary conditions

$$\lim_{\xi\to\pm\infty} \nu(\xi) = \lim_{\xi\to\pm\infty} \nu'(\xi) = \lim_{\xi\to\pm\infty} \nu''(\xi) = 0. \tag{6}$$

To solve (5) in a closed-form, we first write it as

$$\frac{d}{d\xi}\left(\frac{\sigma^2}{2}\nu\nu' + \frac{\sigma^2}{4}\nu^2 - c\nu\right) = 0. \tag{7}$$

Integrate (7) and rearrange to get the standard form (see [6])

$$\nu' = \tfrac{2}{\sigma^2\nu}(c\nu - \tfrac{\sigma^2}{4}\nu^2 + \kappa), \tag{8}$$

where κ is a constant of integration. From (6), $\kappa = 0$. Simplifying (8) further gives

$$-2\tfrac{d\nu}{d\xi} = \nu - \tfrac{4c}{\sigma^2}.$$

Rearranging this equation and integrating gives

$$\nu(\xi) = e^{\frac{\xi_0-\xi}{2}} + \tfrac{4c}{\sigma^2},$$

where ξ_0 is another constant of integration. Hence, the solution to (3) becomes

$$V(x,t) = e^{\frac{x_0-(x-ct)}{2}} + \tfrac{4c}{\sigma^2}.$$

Substitute $\upsilon = \frac{V-1}{4}$. This gives the solution to (2) as

$$\upsilon(x,t) = \tfrac{1}{4}e^{\frac{x_0-(x-ct)}{2}} + \tfrac{c}{\sigma^2} - \tfrac{1}{4}. \tag{9}$$

□

THEOREM 3.1. *If $V(x,t)$ is any positive solution to the nonlinear advection-diffusion-type equation $V_t + \frac{\sigma^2}{2}\left(VV_{xx} + V_x^2 + VV_x\right) = 0$ in $\mathbb{R}\times(0,\infty)$, then*

$$u(s,t) = \begin{cases} -\frac{1}{\rho}\left(\sqrt{s}e^{\frac{s_0+\sigma^2t/4}{2}} + \frac{1}{4}e^{s_0+\sigma^2t/4}\right) < 0 & \text{for} \quad u_{ss} > 0 \\ \frac{1}{\rho}\left(\sqrt{s}e^{\frac{s_0+\sigma^2t/4}{2}} - \frac{1}{4}e^{s_0+\sigma^2t/4}\right) > 0 & \text{for} \quad u_{ss} < 0 \end{cases} \tag{10}$$

solves the nonlinear Black–Scholes equation

$$u_t + \tfrac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0$$

for each $t, \sigma > 0$, $s > \frac{1}{16}e^{s_0+\sigma^2t/4}$, $s_0 \in \mathbb{R}$, and $\rho > 0$.

P r o o f. Substituting $w = \frac{\upsilon}{\rho s}$ and $x = \ln s$ into (9) yields

$$u_{ss} = \tfrac{1}{\rho}\left(\tfrac{1}{4s^{3/2}}e^{\frac{s_0+ct}{2}} + \tfrac{1}{s}\left(\tfrac{c}{\sigma^2} - \tfrac{1}{4}\right)\right). \tag{11}$$

We can rewrite (11) as

$$\rho s u_{ss} = \tfrac{1}{4\sqrt{s}}e^{\frac{s_0+ct}{2}} + \tfrac{c}{\sigma^2} - \tfrac{1}{4}. \tag{12}$$

Apply localizing boundary conditions to (12) to get

$$0 = 0 + \left(\tfrac{c}{\sigma^2} - \tfrac{1}{4}\right).$$

Hence $c = \frac{\sigma^2}{4}$. Plugging $c = \frac{\sigma^2}{4}$ into (11) gives

$$u_{ss} = \begin{cases} \frac{1}{4\rho s^{3/2}} e^{\frac{s_0+\sigma^2 t/4}{2}} > 0 & \text{for} \quad u_{ss} > 0 \\ -\frac{1}{4\rho s^{3/2}} e^{\frac{s_0+\sigma^2 t/4}{2}} < 0 & \text{for} \quad u_{ss} < 0. \end{cases} \tag{13}$$

Integrating (13) twice with respect to s completes the prove. □

Remark 3.1. (**Gamma Positions**) The value of a short call can be obtained when we use

(1) a positive gamma exposure, $u_{ss} > 0$ (see (10)),
(2) a negative gamma exposure, $u_{ss} < 0$, particularly when $s < \frac{1}{16} e^{s_0+\sigma^2 t/4}$.

4. Conclusion

We have studied the hedging of derivatives in illiquid markets. A model where the implementation of a hedging strategy affects the price process of the underlying asset has been considered. Assuming the solution of a forward wave, a classical solution was found for the nonlinear Black–Scholes equation by use of long and short gamma positions. The solution obtained can be used for pricing a European *call* option at time $t > 0$. Negative call option prices serve to show that market frictions can have first-order effects on the prices of securities [7]. Both positive and negative gamma exposures can lead to the value of a short call.

In conclusion, future work will involve finding out how long (short) put price behave in relation to positive (negative) gamma.

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