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LAPLACE TRANSFORM SOLUTIONS OF CAUCHY PROBLEM FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION

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Abstract

In this paper, using Laplace transform we prove the well-posedness of the Cauchy problem for the strongly nonlinear modified Korteweg-de Vries equation on semi-axis in both the focusing and the defocusing case. We applied the outcoming results of this work to another familiar form of the Korteweg–de Vries, that played a significant role in the development of the soliton theory.

Math. Subject Classification: 37K40, 35Q35, 35Q35

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1. Introduction

Over the past several years, there has been a significant number of researches on the topics of partial and ordinary differential equations due to their effectiveness in a variety of pure and applied mathematics domains. Differential equations have unlimited standards and can explain physical models of many phenomena in a wide variety of areas. We are well recognized that finding the exact solution to such an equation is quite difficult, and that the

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exact solution's form is frequently too complicated to be effectively used to numerical calculations. The exact solution of initial value issues for partial and ordinary differential equations can be investigated using the Laplace transform method, Fourier transform method, Green's function approach, and other approximation methods which are all valuable and essential technique (see [1], [2], [3], [4], [5], [6]).

The aim of this work is to develop Laplace transform solutions for the modified Korteweg-de Vries equation on the half-line and applied the results to another type of Korteweg-de Vries equation. This equation is used to represent a wide variety of astrophysical and physical phenomena, including solitary waves and solitons, which propagate with the same shape and constant velocity and remain stable even after mutual collision. Other examples of these phenomena include acoustic waves in enharmonic crystals, slightly interacting waves in shallow water, long internal waves in the ocean, and ion-acoustic waves in plasma.

The Korteweg-de Vries equation can be expressed in different types, for example the following equation

$$v_t + \lambda_1 v_{xx} + 2uu_x = 0, \qquad (1)$$

introduced by Boussinesq (see [9]) for the first time in 1877, and then rediscovered by Korteweg and de Vries (see [18]) in 1895.

In which the equation

$$v_t + \lambda_2 v_{xxxxx} + 2uu_x = 0, \qquad (2)$$

is now as the fifth-order KdV equation or Kawahara equation.

For the Korteweg-de Vries equation, Bubnov studied the initial-boundary value problem in 1979 (see [10]). Several initial boundary value problems of the Korteweg-de Vries equation have been the subject of in-depth research since Bubnov's work (see [7], [11], [13], [14], [19]).

Bona, Sun, Zhang (see [7]) and Colliander, Kenig (see [11]) respectively provided two new, slightly similar approaches to analyse the solvability of the non-homogeneous IBVP of the Korteweg-de Vries equation posed on half line. Kenig, Ponce, Vegas, and Bourgain (see [8], [17]) investigated the existence and uniqueness solution for Cauchy problem of the Korteweg-de Vries equation

$$v_t + vv_x + v_{xx} = 0, u(x,0) = \varphi(x),\, t, x \in R, \qquad (3)$$

on the whole line.

Further, Bona, Sun, Zhang and Homer studied the solvability of the Korteweg-de Vries equation (3) on a finite interval $0 < x < 1$, $t > 0$. We are aware that two distinct approaches to the analysis of IBVPs of dispersive equations have been developed, one by Faminskii (see [13]) and the other by Fokas, Himonas, and Mantzavinos (see [14]).

The paper is organised as follows: in Sec. 2, we prove the well-posedness of the strongly nonlinear modified third-order KdV equation using Laplace transforms and hence using the Green's function to obtain the particular solution. Then in Sec. 3, we provide an example of the most familiar model that used to create an infinite number of Korteweg-de Vries equations. Finally, in conclusion section we list out the obtained solutions as results of using Laplace transform method.

Now, we would rather present the strongly nonlinear modified Korteweg-de Vries equation by Laplace transform solutions as a new approach to prove the well-posedness of Cauchy problem problem for the modified Korteweg-de Vries equation.

2. Laplace transform solutions of the modified Korteweg-de Vries (mKdV) equation

If there are real constants A, B such that the real valued function $f : (0, \infty) \to \mathbb{R}$, $|f(t)| \leq Ae^{Bt}$, $\forall t > 0$, then $f(t)$ is said to be of exponential order and it has its Laplace transform $\mathcal{L}(f) = F(s)$,

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

In which there exists a real number δ such that this integral converges if $\Re(s) > \delta$ and diverges if $\Re(s) < \delta$, where $\Re(s)$ is the real part of s. Moreover, $|F(s)| \to 0$ as $\Re(s) \to \infty$.

Let us investigate how the Laplace transform represents solutions to the strongly nonlinear modified Korteweg-de Vries equation

$$v_t(x,t) + a\varepsilon v^2(x,t)v_x(x,t) + v_{xxx}(x,t) = 0, \quad \varepsilon = \pm 1 \tag{4}$$

$$v(x,0) = \varphi(x), t \in [0,\infty), x \in R, \tag{5}$$

where ε indicates whether the equation is focusing or defocusing, $v(x,t)$ is the real-valued function denotes the average velocity. In which waves steepen due to the nonlinear term, while decay due to the third order term.

We provide the following theorem to prove the well-posedness of the Cauchy problem for the strongly nonlinear modified Korteweg-de Vries equation.

Theorem 2.1. *The solution $v(x,t)$ of the Cauchy problem* (4)-(5) *is*

$$V(x,s) = A(s)e^{\lambda_1 x} + B(s)e^{\lambda_2 x} + C(s)e^{\lambda_3 x} + V_p(x), \tag{6}$$

where $V_p(x)$ denotes the particular solution of problem (4)-(5) *and*

$$V_p(x) = \begin{cases} \int_x^\infty (G(x,\xi)f(\xi)d\xi, & \xi < x \\ \int_{-\infty}^x (G(x,\xi)f(\xi)d\xi, & \xi > x, \end{cases} \tag{7}$$

$$G(x,\xi) = \frac{\Delta}{\Delta_1}, \tag{8}$$

where

$$\Delta = (\lambda_1 - \lambda_3)e^{(\lambda_1+\lambda_3)\xi+\lambda_2 x} + (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)\xi+\lambda_3 x} + (\lambda_3 - \lambda_2)e^{(\lambda_3+\lambda_2)\xi+\lambda_1 x},$$

$$\Delta_1 = \lambda_1(\lambda_3 - \lambda_2)e^{(\lambda_2+\lambda_3)\xi+\lambda_1 x} + \lambda_2(\lambda_1 - \lambda_3)e^{(\lambda_1+\lambda_3)\xi+\lambda_2 x} + \lambda_3(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)\xi+\lambda_3 x},$$

λ_1, λ_2 *and* λ_3 *denote the roots of the equation*

$$\lambda^3 + a\varepsilon v^2\lambda + s = 0.$$

P r o o f. By Applying the Laplace transform to equation (4) with respect to t, yields

$$sV(x,s) - v(x,0) + a\varepsilon v^2(x,t)V_x(x,s) + V_{xxx}(x,s) = 0.$$

Reorganizing the equation, we get

$$V_{xxx}(x,s) + a\varepsilon v^2(x,t)V_x(x,s) + sV(x,s) = \varphi(x), \tag{9}$$

where $V(x,s) = \mathcal{L}[v(x,t)]$.

From (9), the cubic auxiliary equation takes the form

$$\lambda^3 + a\varepsilon v^2\lambda + s = 0, \tag{10}$$

with roots λ_1, λ_2 and λ_3.

The general solution of equation (9) can be formulated in the following form:

$$V(x,s) = A(s)e^{\lambda_1 x} + B(s)e^{\lambda_2 x} + C(s)e^{\lambda_3 x} + V_p(x), \tag{11}$$

because of equation (9) has only a first and third partial derivatives with respect to x.

In which

$$V_p(x) = \int_{-\infty}^{x} G(x,\xi)f(\xi)d\xi + \int_{x}^{\infty} G(x,\xi)f(\xi)d\xi,$$

is a particular solution of the problem (4)-(5) with the Green's function $G(x,\xi) = \frac{\Delta}{\Delta_1}$, where

$$\Delta = \begin{vmatrix} e^{\lambda_1\xi} & e^{\lambda_2\xi} & e^{\lambda_3\xi} \\ \lambda_1 e^{\lambda_1\xi} & \lambda_2 e^{\lambda_2\xi} & \lambda_3 e^{\lambda_3\xi} \\ e^{\lambda_1 x} & e^{\lambda_2 x} & e^{\lambda_3 x} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} e^{\lambda_1\xi} & e^{\lambda_2\xi} & e^{\lambda_3\xi} \\ \lambda_1 e^{\lambda_1\xi} & \lambda_2 e^{\lambda_2\xi} & \lambda_3 e^{\lambda_3\xi} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \lambda_3 e^{\lambda_3 x} \end{vmatrix}$$

and λ_1, λ_2 and λ_3 are roots of the equation (10). The theorem is proved. □

3. Example

It has been considered that the KdV equation describes the a variety of physical problems such as plasma waves, magnetohydrodynamic waves, and waves with extended wave lengths.

In this example, we investigate the Cauchy problem for the important strongly nonlinear third-order KdV equation:

$$v_t + 6\varepsilon v^2 v_x + v_{xxxx} = 0, \;\; \varepsilon = \pm 1, \tag{12}$$

$$v(x,0) = v_0(x), \;\; t \in [0,\infty), \;\; x \in R. \tag{13}$$

Problem (12)-(13) provides the most familiar model that used to create an infinite number of Korteweg-de Vries (KdV) equation conservation laws (see [21]), leading to the identification of the KdV equation's Lax pair and the development of the Inverse Scattering Transform (IST) (see [16]). Another notable feature of the modified Korteweg–de Vries equation (mKdV) equation is its peculiar soliton behavior or breathers that played a significant role in the development of the soliton theory.

Problem (12)-(13) has been investigated by some other methods via review solutions to the mKdV equation in terms of Wronskians (see [22]), the method of commuting flows (see [15]), separation of variables method (see [20]) and the inverse scattering transform method (see [12]).

Now, we apply Laplace transform approach to solve the Cauchy problem (12)-(13) as follows

$$V_{xxx}(x,s) + 6\varepsilon v^2(x,t) V_x(x,s) + sV(x,s) = \varphi(x). \tag{14}$$

The cubic polynomial auxiliary equation of equation (14) takes the form

$$\lambda^3 + 6\varepsilon v^2 \lambda + s = 0, \tag{15}$$

with the roots λ_1, λ_2 and λ_3 corresponding to its discriminant $32\varepsilon^3 v^6 + s^2$.

If $32\varepsilon^3 v^6 + s^2 > 0$, then the unique real solution of equation (15) is obtained from:

$$\lambda_1 = \sqrt[3]{-\frac{s}{2} + \sqrt{\frac{s^2}{4} + 8\varepsilon^3 v^6}} + \sqrt[3]{-\frac{s}{2} - \sqrt{\frac{s^2}{4} + 8\varepsilon^3 v^6}}.$$

If $32\varepsilon^3 v^6 + s^2 < 0$, the three real solutions of (15) are given by:

$$\lambda_1 = 2\sqrt{-2\varepsilon v^2} \cos\left[\frac{Arc\cos\left(\frac{s}{4\varepsilon v^2}\sqrt{-\frac{1}{2\varepsilon v^2}}\right) + 2\pi}{3}\right],$$

$$\lambda_2 = 2\sqrt{-2\varepsilon v^2} \cos\left[\frac{Arc\cos\left(\frac{s}{4\varepsilon v^2}\sqrt{-\frac{1}{2\varepsilon v^2}}\right) + 4\pi}{3}\right],$$

$$\lambda_3 = 2\sqrt{-2\varepsilon v^2}\cos\left[\frac{Arc\cos\left(\frac{s}{4\varepsilon v^2}\sqrt{-\frac{1}{2\varepsilon v^2}}\right)}{3}\right].$$

If $32\varepsilon^3 v^6 + s^2$, the equation (15) has three real roots with two the same. Hence

$$V(x,s) = A(s)e^{\lambda_1 x} + B(s)e^{\lambda_2 x} + C(s)e^{\lambda_3 x} + \int_{-\infty}^{+\infty} G(x,\xi)f(\xi)d\xi,$$

$G(x,\xi)$ is calculated in Theorem 2.1.

4. Conclusion

The well-posedness of the Cauchy problem of the strongly nonlinear mKdV equation proved using the Laplace transform method. It is possible to show that the Laplace transform approach is an extremely efficient and successful approach for obtaining the exact solutions for many problems of differential equations. We provided an example to indicate how the Laplace transform represents the solutions to well-known class of KdV equations.

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