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A GENERALIZED q-GRÜSS INEQUALITY INVOLVING THE GENERALIZED KOBER TYPE FRACTIONAL q-INTEGRAL OPERATOR

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Abstract

In this paper we consider the generalized fractional q-integral operator of Kober type, which contains the basic analogue of the Fox-Wright hypergeometric function to derive a new fractional q-integral inequality of Grüss type, for synchronous functions and absolutely continuous q-functions. The results obtained by Kalla and Rao, Secer et al., Zhu et al., Dahmani and Benzidane, Belarbi and Dahmani are particular cases of our results.

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1. **Introduction**

In 2020 Castillo and Galué [3] presented the generalized fractional q-integral operator of the Kober type, which contains the basic analogue of the Fox-Wright hypergeometric function, in the following form

$$I_q^n\left[\begin{array}{l}\rho,\ (M,1)\,,\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]f(x)=\frac{x^{-\rho-1}}{\Gamma_q(M+1)}$$
$$\times\int_0^x t^\rho\,{}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1)\,,...,(\alpha_{r-1},A_{r-1})\,,(-M,1)\\ (\beta_1,B_1)\,,...,(\beta_s,B_s)\end{array}\middle|\,q,q^n\frac{t}{x}\right]f(t)\,d_qt, \quad (1)$$

$$A_i,B_j\in\mathbb{R}^+,Re\,(\alpha_i)>0,\ Re\,(\beta_j)>0;i=1,...,r-1,j=1,...,s,$$

$$\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,M\in\mathbb{N}_0,n\in\mathbb{N},\rho\in\mathbb{C},\ 0<q<1,\ \left|\frac{t}{x}\right|<1,$$

where the function ${}_r\psi_s^*(\cdot)$ appearing as a kernel for the operator (1) is the basic analogue of the Fox-Wright hypergeometric function defined by

$${}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1)\,,...,(\alpha_{r-1},A_{r-1})\,,(-M,1)\\ (\beta_1,B_1)\,,...,(\beta_s,B_s)\end{array}\middle|\,q,z\right]$$
$$=\sum_{k=0}^{M}\frac{\left(q^{-M};q\right)_k}{(q;q)_k}\frac{\prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_ik}}{\prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_jk}}\left[(1-q)^k\,q^{\frac{k(k-1)}{2}}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i}z^k, \quad (2)$$

$$A_i,B_j\in\mathbb{R}^+,M\in\mathbb{N}_0,Re\,(\alpha_i)>0,\ Re\,(\beta_j)>0;i=1,...,r-1,$$

$$j=1,...,s,\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,0<q<1,\ |z|<1.$$

In (1), when $r=s+1$, each $A_i=B_i=1$, and taking $\alpha_i=\beta_i+m_i$, $m_i\in\mathbb{N}_0$; $i=1,...,s$ reduces to

$$L_q^n\{M,\beta_1,\beta_2,...,\beta_s,\rho,m_1,m_2,...,m_s;f(x)\}, \quad (3)$$

$$M,m_i\in\mathbb{N}_0,\ \mathrm{Re}\,(\beta_i)\neq 0,-1,-2,...,;i=1,...,s,n\in\mathbb{N},\rho\in\mathbb{C},$$
$$0<q<1,\ \left|\frac{t}{x}\right|<1.$$

Here (3) is the fractional q-integral operator established by Galué in [11].

A large number of researchers have used fractional integral operators and fractional q-integral operators to establish new q-integral inequalities, including the following: In 2004 Gauchman [14] investigated the basic analogues of some classical integral inequalities and used them to estimate the sum of some convergent series. Öğünmez and Özkan in 2011, [20], established some fractional q-integral inequalities on the specific time scales $T_{t_0} = \{t : t = t_0 q^n, n \in \mathbb{N}_0\} \cup \{0\}$, where $t_0 \in \mathbb{R}$, and $0 < q < 1$, which generalize those established by Belarbi and Dahmani in 2010 [2]. In 2012 Sulaiman [23] established some fractional q-integral inequalities for the product of two and three functions using the Riemann-Liouville fractional q-integral operator, which generalize the results established in [2] and [20]. In 2012 Dahmani and Benzidane [6] also considered the set T_{t_0} to generate some fractional q-integral inequalities of Grüss type. Zhu et al. [24] recently used a fractional q-integral on the specific time scales T_{t_0} to establish some Grüss-type fractional q-integral inequalities, for one or two fractional parameters. Later, Galué in 2014 [12] used the generalized Erdélyi-Kober fractional q-integral operator [10] to derive some fractional q-integral inequalities, for the product of three functions, which generalize the previous results. Recently Choi et al. [4] used the generalized Erdélyi-Kober fractional q-integral operator established in [10] with two deformation parameters and established fractional q-integral inequalities for the product of four and five functions, which generalize the results established by Galué in [12], Öğünmez and Özkan in [20], Belarbi and Dahmani in [2].

On the other hand, fractional integral inequalities have many applications, they are very useful to establish the uniqueness of the solutions of fractional boundary value problems, and in partial fractional differential equations. They also provide upper and lower bounds to the solutions of the aforementioned equations, [21]. Also, there is a diversity of applications in mathematics, statistics and physics, [22]. Particularly in q-analysis, various applications have been established in partition theories, combinatorics, exactly solvable models in statistical mechanics, computational algebra, theory of geometric functions, optimal control problems, q-differential and q-integral equations [1], [13], [16], [19].

In this work a new Gruss-type inequality is established and some special cases are derived.

1.1. **Notions of fractional integral inequalities.** In this section we consider some definitions, the Cauchy-Schwarz inequality for q-double integrals, the Grüss inequality, to derive our results. Also some recent results, related to fractional integral and fractional q-integral inequalities are presented, which are useful to emphasize our results.

Definition 1.1. Two functions f and g are synchronous in $[a,b]$, if $(f(x)-f(y))(g(x)-g(y))\geq 0$, for all $x,y\in[a,b]$, [2].

Definition 1.2. If $q\in\mathbb{R}$ is fixed, a subset A of $\mathbb{C}$ is called q-geometric if $qz\in A$ whenever$z\in A$, [1].

Definition 1.3. A function f which is defined on a q-geometric set A, $0\in A$, is said to be q-regular at zero if $\lim\limits_{n\to\infty} f(zq^n)=f(0)$ for all $z\in A$, [1].

Definition 1.4. A function f defined on $[0,a]$ is called q-absolutely continuous, that is $f\in AC_q[0,a]$, if f is q-regular at zero, and there exists $K>0$ such that $\sum\limits_{j=0}^{\infty}\left|f\left(tq^j\right)-f\left(tq^{j+1}\right)\right|\leq K$ for all $t\in[qa,a]$, [1].

1.1.1. **Cauchy–Schwarz inequality for double q-integrals.** Zhu et al. [24] considered the set $T_{t_0}=\{t:t=t_0q^n,\ n\in\mathbb{N}_0\}\cup\{0\}$, where $t_0\in\mathbb{R}$, and $0<q<1$, and established the basic analogue of the Cauchy-Schwarz inequality for double integrals, as follows:

Let $f(x,y)$, $g(x,y)$ and $h(x,y)$ be three functions defined on $\mathbf{T}_{t_0}^2$ with $h(x,y)\geq 0$, then

$$\left(\int\limits_0^t\int\limits_0^t h(x,y)\,f(x,y)\,g(x,y)\,d_qxd_qy\right)^2\leq\int\limits_0^t\int\limits_0^t h(x,y)\,f^2(x,y)\,d_qxd_qy \times\int\limits_0^t\int\limits_0^t h(x,y)\,g^2(x,y)\,d_qxd_qy. \tag{4}$$

1.1.2. **Grüss inequality:** In 1935 Grüss established the inequality that provides an estimate of the difference between the integral of the product of two functions and the product of their integrals, as follows

$$\left|\frac{1}{(b-a)}\int_a^b f(x)\,g(x)\,dx-\frac{1}{(b-a)}\int_a^b f(x)\,dx\cdot\frac{1}{(b-a)}\int_a^b g(x)\,dx\right| \leq\frac{1}{4}(M-m)(P-p), \tag{5}$$

where f and g are two synchronous integrable functions on $[a,b]$ satisfying the conditions

$$m\leq f(x)\leq M,\ \ p\leq g(x)\leq P;\ \ m,M,p,P\in\mathbb{R},x\in[a,b]. \tag{6}$$

Many researchers have established new generalizations of the Grüss inequality [5], [6], [7, 8, 9], [14], [18], [24]. In 2010, Dahmani et al. [5] generalized the inequality (5) using the Riemann-Liouville fractional integral as follows:

Let f and g be bounded integrable functions defined on $[0,\infty)$, such that $m \leq f(t) \leq M,\ p \leq g(t) \leq P;\ m, M, p, P \in \mathbb{R}, t \in [0,\infty)$, then for $\alpha > 0$

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha} f(t)\, g(t) - J^{\alpha} f(t)\, J^{\alpha} g(t)\right| \leq \left(\frac{t^{\alpha}}{2\Gamma(\alpha+1)}\right)^{2} (M-m)(P-p), \tag{7}$$

where $J^{\alpha} f(t)$ is the fractional integral operator of Riemann-Liouville type of order $\alpha \geq 0$, for a function $f \in C_{\mu}$ $(\mu \geq -1)$ defined by

$$J^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} \varphi(t)\, dt, \quad \operatorname{Re}(\alpha) > 0,$$

with $C_{\mu} = \left\{ f(x) = x^{p} \tilde{f}(x);\ p > \alpha,\ \tilde{f} \in C[0,\infty) \right\}$.

2. Main results

The following theorem will be useful to obtain the Grüss type inequality.

THEOREM 2.1. *Let h be a q-absolutely continuous function on $[0,\infty)$ and $p, P \in \mathbb{R}$, such that $p \leq h(x) \leq P$, for all $x \in [0,\infty)$, then*

$$\begin{aligned}
& I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)\, I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] \\
& h^2(x) - \left(I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] h(x) \right)^2 \\
& = I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (P - h(x)) \\
& \times I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (h(x) - p) - \\
& I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)\, I_q^n \left[\begin{array}{l} \rho,\ (M,1),\ (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] \\
& \times (P - h(x))(h(x) - p),
\end{aligned} \tag{8}$$

$$A_i, B_j \in \mathbb{R}^{+}, M \in \mathbb{N}_0, \operatorname{Re}(\alpha_i) > 0, \operatorname{Re}(\beta_j) > 0; i = 1, ..., r-1,$$

$$j = 1, ..., s, n \in \mathbb{N}, \rho \in \mathbb{C}, 0 < q < 1, \textstyle\sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i \geq 0, x > 0.$$

P r o o f. Let h be a q-absolutely continuous function on $[0,\infty)$ and $p, P \in \mathbb{R}$, such that $p \leq h(x) \leq P$, for all $x > 0$.

On the other hand, Kalla [18] established the following result, valid for any $u, v \in [0,\infty)$:

$$(P-h(u))(h(v)-p)+(P-h(v))(h(u)-p)-(P-h(u)) \tag{9}$$

$$\times(h(u)-p)-(P-h(v))(h(v)-p) \tag{10}$$

$$=h^2(u)+h^2(v)-2h(u)h(v). \tag{11}$$

Multiplying both sides of (11) by

$$\frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)}\;{}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\(\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\, q, q^n\frac{u}{x}\right],$$

$$A_i, B_j \in \mathbb{R}^+, M \in \mathbb{N}_0, \operatorname{Re}(\alpha_i)>0, \operatorname{Re}(\beta_j)>0;$$

$$i=1,...,r-1, j=1,...,s,$$

$$n\in\mathbb{N}, \rho\in\mathbb{C}, 0<q<1, \textstyle\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0, u\in(0,x), x>0,$$

we get

$$\frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)}\;{}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ \\(\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\, q, q^n\frac{u}{x}\right]$$

$$\times\{(P-h(u))(h(v)-p)+(P-h(v))(h(u)-p)$$

$$-(P-h(u))(h(u)-p)-(P-h(v))(h(v)-p)\}$$

$$=\frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)}\;{}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ \\(\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\, q, q^n\frac{u}{x}\right]$$

$$\times\left\{h^2(u)+h^2(v)-2h(u)h(v)\right\}.$$

Developing and later integrating the variable u from 0 to x we have

$$\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ \\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right]$$

$$\times(P-h(u))(h(v)-p)\,d_qu+\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,\\ (\beta_1,B_1),...,\end{array}\right.$$

$$\left.\begin{array}{c}(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right](P-h(v))(h(u)-p)\,d_qu-$$

$$\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right]$$

$$\times(P-h(u))(h(u)-p)\,d_qu-\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,\\ (\beta_1,B_1),...,\end{array}\right.$$

$$\left.\begin{array}{c}(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right](P-h(v))(h(v)-p)\,d_qu$$

$$=\frac{x^{-\rho-1}}{\Gamma_q(M+1)}$$

$$\times\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right]$$

$$\times h^2(u)\,d_qu+\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,\\ (\beta_1,B_1),...,\end{array}\right.$$

$$\left.\begin{array}{c}(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right]h^2(v)\,d_qu-2\frac{x^{-\rho-1}}{\Gamma_q(M+1)}$$

$$\times\int_0^x u^{\rho}\ {}_r\psi_s^*\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle| q,q^n\frac{u}{x}\right]$$

$$\times h(u)\,h(v)\,d_qu.$$

Using (1) yields

$$\begin{aligned}
&I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ \\ (\beta_j,B_j)_{1,s}\end{array}\right](P-h(x))(h(v)-p)+\\
&I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right](h(x)-p)(P-h(v))-\\
&I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right](P-h(x))(h(x)-p)-\\
&I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right](1)(P-h(v))(h(v)-p)\\
&=I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]h^2(x)\\
&+I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right](1)\\
&\times h^2(v)-2I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]h(x)h(v).
\end{aligned}$$

Now, multiplying by

$$\frac{x^{-\rho-1}v^{\rho}}{\Gamma_q(M+1)}\,{}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\,q,q^n\frac{v}{x}\right],$$

$$A_i,B_j\in\mathbb{R}^+,M\in\mathbb{N}_0,\mathrm{Re}(\alpha_i)>0,\mathrm{Re}(\beta_j)>0;$$

$$i=1,...,r-1,j=1,...,s,$$

$$n\in\mathbb{N},\rho\in\mathbb{C},0<q<1,\textstyle\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,v\in(0,x),x>0,$$

and integrating over v from 0 to x we obtain

$$\begin{aligned}
&I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right](P-h(x))\\
&\times I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\ \\ (\beta_j,B_j)_{1,s}\end{array}\right](h(x)-p)
\end{aligned}$$

$$+ I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (h(x) - p)$$

$$\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (P - h(x))$$

$$- I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (P - h(x)) (h(x) - p)$$

$$\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)$$

$$- I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)$$

$$\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (P - h(x)) (h(x) - p)$$

$$= I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] h^2(x)$$

$$I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) +$$

$$I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)\, I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] h^2(x)$$

$$- 2 I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] h(x)$$

$$\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] h(x).$$

The result of Theorem 5 is obtained. □

By means of the following theorem we introduce the new fractional q-integral inequality of Grüss type.

THEOREM 2.2. *Let f and g be q-absolutely continuous functions on $[0, \infty)$, $l, L, p, P \in \mathbb{R}$, such that $l \leq f(x) \leq L$, $p \leq g(x) \leq P$ and $I_q^n(\cdot)$ a fractional q-integral operator as defined by (1), then*

$$\left| I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)\, I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] \right.$$

$$\times f(x)\, g(x) - I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x)$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x) \Bigg|$$

$$\leq \left(\frac{I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)}{2} \right)^2 (L-l)(P-p), \qquad (12)$$

$$A_i, B_j \in \mathbb{R}^+, M \in \mathbb{N}_0, n \in \mathbb{N}, \rho \in \mathbb{C}, \mathrm{Re}(\alpha_i) > 0,\ \mathrm{Re}(\beta_j) > 0;$$
$$i = 1, ..., r-1, j = 1, ..., s, 0 < q < 1,$$
$$\textstyle\sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i \geq 0, x > 0.$$

P r o o f. Let f and g be q-absolutely continuous functions on $[0,\infty)$, $l, L, p, P \in \mathbb{R}$, such that $l \leq f(x) \leq L$, $p \leq g(x) \leq P$ and any $u, v \in (0,x)$ the following function is defined:

$$A(u,v) = (f(u) - f(v))(g(u) - g(v)). \qquad (13)$$

Considering

$$F(u,x) = \frac{x^{-\rho-1} u^{\rho}}{\Gamma_q(M+1)}$$
$$\times {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_{r-1}, A_{r-1}), (-M,1) \\ (\beta_1, B_1), \ldots, (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right]$$

and

$$G(v,x) = \frac{x^{-\rho-1} v^{\rho}}{\Gamma_q(M+1)}$$
$$\times {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_{r-1}, A_{r-1}), (-M,1) \\ (\beta_1, B_1), \ldots, (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right],$$

$$A_i, B_j \in \mathbb{R}^+, M \in \mathbb{N}_0, Re(\alpha_i) > 0, Re(\beta_j) > 0; i = 1, \ldots, r-1,$$
$$j = 1, \ldots, s, \sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i \geq 0, 0 < q < 1, u, v \in (0,x), x > 0.$$

Multiplying both sides of (13) by $F(u,x)\,G(v,x)$, we have

$$F(u,x)\,G(v,x)\,A(u,v)$$

$$= \frac{x^{-\rho-1} u^{\rho}}{\Gamma_q(M+1)}\, {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M,1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right]$$
$$\times \frac{x^{-\rho-1} v^{\rho}}{\Gamma_q(M+1)}\, {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M,1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right]$$
$$\times (f(u) - f(v))(g(u) - g(v)).$$

Integrating both u and v from 0 to x we have

$$\begin{aligned}
&\int_0^x \int_0^x F(u,x)\, G(v,x)\, A(u,v)\, d_q u d_q v \\
&= \int_0^x \int_0^x \frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)} \\
&\times\ {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right] \\
&\times \frac{x^{-\rho-1}v^{\rho}}{\Gamma_q(M+1)} {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right] \\
&\times (f(u) - f(v))(g(u) - g(v))\, d_q u d_q v.
\end{aligned}$$

The square of each member of the equality is calculated

$$\left(\int_0^x \int_0^x F(u,x)\, G(v,x)\, A(u,v)\, d_q u d_q v \right)^2 \tag{14}$$

$$= \left(\int_0^x \int_0^x \frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)} \right. \tag{15}$$

$$\begin{aligned}
&{}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right] \\
&\times \frac{x^{-\rho-1}v^{\rho}}{\Gamma_q(M+1)}\ {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right] \\
&\left. \times (f(u) - f(v))(g(u) - g(v))\, d_q u d_q v \right)^2.
\end{aligned} \tag{16}$$

Applying the Cauchy-Schwarz inequality for double q-integrals given in (4) to the right-hand side of (16) yields

$$\begin{aligned}
&\left(\int_0^x \int_0^x \frac{x^{-\rho-1}u^{\rho}}{\Gamma_q(M+1)} \right. \\
&\times\ {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right] \\
&\times \frac{x^{-\rho-1}v^{\rho}}{\Gamma_q(M+1)}\ {}_r\psi_s^* \left[\begin{array}{c} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right] \\
&\left. \times (f(u) - f(v))(g(u) - g(v))\, d_q u d_q v \right)^2
\end{aligned}$$

$$\leq \left\{ \int_0^x \int_0^x \frac{x^{-\rho-1} u^{\rho}}{\Gamma_q (M+1)} \right. \tag{17}$$

$$\tag{18}$$

$$\times \; {}_r\psi_s^* \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right]$$

$$\tag{19}$$

$$\times \frac{x^{-\rho-1} v^{\rho}}{\Gamma_q (M+1)} {}_r\psi_s^* \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right]$$

$$\tag{20}$$

$$\left. \times (f(u) - f(v))^2 \, d_q u d_q v \right\}$$

$$\tag{21}$$

$$\times \left\{ \int_0^x \int_0^x \frac{x^{-\rho-1} u^{\rho}}{\Gamma_q (M+1)} \right. \tag{22}$$

$$\tag{23}$$

$$\times \; {}_r\psi_s^* \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{u}{x} \right]$$

$$\tag{24}$$

$$\times \frac{x^{-\rho-1} v^{\rho}}{\Gamma_q (M+1)} {}_r\psi_s^* \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_{r-1}, A_{r-1}), (-M, 1) \\ (\beta_1, B_1), ..., (\beta_s, B_s) \end{array} \middle| q, q^n \frac{v}{x} \right]$$

$$\tag{25}$$

$$\left. \times (g(u) - g(v))^2 \, d_q u d_q v \right\}. \tag{26}$$

After developing and applying (1) on both sides of (26), we have

$$\left\{ I_q^n \left[\begin{array}{c} \rho, (M, 1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right. \tag{27}$$

$$\tag{28}$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M, 1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) g(x)$$

$$\tag{29}$$

$$-I_q^n \left[\begin{array}{c} \rho, (M, 1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) \tag{30}$$

$$I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]g(x)^2$$

$$\leq\left(I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\right. \tag{31}$$

$$I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]f^2(x)$$

$$\left.-\left(I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]f(x)\right)^2\right) \tag{32}$$

$$\left(I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\right.$$

$$\times I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]g^2(x) \tag{33}$$

$$-\left(I_q^n\left[\begin{array}{c}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]g(x)\right)^2. \tag{34}$$

By hypothesis, the following results are true

$$(L-f(x))(f(x)-l))\geq 0,(P-g(x))(g(x)-p))\geq 0,$$

then

$$I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right] \tag{35}$$

$$\times(L-f(x))(f(x)-l))\geq 0, \tag{36}$$

$$I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right] \tag{37}$$

$$\times(P-g(x))(g(x)-p))\geq 0. \tag{38}$$

After suppressing in Theorem 5 the quantities given in (38), respectively, the following results are obtained:

$$I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]f^2(x) \tag{39}$$

$$-\left(I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]f(x)\right)^2 \tag{40}$$

$$\leq I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right] \tag{41}$$

$$\times(L-f(x))\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](f(x)-l) \tag{42}$$

and

$$I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](1)\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]g^2(x) \tag{43}$$

$$-\left(I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]g(x)\right)^2 \tag{44}$$

$$\leq I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right] \tag{45}$$

$$\times(P-g(x))\,I_q^n\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right](g(x)-p). \tag{46}$$

Using (34) and the inequalities (42) and (46), we get

$$\Bigg\{ I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) g(x)$$

$$- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x)$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x)^{2}$$

$$\leq \Bigg(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (L - f(x))$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (f(x) - l) \Bigg)$$

$$\times \Bigg(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (P - g(x))$$

$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (g(x) - p) \Bigg),$$

then

$$\left\{ I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right.$$
$$\times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) g(x)$$
$$- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x)$$
$$\left. \times I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x) \right\}^2$$
$$\leq \left(L I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right.$$
$$- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x)$$
$$\times \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) \right.$$
$$- l I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)$$
$$\times \left(P I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right.$$
$$- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x)$$
$$\times \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x) \right.$$
$$- p I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1).$$

Using the elementary inequality $4ab \leq (a+b)^2$; $a, b \in \mathbb{R}$, we have:

$$4 \left(L I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right.$$
$$- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x)$$
$$\times \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) \right.$$
$$- l I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1)$$

$$\leq \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) (L-l) \right)^2$$

and

$$\begin{aligned} &4 \left(P I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right. \\ &- I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x) \\ &\times \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x) \right. \\ &- p I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \\ &\leq \left(I_q^n \left[\begin{array}{c} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) (P-p) \right)^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\{ I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) \right. \\ &\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) g(x) \\ &- I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] f(x) \\ &\times I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] g(x)^2 \\ &\leq \frac{1}{16} \left(I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) (L-l) \right)^2 \\ &\times \left(I_q^n \left[\begin{array}{l} \rho, (M,1), (\alpha_i, A_i)_{1,r-1} \\ (\beta_j, B_j)_{1,s} \end{array} \right] (1) (P-p) \right)^2 . \end{aligned}$$

After extracting the square root, we get (12). □

For $r = s+1$, $A_i = B_i = 1$ and $\alpha_i = \beta_i + m_i$ in (12), after using (3) the following result is deduced:

COROLLARY 2.1. *Let f and g q-absolutely continuous functions on $[0,\infty)$, $l, L, p, P \in \mathbb{R}$, such that $l \leq f(x) \leq L$, $p \leq g(x) \leq P$ and $L_q^n(\cdot)$ a fractional* *q-integral operator as defined by (3), then*

$$\Big| L_q^n\{M,\beta_1,\beta_2,...,\beta_s,\rho,m_1,m_2,...,m_s;1\} L_q^n\left\{\{M,\beta_1,\beta_2,...,\beta_s,\rho,\right.$$
$$m_1,m_2,...,m_s; f(x)\, g(x)\} - L_q^n\left\{\{M,\beta_1,\beta_2,...,\beta_s,\rho,m_1,m_2,...,m_s;\right.$$
$$f(x)\}\, L_q^n\{M,\beta_1,\beta_2,...,\beta_s,\rho,m_1,m_2,...,m_s; g(x)\}\Big|$$
$$\leq \left(\frac{L_q^n\{M,\beta_1,\beta_2,...,\beta_s,\rho,m_1,m_2,...,m_s;1\}}{2}\right)^2 (L-l)(P-p) \tag{47}$$
$$M, m_i \in \mathbb{N}_0,\ \operatorname{Re}(\beta_i) \neq 0,-1,-2,...; i=1,...,s, n\in\mathbb{N}, \rho\in\mathbb{C},$$
$$0<q<1,\ x>0.$$

For $r = s+1, A_i = B_i,\ \alpha_i = \beta_i; i = 1,...,s, n = M, \rho = 0$ in (12) we deduce the following result:

Corollary 2.2. *Let f and g q-absolutely continuous functions on $[0,\infty)$, $l,L,p,P \in \mathbb{R}$, such that $l \leq f(x) \leq L$, $p \leq g(x) \leq P$ and $J_q^{M+1} f(x)$ is a fractional q-integral operator of Riemann-Liouville type, then*

$$\left|\frac{x^{M+1}}{\Gamma_q(M+2)} J_q^{M+1} f(x)\, g(x) - J_q^{M+1} f(x)\, J_q^{M+1} g(x)\right|$$
$$\leq \left(\frac{x^{M+1}}{2\Gamma_q(M+2)}\right)^2 (L-l)(P-p), \tag{48}$$
$$x>0, 0<q<1,\ M\in\mathbb{N}.$$

For $r = s+1,\ A_i = B_i,\ \alpha_i = \beta_i; i = 1,...,s, n = M+1$ in (12) we get

Corollary 2.3. *Let f and g q-absolutely continuous functions on $[0,\infty)$, $l,L,p,P \in \mathbb{R}$, such that $l \leq f(x) \leq L$, $p \leq g(x) \leq P$ and $I_q^{\rho,M+1} f(x)$ is a fractional q-integral operator of Kober type, then*

$$\left|\frac{\Gamma_q(\rho+1)}{\Gamma_q(M+\rho+2)} I_q^{\rho,M+1} f(x)\, g(x) - I_q^{\rho,M+1} f(x)\, I_q^{\rho,M+1} g(x)\right|$$
$$\leq \left(\frac{\Gamma_q(\rho+1)}{2\Gamma_q(M+\rho+2)}\right)^2 (L-l)(P-p) \tag{49}$$
$$x>0, 0<q<1,\ M\in\mathbb{N}_0, \operatorname{Re}(\rho)>-1.$$

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