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EXACT SOLUTIONS FOR NONLINEAR PDES VIA HERMITE POLYNOMIALS

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Abstract

In this note we construct exact solutions for nonlinear partial differential equations by means of Hermite polynomials. In particular, we consider generalized Burgers equations, that can be linearized by means of a Cole-Hopf-type transform. We show that it is possible to construct an interesting particular solution by using Hermite polynomials. We also consider a class of nonlinear equations admitting isochronous solutions. The main aim is to show that Hermite polynomials can be used to construct exact solutions for a wide class of nonlinear integro-differential equations including nonlinear fractional equations.

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1. Introduction

It is well-known that the Kampé de Fériet Hermite polynomials play a relevant role in the theory of linear differential equations. Indeed, the solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}, \tag{1}$$

under the condition $u(x,0) = x^n$ is given by the two variable Hermite polynomials of Kampé de Fériet

$$H_n(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{(n-2r)!r!}. \tag{2}$$

This means that these two variable Hermite polynomials are useful to construct exact solutions for linear partial differential equation, in particular diffusive equations.

In this note we use this interesting result in order to show that it is possible to construct interesting solutions for a wide class of nonlinear equations by using two variable Hermite polynomials.

First of all, we consider generalized Burgers type equations with variable coefficients. It is well-known that these equations are directly related to diffusion equations by means of the Cole-Hopf transform (see e.g. [8], [9] and the references therein). Therefore, it is possible to construct explicit solutions involving a ratio of two-variable Hermite polynomials.

We also consider (2+1) dimensional nonlinear equations admitting interesting completly periodic solutions. We show that also in the case of nonlinear fractional equations particular solutions can be obtained in this way. Many other cases can be treated by using a similar scheme, i.e. considering nonlinear equations that can are related in someway to the master equation (1).

2. The generalized Burgers equation

The Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

is one of the most celebrated example of nonlinear equations that can be linearized to the linear diffusive equation by means of the so-called Cole-Hopf transform. There are many generalizations of this equation involving for example variable coefficients or non-homogeneous terms. We refer for example to the recent paper [9] and the references therein.

Here we consider the following non-homogeneous Burgers type equation

$$\frac{\partial u}{\partial t} - A(t)\frac{\partial u^2}{\partial x} = \frac{\partial^2 u}{\partial x^2} + A(t)u, \tag{4}$$

where the variable coefficient $A(t)$ is a suitable function such that $A(t) \neq 0$ for all $t \geq 0$. By means of the Cole-Hopf-type transform

$$u = \frac{1}{A(t)}\frac{\partial}{\partial x}\ln\psi(x,t) \tag{5}$$

we have that

$$-\frac{\dot{A}}{A^2}\cdot\frac{\partial_x\psi}{\psi} + \frac{1}{A}\partial_x\left(\frac{\partial_t\psi}{\psi} - \frac{\partial_{xx}\psi}{\psi}\right) = \frac{\partial_x\psi}{\psi}, \tag{6}$$

where we denoted by $\dot{A} = dA/dt$ the first order time-derivative of the function $A(t)$.

Under the condition that $A(t)$ solves the nonlinear ODE

$$\dot{A} = -A^2, \tag{7}$$

we have that the equation (4) is transformed in the linear diffusive equation

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2}. \tag{8}$$

Therefore, we can construct particular explicit solution for this equation by means of the Hermite polynamials.

Observing that

$$\frac{d}{dx}H_n(x,t) = nH_{n-1}(x,t), \tag{9}$$

we have that the function

$$u(x,t) = \frac{1}{A(t)}\frac{nH_{n-1}(x,t)}{H_n(x,t)}, \tag{10}$$

provides an exact solution for the equation (4) if A(t) satisfy the ODE (7) and $A(t) \neq 0$ for all $t > 0$. A similar construction was used for the classical Burgers equation with constant coefficient in [3] and more recently in [4].

A simple example is given by $A(t) = 1/(1+t)$, i.e. with this choice the equation (4) admits the solution

$$u(x,t) = (t+1)\frac{nH_{n-1}(x,t)}{H_n(x,t)}. \tag{11}$$

A similar scheme can be used to construct solutions based on Hermite polynomials for modified Burgers equation linearizable to the diffusion equation.

3. Nonlinear (2+1)-dimensional partial differential equations

In the recent paper [5], the authors considered isochronous PDEs involving Laguerre derivatives in space (see [2] about Laguerre derivatives). Inspired by these works, here we show that is possible to construct a completely periodic solution for the nonlinear (2+1)-dimensional PDE

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} \right). \tag{12}$$

This equation admits the completely periodic separating variable solution

$$u(x,y,t) = \exp\left(\frac{i\omega}{\alpha} t \right) H_n(x,y). \tag{13}$$

Moreover, recalling that the higher order Hermite polynomials

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{[\frac{n}{m}]} \frac{y^r x^{n-mr}}{(n-mr)!r!} \tag{14}$$

is a solution of the higher order PDE

$$\frac{\partial u}{\partial y} - \frac{\partial^m u}{\partial x^m} = 0, \quad m > 2, \tag{15}$$

then we have that the function

$$u(x,y,t) = \exp\left(\frac{i\omega}{\alpha} t \right) H_n^{(m)}(x,y) \tag{16}$$

is a solution for the equation

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial u}{\partial y} - \frac{\partial^m u}{\partial x^m} \right), \quad m > 2. \tag{17}$$

Also in this case, this simple scheme can be adapted to solve a wide class of nonlinear differential equations.

We observe that also the time-fractional generalization admits and explicit exact solution involving Hermite polynomials. But in this case the property of completely periodicity is lost. Indeed, the nonlinear time-fractional differential equation

$$\frac{\partial^{\nu} u}{\partial t^{\nu}} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} \right), \tag{18}$$

admits the solution

$$u(x,y,t) = E_{\nu}\left(\frac{i\omega}{\alpha} t^{\nu} \right) H_n(x,y), \tag{19}$$

where

$$E_{\nu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\nu k + 1)}, \tag{20}$$

is the well-known one-parameter Mittag-Leffler function. In the equation (18) we denoted by $\partial^\nu/\partial t^\nu$ the time-fractional partial derivative of order $\nu \in (0,1)$ in the sense of Caputo, see e.g. [6] for details.

We observe that, for a given linear integro-differential operator acting only on the time variable, namely $\widehat{O}_t$, we can construct a solution for the equation

$$\widehat{O}_t u - \frac{i\omega}{\alpha} u = a u^\alpha \left(\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} \right), \tag{21}$$

involving the Hermite polynomials. Indeed, if $f(t)$ is a solution of the linear equation

$$\widehat{O}_t f = \frac{i\omega}{\alpha} f, \tag{22}$$

then the function

$$u(x,y,t) = f(t) \cdot H_n(x,y), \tag{23}$$

is obviously a solution for the nonlinear equation (21).

There is another interesting case of (3+1)-dimensional nonlinear PDE that admits exact solutions involving Hermite polynomials.

Let us consider the non-linear diffusive equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^k}{\partial y^2} + \frac{\partial^2 u^m}{\partial z^2}, \quad k, m > 0. \tag{24}$$

In this case, a rather trivial exact solution is given by

$$u(x,y,z,t) = H_n(x,t) y^{1/k} z^{1/m}. \tag{25}$$

This basic construction can be adapted to other cases, showing the relevance of the two variable Hermite polynomials to obtain exact solutions for linear and nonlinear partial differential equations. The space-fractional counterpart of this equation can be easily treated in a similar way.

By using the previous results, we can easily construct and exact solution for the (3+1)-dimensional non-linear differential equation

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^\alpha \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} + \frac{\partial^2 u^m}{\partial z^2} \right), \quad m > 0 \tag{26}$$

that is given by

$$u(x,y,z,t) = e^{\frac{i\omega}{\alpha} t} z^{1/m} H_n(x,y). \tag{27}$$

The generalization to the high dimension can be easily obtained following the same scheme.

Moreover, in the same way, we can consider the (3+1)-dimensional equation

$$\widehat{O}_t u - \frac{i\omega}{\alpha} u = a u^\alpha \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} + \frac{\partial^2 u^m}{\partial z^2} \right), \quad m > 0, \tag{28}$$

where $\widehat{O}_t$ is a linear integro-differential operator. In this case a solution is given by

$$u(x, y, z, t) = f(t) z^{1/m} H_n(x, y), \tag{29}$$

where f is a solution of the equation $\widehat{O}_t f = \frac{i\omega}{\alpha} f$.

Concluding, the main aim of this note is to underline the potential utility of the two variable Hermite polynomials to construct explicit particular solutions for nonlinear PDEs. The most relevant cases are given by Burgers-type equations linearizable to the diffusion equation by means of the Cole-Hopf transform (see also [3] and [4]). Then, we provide other simple examples with solutions that can be constructed by means of the separation of variables. This method is widely used to solve nonlinear PDEs, we refer for example to the recent monograph [7].

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