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# HJORTNAES INTEGRALS INVOLVING THE DI- AND TRILOGARITHMS

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**Abstract**

At the 1953 Scandinavian Mathematical Congress, M. Hjortnaes presented the following transformation of $\sum_{k=1}^{\infty} 1/k^3 = \zeta(3)$ to a definite integral: $\zeta(3) = 10 \int_0^{\log[(1+\sqrt{5})/2]} t^2 \coth t \, dt$. We show that a similar formula holds for $\zeta(2)$, namely $\zeta(2) = \dfrac{10}{3} \int_0^{\log[(1+\sqrt{5})/2]} t \coth t \, dt$, and refer to the above and related integrals as Hjortnaes integrals. Based on the known exact values of the first two polylogarithms, we derive corresponding Hjortnaes integrals and also some series representations.

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**Key Words and Phrases**: Hjortnaes integral for $\zeta(3)$; Riemann Zeta function; dilogarithm; trilogarithm; golden ratio

## 1. Introduction

The dilogarithm $\mathrm{Li}_2(z)$ and the trilogarithm $\mathrm{Li}_3(z)$ are defined initially for $|z| \leq 1$ by the absolutely convergent series

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$$\mathrm{Li}_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \cdots + \frac{z^n}{n^2} + \cdots, \quad \mathrm{Li}_3(z) = \frac{z}{1^3} + \frac{z^2}{2^3} + \cdots + \frac{z^n}{n^3} + \cdots,$$

and then continued analytically to the whole complex plane via

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt, \quad \mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\,dt.$$

As usual, we denote by $\zeta(z)$ the Riemann Zeta function which is the analytic continuation of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1$ to all complex $z \neq 1$.

With a remarkable tour-de-force, Euler has shown that $\zeta(2) = \frac{\pi^2}{6} = \mathrm{Li}_2(1)$ (the Basel Problem), $\zeta(4) = \frac{\pi^4}{90}$, and in general

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n}}{2(2n)!}(2\pi)^{2n}, \tag{1}$$

where $B_{2n}$ are the Bernoulli numbers.

At the Twelfth Scandinavian Mathematical Congress in 1953, Margrethe Munthe Hjortnaes presents (see [1]) an interesting transformation of the series $\sum_{k=1}^{\infty} 1/k^3$ to a definite integral, namely the relation

$$\zeta(3) = 10 \int_0^{\log[(1+\sqrt{5})/2]} t^2 \coth t\,dt, \tag{2}$$

which she proves through utilization of Euler's transformation of series. As far as we can tell, it was Lewin [2] who observed that (2) can be proved using the trilogarithm. Motivated by these results, we prove that a similar representation holds for $\zeta(2)$ :

$$\zeta(2) = \frac{10}{3} \int_0^{\log[(1+\sqrt{5})/2]} t \coth t\,dt, \tag{3}$$

and study more generally the integrals:

$$\int_0^{\log \alpha} t \coth t\,dt, \quad \alpha \in \{\phi, \sqrt{\phi}, \sqrt{2}\},$$

$$\int_0^{\log \beta} t^2 \coth t\,dt, \quad \beta \in \{\phi, \sqrt{2}\},$$

where as usual we denote by $\phi$ the golden ratio $\frac{1+\sqrt{5}}{2}$. The above integrals give rise to some additional formulas stemming from the known values for $\mathrm{Li}_2$

and $\mathrm{Li}_3$ and also to some interesting series. All necessary preliminaries can be found in [2], but we shall reprove some results for completeness.

## 2. **The results**

For the proof of formula (3) one needs the exact value

$$\mathrm{Li}_2\Big(\frac{1}{\phi}\Big) \;=\; \frac{3}{5}\mathrm{Li}_2(1) - \log^2(\phi). \tag{4}$$

Recall the main functional equations for the dilogarithm:

1) $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \mathrm{Li}_2(1) - \log(x)\log(1-x)$ (Reflection Formula);

2) $\mathrm{Li}_2(-x) + \mathrm{Li}_2\Big(\dfrac{x}{x+1}\Big) = -\dfrac{1}{2}\log^2(1+x)$ (Landen's Formula);

3) $\frac{1}{2}\mathrm{Li}_2\big(x^2\big) = \mathrm{Li}_2(x) + \mathrm{Li}_2(-x)$ (Factorization Formula).

Eliminating $\mathrm{Li}_2(-x)$ from Landen's formula and the Factorization theorem gives

$$\mathrm{Li}_2\Big(\frac{x}{x+1}\Big) + \tfrac{1}{2}\mathrm{Li}_2\big(x^2\big) - \mathrm{Li}_2(x) \;=\; -\tfrac{1}{2}\log^2(1+x)\,. \tag{5}$$

Note the similarity between (5) and the three-term Landen's relation for $\mathrm{Li}_3$ [2, p. 155, (6.10) & (6.4)].

The solution to $\dfrac{x}{x+1} = x^2$ is $x = \dfrac{1}{\phi}$, and one obtains

$$\tfrac{3}{2}\mathrm{Li}_2\Big(\frac{1}{\phi^2}\Big) \;-\; \mathrm{Li}_2\Big(\frac{1}{\phi}\Big) \;=\; -\tfrac{1}{2}\log^2(\phi)\,.$$

Taking $x = \dfrac{1}{\phi^2}$ in the reflection formula gives

$$\mathrm{Li}_2\Big(\frac{1}{\phi^2}\Big) \;=\; -\,\mathrm{Li}_2\Big(\frac{1}{\phi}\Big) \;+\; \mathrm{Li}_2(1) \;-\; \log\Big(\frac{1}{\phi^2}\Big)\log\Big(\frac{1}{\phi}\Big),$$

which, when substituted in the previous equation reduces it to (4). Note that (4) does not depend on the fact that $\mathrm{Li}_2(1) = \dfrac{\pi^2}{6}$.

Take the integral $\displaystyle\int_0^{\log\alpha} t\coth t\,dt$, rewrite $\coth t = \dfrac{1+e^{-2t}}{1-e^{-2t}}$, and change the variable to $x = 1-e^{-2t}$. This gives

$$-\frac{1}{4}\int_0^A \log(1-x)\Big[\frac{2}{x} + \frac{1}{1-x}\Big]\,dx\,, \tag{6}$$

where $\alpha \in \{\phi, \sqrt{\phi}, \sqrt{2}\} \;\Rightarrow\; A \in \Big\{\dfrac{1}{\phi}, \dfrac{1}{\phi^2}, \dfrac{1}{2}\Big\}$, in the same order.

The integration in (6) is easily performed, and the result is

$$\int_0^{\log\alpha} t\coth t\,dt = \frac{1}{2}\mathrm{Li}_2(A) + \frac{1}{2}\log^2(\alpha). \tag{7}$$

Similarly (cf. [2, (6.15)], the integral $\int_0^{\log\beta} t^2\coth t\,dt$, $\beta \in \{\phi, \sqrt{2}\}$ is evaluated in terms of trilogarithms and dilogarithms.

From the exact values for $\mathrm{Li}_2$ and $\mathrm{Li}_3$ (given in Appendix 1), one readily obtains the following formulas:

$$\int_0^{\log\phi} t\coth t\,dt = \frac{\pi^2}{20} = \frac{3}{10}\zeta(2), \tag{8}$$

$$\int_0^{\log\sqrt{\phi}} t\coth t\,dt = \frac{\pi^2}{30} - \frac{3}{8}\log^2\phi, \tag{9}$$

$$\int_0^{\log\sqrt{2}} t\coth t\,dt = \frac{\pi^2}{24} - \frac{1}{8}\log^2 2, \tag{10}$$

and

$$\int_0^{\log\phi} t^2\coth t\,dt = \frac{1}{10}\zeta(3), \tag{11}$$

$$\int_0^{\log\sqrt{2}} t^2\coth t\,dt = \frac{1}{16}\zeta(3) - \frac{1}{24}\log^3 2. \tag{12}$$

Next, we recall the power series expansion:

$$z\coth z = 2\sum_{n=0}^{\infty}\zeta(2n)\Big(\frac{z}{\pi}\Big)^{2n}, \quad |z| < \pi.$$

Integrating $\int_0^{\log\alpha} t\coth t\,dt$ and $\int_0^{\log\beta} t^2\coth t\,dt$ termwise, and using Euler's relation (1), we obtain the interesting sums

$$\sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}\frac{(\log\phi)^{2n+1}}{2n+1} = \frac{\pi^2}{20},$$

$$\sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}\frac{(\log\phi)^{2n+2}}{2n+2} = \frac{\zeta(3)}{10},$$

and, after some rearrangement,

$$B_0\frac{\log\phi}{1!} - 3B_1\frac{(\log\phi)^2}{2!} + B_2\frac{(\log\phi)^3}{3!} + \cdots + B_n\frac{(\log\phi)^{n+1}}{(n+1)!} + \cdots = \frac{\pi^2}{15},$$

$$B_0\frac{\log 2}{1!} - B_1\frac{(\log 2)^2}{2!} + B_2\frac{(\log 2)^3}{3!} + \cdots + B_n\frac{(\log 2)^{n+1}}{(n+1)!} + \cdots = \frac{\pi^2}{12},$$

$$B_0\frac{(\log 2)^2}{0!\cdot 2} - B_1\frac{(\log 2)^3}{1!\cdot 3} + B_2\frac{(\log 2)^4}{2!\cdot 4} + \cdots + B_n\frac{(\log 2)^{n+2}}{n!\cdot (n+2)} + \cdots = \frac{\zeta(3)}{4}.$$

The last three series are notable for involving all of the Bernoulli numbers and not only the even-indexed ones. The interested reader may see [3] for some other Zeta series having the golden ratio in the coefficients, and the recent paper [4] for a one-line solution to the Basel problem. The book by Srivastava and Choi [5] may also be recommended as it contains a rich collection of zeta series summing to various mathematical constants.

**Appendix 1.**

Exact values for $\mathrm{Li}_2$ and $\mathrm{Li}_3$ :

$$\mathrm{Li}_2\Big(\frac{\sqrt{5}-1}{2}\Big) = \mathrm{Li}_2\Big(\frac{1}{\phi}\Big) = \frac{\pi^2}{10} - \log^2(\phi),$$

$$\mathrm{Li}_2\Big(\frac{3-\sqrt{5}}{2}\Big) = \mathrm{Li}_2\Big(\frac{1}{\phi^2}\Big) = \frac{\pi^2}{15} - \log^2(\phi),$$

$$\mathrm{Li}_2\Big(\frac{1}{2}\Big) = \frac{1}{2}\mathrm{Li}_2(1) - \frac{1}{2}\log^2(2) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),$$

and

$$\mathrm{Li}_3\Big(\frac{1}{\phi^2}\Big) = \frac{4}{5}\mathrm{Li}_3(1) + \frac{2}{3}\log^3(\phi) - \frac{2}{15}\pi^2\log(\phi),$$

$$\mathrm{Li}_3\Big(\frac{1}{2}\Big) = \frac{7}{8}\mathrm{Li}_3(1) - \frac{1}{12}\pi^2\log 2 + \frac{1}{6}\log^3(2).$$

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## References

[1] M. Munthe Hjortnaes, Overføring av rekken $\sum_{k=1}^{\infty}(1/k^3)$ til et bestemt integral, *Tolfte Skandinaviska Matematikerkongressen, Lund, 1953* (1954), 211–213.

[2] L. Lewin, *Polylogarithms and Associated Functions*, Elsevier, North-Holland, New York/London/Amsterdam, 1981.

[3] L. Markov, A functional expansion and a new set of rapidly convergent series involving Zeta values, *Stud. Comput. Intelligence* **793** (2019), 267–276.

[4] L. Markov, Two short proofs of the formula $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$, *Math. Gazette* **106** (2022), 28–31.

[5] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.