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ON THE CONTROL PROBLEM ASSOCIATED WITH A PSEUDO-PARABOLIC TYPE EQUATION IN AN ONE-DIMENSIONAL DOMAIN

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Abstract

In previous works, the control problem for the initial-boundary value problem in the interval for such a pseudo-parabolic type equation was studied. That is, the Dirichlet boundary value problem was considered. In this work we consider control problem for a homogeneous pseudo-parabolic type equation. In the part of the bound of the given region it is given value of the derivative of the solution with the respect to the normal and it is required to find control to get the average value of solution. By the Laplace transform method it is proved that like this control exists.

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Key Words and Phrases: Pseudo-parabolic equation, admissible control, initial-boundary value problem, integral equation, Laplace transform

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1. Introduction

Consider the pseudo-parabolic equation in the domain $\Omega = \{(x,t) : 0 < x < a, \quad t > 0\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} + \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in \Omega, \tag{1}$$

with boundary value conditions

$$u_x(0,t) = -h(t), \quad u_x(a,t) + \gamma\, u(a,t) = 0, \quad t > 0, \tag{2}$$

and initial value condition

$$u(x,0) = 0, \quad 0 \le x \le a, \tag{3}$$

where $\gamma = \text{const} > 0$.

The equation (1) was called the pseudo-parabolic equation by "R.E. Showalter and T.W. Ting" (see, [1]) from the following considerations: a) correctly posed initial boundary value problems for a parabolic equation are also correctly posed for equation (1), b) in some cases, the solution of the initial-boundary value problem can be obtained as the limit of the corresponding solution of the problem for pseudo-parabolic equations.

The condition (2) means that there is a magnitude of output given by a measurable real-valued function $h(t)$.

Definition 1.1. If function $h(t) \in W_2^1(R_+)$ satisfies the conditions $h(0) = 0$ and $|h(t)| \le 1$, we say that this function is an *admissible control.*

Problem A. *For the given function $f(t)$ Problem A consists looking for the admissible control $h(t)$ such that the solution $u(x,t)$ of the initial-boundary problem* (1)-(3) *exists and for all $t \ge 0$ satisfies the equation*

$$\int_0^a u(x,t)\, dx = f(t). \tag{4}$$

The tasks of impulse control, i.e. the case of delta-like distribution for systems with distributed parameters was the subject of study in works [2, 3]. One of the models is the theory of incompressible simple fluids with decaying memory, which can be described by equation (1) (see [4]). In [5], stability, uniqueness, and availability of solutions of some classical problems for the considered equation were studied (see also [6, 7]). Point control problems for parabolic and pseudo-parabolic equations were considered.

It is known that the problem of optimal control for parabolic type equations was first studied in [8] and [9]. In [10, 11, 12], boundary control problems for parabolic equation were studied in an $n-$dimensional domain with a piecewise smooth boundary. In these works, an estimate was found for the minimum time required to reach a given average temperature. The latest results on boundary control problems for parabolic type equations are studied in works [13, 14, 15, 16, 17, 18]. More information on the optimal control problems for distributed parameter systems is given in [19] and in the monographs [20, 21, 22].

In [23, 24, 25], boundary control problems for pseudo-parabolic type equations were studied. These works mainly deal with the boundary control problem for the Dirichlet problem.

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [26]. The practical approaches to optimal control of the heat conduction equation are described in publications like [27].

In previous works, boundary control problems for pseudo-parabolic equation were also considered. For example, the control problem of the inhomogeneous pseudo-parabolic equation was studied in [24]. In our work, it is proved that the control function exists when the boundary condition is type 3. The problem presented in Section 2 is reduced to a Volterra integral equation of the second kind. Finally, the desired value for the kernel is obtained, and the existence of the control function is proved using the Laplace transform method in Section 3.

2. Integral equation

We now consider the eigenvalue problem

$$\begin{cases} X_k''(x) = -\mu_k X_k(x), \quad 0 < x < a, \\ X_k'(0) = 0, \quad X_k'(a) + \gamma X_k(a) = 0, \quad 0 \leq x \leq a, \end{cases}$$

where $\gamma = \text{const} > 0$.

We set $\sqrt{\mu_k} = \lambda_k$. Then we have $X_k(x) = \cos \lambda_k x$, $k = 1, 2, \ldots$. Here, eigenvalue λ_k is the solution of this equation $\lambda_k \cdot \tan \lambda_k a = \gamma$ (see, [28]).

Definition 2.1. By the solution of the problem (1)–(3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \frac{(a-x)^2}{2a}\, h(t) - w(x,t), \tag{5}$$

where the function $w(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$, $w_x \in C(\bar{\Omega})$ is the solution to the problem:

$$w_t - w_{xxt} - w_{xx} = \frac{(a-x)^2}{2a}\, h'(t) - \frac{1}{a} h(t) - \frac{1}{a} h'(t),$$

with boundary conditions

$$w_x(0,t) = 0, \quad w_x(a,t) + \gamma\, w(a,t) = 0,$$

and initial condition $w(x,0)=0$.

Then we have (see, [28, 29])

$$w(x,t)=\sum_{k=1}^{\infty}\frac{\cos\lambda_k x}{1+\lambda_k^2}\int_0^t e^{-q_k\,(t-s)}\big(h'(s)\,b_k-h(s)\,c_k\big)\,ds, \tag{6}$$

where $q_k=\frac{\lambda_k^2}{1+\lambda_k^2}$ and

$$b_k=\frac{2}{a\,\lambda_k^2}-\frac{2(1+\lambda_k^2)\,\sin\lambda_k a}{a^2\,\lambda_k^3},\quad c_k=\frac{2}{a^2\,\lambda_k}\sin\lambda_k a.$$

By (5) and (6) we get the solution of the initial-boundary problem (1)–(3):

$$u(x,t)=\frac{(a-x)^2}{2a}\,h(t)$$

$$-\sum_{k=1}^{\infty}\frac{\cos\lambda_k x}{1+\lambda_k^2}\int_0^t e^{-q_k\,(t-s)}\big(h'(s)\,b_k-h(s)\,c_k\big)\,ds. \tag{7}$$

By (4) and (7) we can write

$$f(t)=\int_0^a u(x,t)dx=h(t)\int_0^a\frac{(a-x)^2}{2a}\,dx$$

$$-\sum_{k=1}^{\infty}\frac{b_k\,\sin\lambda_k a}{\lambda_k(1+\lambda_k^2)}\int_0^t e^{-q_k\,(t-s)}\,dh(s)$$

$$+\sum_{k=1}^{\infty}\frac{c_k\,\sin\lambda_k a}{\lambda_k(1+\lambda_k^2)}\int_0^t e^{-q_k\,(t-s)}h(s)\,ds.$$

Note that

$$\int_0^a\frac{(a-x)^2}{2a}\,dx=\sum_{k=1}^{\infty}\left(\frac{2}{a\,\lambda_k^2}-\frac{2\,\sin\lambda_k a}{a^2\lambda_k^3}\right)\frac{\sin\lambda_k a}{\lambda_k}.$$

Considering the above and $h(0)=0$, we get the following

$$f(t)=h(t)\,\sum_{k=1}^{\infty}\frac{2\,\sin\lambda_k a}{a\lambda_k(1+\lambda_k^2)}+\sum_{k=1}^{\infty}\frac{2\,\sin\lambda_k a}{a\lambda_k(1+\lambda_k^2)^2}\int_0^t e^{-q_k\,(t-s)}h(s)\,ds.$$

Set

$$\alpha=\sum_{k=1}^{\infty}\frac{2\,\sin\lambda_k a}{a\lambda_k(1+\lambda_k^2)},\quad \beta_k=\frac{2\,\sin\lambda_k a}{a\lambda_k(1+\lambda_k^2)^2}.$$

and

$$L(t) = \sum_{k=1}^{\infty} \beta_k \, e^{-q_k \, t}, \quad t > 0. \tag{8}$$

Then we have the following Volterra integral equation

$$\alpha \, h(t) + \int_0^t L(t-s) \, h(s) \, ds = f(t). \tag{9}$$

It is clear that $\sin \lambda_k a > 0$. Indeed, according to $\lambda_k \cdot \tan \lambda_k a = \gamma$, we get

$$\sin \lambda_k a = \frac{\gamma}{\sqrt{\lambda_k^2 + \gamma^2}}, \quad \text{where} \quad \gamma = \text{const} > 0.$$

PROPOSITION 2.1. *A function $L(t)$ defined by (8) is continuous on the half-line $t \geq 0$.*

P r o o f. Indeed, according to (8), we can write

$$0 < L(t) \leq \text{const} \sum_{k=1}^{\infty} \beta_k.$$

□

Denote by $W(M)$ the set of function $f \in W_2^2(-\infty, +\infty)$, $f(t) = 0$ for $t \leq 0$ which satisfies the condition

$$\|f\|_{W_2^2(R_+)} \leq M.$$

THEOREM 2.1. *There exists $M > 0$ such that for any function $f \in W(M)$ the solution $h(t)$ of the equation (9) exists, and satisfies condition*

$$|h(t)| \ \leq \ 1.$$

3. **Proof of the Theorem 2.1**

We write integral equation (9)

$$\alpha \, h(t) + \int_0^t L(t-s) \, h(s) ds = f(t), \quad t > 0.$$

By definition of the Laplace transform we have

$$\widetilde{h}(p) = \int_0^{\infty} e^{-pt} \, h(t) \, dt.$$

Applying the Laplace transform to the second kind Volterra integral equation (9) and taking into account the properties of the transform convolution we get

$$\widetilde{f}(p) = \alpha\, \widetilde{h}(p) + \widetilde{L}(p)\, \widetilde{h}(p).$$

Consequently, we obtain

$$\widetilde{h}(p) = \frac{\widetilde{f}(p)}{\alpha + \widetilde{L}(p)}, \quad \text{where} \;\; p = \delta + i\xi, \quad \delta > 0,$$

and

$$h(t) = \frac{1}{2\pi i} \int\limits_{\delta - i\xi}^{\delta + i\xi} \frac{\widetilde{f}(p)}{\alpha + \widetilde{L}(p)}\, e^{pt} dp$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\widetilde{f}(\delta + i\xi)}{\alpha + \widetilde{L}(\delta + i\xi)}\, e^{(\delta + i\xi)t} d\xi. \tag{10}$$

Then we can write

$$\widetilde{L}(p) = \int\limits_0^\infty L(t)\, e^{-pt}\, dt$$

$$= \sum_{k=1}^\infty \beta_k \int\limits_0^\infty e^{-(p+q_k)t}\, dt = \sum_{k=1}^\infty \frac{\beta_k}{p + q_k},$$

where

$$\widetilde{L}(\delta + i\xi) = \sum_{k=1}^\infty \frac{\beta_k}{\delta + q_k + i\xi} = \sum_{k=1}^\infty \frac{\beta_k\, (\delta + q_k)}{(\delta + q_k)^2 + \xi^2} - i\xi \sum_{k=1}^\infty \frac{\beta_k}{(\delta + q_k)^2 + \xi^2}.$$

We know that

$$(\delta + q_k)^2 + \xi^2 \le [(\delta + q_k)^2 + 1](1 + \xi^2),$$

and we have the inequality

$$\frac{1}{(\delta + q_k)^2 + \xi^2} \ge \frac{1}{1 + \xi^2}\, \frac{1}{(\delta + q_k)^2 + 1}. \tag{11}$$

Consequently, according to (11) we can obtain the estimates

$$|\mathrm{Re}(\alpha + \widetilde{L}(\delta + i\xi))| = \alpha + \sum_{k=1}^\infty \frac{\beta_k\, (\delta + q_k)}{(\delta + q_k)^2 + \xi^2}$$

$$\ge \frac{1}{1 + \xi^2} \sum_{k=1}^\infty \frac{\beta_k\, (\delta + q_k)}{(\delta + q_k)^2 + 1} = \frac{C_{1\delta}}{1 + \xi^2}, \tag{12}$$

and

$$|\mathrm{Im}(\alpha+\widetilde{L}(\delta+i\xi))| = |\xi|\sum_{k=1}^{\infty}\frac{\beta_k}{(\delta+q_k)^2+\xi^2}$$

$$\geq \frac{|\xi|}{1+\xi^2}\sum_{k=1}^{\infty}\frac{\beta_k}{(\delta+q_k)^2+1} = \frac{C_{2\delta}\,|\xi|}{1+\xi^2}, \tag{13}$$

where $C_{1\delta}$, $C_{2\delta}$ as follows

$$C_{1\delta} = \sum_{k=1}^{\infty}\frac{\beta_k\,(\delta+q_k)}{(\delta+q_k)^2+1},\quad C_{2\delta} = \sum_{k=1}^{\infty}\frac{\beta_k}{(\delta+q_k)^2+1}.$$

From (12) and (13), we have the estimate

$$|\alpha+\widetilde{L}(\delta+i\xi)|^2 = |\mathrm{Re}(\alpha+\widetilde{L}(\delta+i\xi))|^2 + |\mathrm{Im}(\alpha+\widetilde{L}(\delta+i\xi))|^2$$

$$\geq \frac{\min(C_{1\delta}^2, C_{2\delta}^2)}{1+\xi^2},$$

and

$$|\alpha+\widetilde{L}(\delta+i\xi)| \geq \frac{C_\delta}{\sqrt{1+\xi^2}}, \tag{14}$$

where $C_\delta = \min(C_{1\delta}, C_{2\delta})$.

Then, by passing to the limit at $\delta \to 0$ from (10), we can obtain the equality

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{f}(i\xi)}{\alpha+\widetilde{L}(i\xi)}\,e^{i\xi t}d\xi. \tag{15}$$

□

THEOREM 3.1. *Let $f(t) \in W(M)$. Then for the image of the function $f(t)$ the following inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{f}(i\xi)|\sqrt{1+\xi^2}d\xi \leq C\,\|f\|_{W_2^2(R_+)},$$

is valid.

P r o o f. It is known that the following relation is valid for the Laplace transform of the function $f(t)$:

$$(\delta+i\xi)\,\widetilde{f}(\delta+i\xi) = \int_0^{\infty} e^{-(\delta+i\xi)t}\,f'(t)\,dt,$$

and for $\delta \to 0$ we have

$$i\xi\,\widetilde{f}(i\xi) = \int\limits_0^\infty e^{-i\xi t}\,f'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{f}(i\xi) = \int\limits_0^\infty e^{-i\xi t}\,f''(t)\,dt.$$

Then we have

$$\int\limits_{-\infty}^{+\infty} |\widetilde{f}(i\xi)|^2(1+\xi^2)^2 d\xi \le C\,\|f\|^2_{W_2^2(R_+)}. \tag{16}$$

Consequently, according to (16) we get the following estimate

$$\int\limits_{-\infty}^{+\infty} |\widetilde{f}(i\xi)|\sqrt{1+\xi^2}d\xi = \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{f}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}}$$

$$\le \Big(\int\limits_{-\infty}^{+\infty} |\widetilde{f}(i\xi)|^2(1+\xi^2)^2 d\xi\Big)^{1/2}\Big(\int\limits_{-\infty}^{+\infty} \frac{1}{1+\xi^2}d\xi\Big)^{1/2} \le C\,\|f\|_{W_2^2(R_+)}.$$

□

P r o o f o f T h e o r e m 2.1. We prove that $h \in W_2^1(R_+)$. Indeed, according to (14) and (15), we obtain

$$\int\limits_{-\infty}^{+\infty} |\widetilde{h}(\xi)|^2(1+|\xi|^2)\,d\xi \;=\; \int\limits_{-\infty}^{+\infty} \left|\frac{\widetilde{f}(i\xi)}{\alpha+\widetilde{L}(i\xi)}\right|^2 (1+|\xi|^2)\,d\xi$$

$$\le C\int\limits_{-\infty}^{+\infty} |\widetilde{f}(i\xi)|^2(1+|\xi|^2)^2\,d\xi \;=\; C\|f\|^2_{W_2^2(R)}.$$

Further,

$$|h(t)-h(s)| \;=\; \left|\int\limits_s^t h'(\tau)\,d\tau\right| \;\le\; \|h'\|_{L_2}\sqrt{t-s}.$$

Hence, $h \in \mathrm{Lip}\,\alpha$, where $\alpha = 1/2$. Then from (14), (15) and Theorem 3.1, we have

$$|h(t)| \le \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{|\widetilde{f}(i\xi)|}{|\alpha+\widetilde{L}(i\xi)|}d\xi \le \frac{1}{2\pi C_0}\int\limits_{-\infty}^{+\infty} |\widetilde{f}(i\xi)|\sqrt{1+\xi^2}d\xi$$

$$\leq \frac{C}{2\pi C_0}\|f\|_{W_2^2(R_+)} \leq \frac{C\,M}{2\pi C_0} = 1,$$

as M we took $M = \frac{2\pi C_0}{C}$.

Thus Theorem 2.1 is proved. □

4. **Conclusions**

In this work, we considered a control problem for a pseudo-parabolic type equation in an interval. The existence of such a control function was proved using the method of Laplace transform. We believe that this boundary control problem can be considered later on rectangular and smooth domain.

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