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A PROBLEM FOR A THREE-DIMENSIONAL EQUATION OF MIXED TYPE WITH SINGULAR COEFFICIENT

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Abstract

In this paper, we study the spatial Tricomi problem for a three-dimensional equation of mixed type with a singular coefficient in a domain whose elliptical part is a quarter of a cylinder, and whose hyperbolic part is a triangular right prism. The study of the problem is carried out using the method of separation of variables and spectral analysis. The solution to the considered problem is constructed as a sum of a double series. To justify the uniform convergence of the constructed series, asymptotic estimates of the Bessel and Gauss functions were used. On their basis, estimates were obtained for each member of the series, which made it possible to prove the convergence of the resulting series and its derivatives up to the second order inclusive, as well as the existence theorem in the class of regular solutions.

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1. **Introduction. Problem statement**

The study of boundary value problems for mixed-type equations is one of the central problems of the theory of partial differential equations of its applied importance. For the first time, F.I. Frankl [1] found important applications of these problems in gas dynamics, and I.N. Vekua [2] pointed out the importance of the problem of mixed-type equations in solving problems arising in the momentless theory of shells.

So far, the studies of boundary value problems for mixed-type equations with singular coefficients have been carried out mainly in the case of two independent variables. However, such problems in three-dimensional domains remain poorly studied.

The Tricomi problem for a mixed elliptic-hyperbolic equation in three-dimensional space using the method of integral Fourier transform was first studied in [3]. After this work, a number of works appeared in which boundary value problems for various elliptic-huperbolic equations in three-dimensional domains were considered (see, for example, [4], [5], [6], [7], [8], [9], [10], [11], [12]).

In this paper, we study the spatial Tricomi problem for a three-dimensional mixed type equation with singular coefficient in a region whose elliptic part is a quarter cylinder and whose hyperbolic part is a triangular straight prism.

Let $\Omega = \{(x,y,z) : (x,y) \in \Delta,\ z \in (0,c)\}$, where Δ is the finite one-connected domain of the plane xOy, bounded for $y \geq 0$ by the arc $\bar{\sigma}_0 = \left\{(x,y) : x^2+y^2=1,\ x \geq 0,\ y \geq 0\right\}$ and segment $\overline{OM} = \{(x,y) : x=0,\ 0 \leq y \leq 1\}$ and for $y \leq 0$ by segments $\overline{OQ} = \{(x,y) : x+y=0,\ 0 \leq x \leq 1/2\}$ and $\overline{QP} = \{(x,y) : x-y=1,\ \ 1/2 \leq x \leq 1\}$, $O=O\,(0,0)$, $M=M\,(0,1)$, $P=P\,(1,0)$, $Q=Q\,(1/2,-1/2)$.

Let us introduce the notations: $\Omega_0 = \Omega \cap (y>0)$, $\Omega_1 = \Omega \cap (y<0)$, $\Delta_0 = \Delta \cap (y>0)$, $\Delta_1 = \Delta \cap (y<0)$, $S_0 = \{(x,y,z) : \sigma_0 \times (0,c)\}$, $S_1 = \{(x,y,z) : OM \times (0,c)\}$, $\bar{S}_2 = \left\{(x,y,z) : \overline{OQ} \times [0,c]\right\}$, $\bar{S}_3 = \left\{(x,y,z) : \bar{\Omega} \cap (z=0)\right\}$, $\bar{S}_4 = \left\{(x,y,z) : \bar{\Omega} \cap (z=c)\right\}$.

In the domain Ω consider the equation

$$U_{xx} + (\mathrm{sgn} y)U_{yy} + U_{zz} + \frac{2\gamma}{z}U_z = 0, \tag{1}$$

where γ is parameter such that $\gamma \in (0,1/2)$.

In the domain Ω equation (1) belongs to a mixed type, namely in the domain Ω_0 elliptic type, and in the domain Ω_1 – hyperbolic type, and $z=0$ are the planes of singularity of the equation, and when passing through the rectangle $\bar{\Omega}_0 \cap \bar{\Omega}_1$ the equation changes its type.

We investigate the following problem for equation (1) in the domain Ω.

Problem T (Tricomi problem). *Find a function $U(x,y,z)$, satisfying in the domain Ω equation (1) and the following conditions:*

$$U \in C\left(\bar{\Omega}\right) \cap C^{2,2,2}_{x,y,z}\left(\Omega_0 \cup \Omega_1\right), \ U_x,\ U_y,\ z^{2\gamma}U_z \in C\left(\bar{\Omega}_0\right), \tag{2}$$

$$\left.U(x,y,z)\right|_{S_0} = F(x,y,z), \tag{3}$$

$$\left.U(x,y,z)\right|_{S_1} = 0,\ \left.U(x,y,z)\right|_{\bar{S}_2} = 0, \tag{4}$$

$$\left.U(x,y,z)\right|_{\bar{S}_3} = 0,\ \left.U(x,y,z)\right|_{\bar{S}_4} = 0, \tag{5}$$

as well as the gluing condition

$$U_y(x,-0,z) = U_y(x,+0,z), \quad x \in (0,1), \ z \in (0,c), \tag{6}$$

where $F(x,y,z)$ is a given function.

Note that the considered problem at $\gamma = 0$ was studied in [13].

2. Construction of particular solutions of equation (1) in the domain of hyperbolicity and ellipticity of the equation

We find nontrivial solutions of equation (1) satisfying conditions (4) and (5). Separating the variables by the formula $U(x,y,z) = w(x,y)\,Z(z)$, from equation (1) and boundary conditions (4) and (5), we obtain the following problems:

$$w_{xx} + (\mathrm{sgn} y) w_{yy} - \lambda w = 0, \quad (x,y) \in \Delta \cap \{x > 0\}, \tag{7}$$

$$w(0,y) = 0, \ y \in (0,1); \quad w(x,-x) = 0, \ x \in [0,1/2], \tag{8}$$

$$Z''(z) + \frac{2\gamma}{z} Z'(z) + \lambda Z(z) = 0, Z(0) = 0,\ Z(c) = 0, z \in (0,c). \tag{9}$$

Problem (9) has nontrivial solutions of the form [14], [15], [16]

$$Z_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z/c), \ m \in N, \tag{10}$$

where $J_l(z)$ is the Bessel function [17] and σ_m m is the positive root of the equation, $J_{1/2-\gamma}\left(\sqrt{\lambda}c\right) = 0$, $\lambda_m = (\sigma_m/c)^2$, $m \in N$.

According to [17], the system of eigenfunctions (10) is orthogonal and complete in space $L_2(0,c)$ with the weight $z^{2\gamma}$.

Now, consider the problem $\{(7),(8)\}$ when $\lambda = \lambda_m$ in the domain Δ_1, i.e., consider the following problem:

$$w_{xx} - w_{yy} - \lambda_m w = 0, \quad (x,y) \in \Delta_1, \tag{11}$$

$$w(x,-x) = 0, \quad x \in [0,1/2]. \tag{12}$$

We search the solution to this problem in the following form

$$w(x,y)=X(\xi)\,Y(\eta)\,, \tag{13}$$

where $\xi=\sqrt{x^2-y^2}$, $\eta=x^2/\xi^2$.

Then, with respect to the functions $X(\xi)$ and $Y(\eta)$ we obtain the following conditions, $X(0)=0$, $\left|\lim\limits_{\eta\to+\infty}Y(\eta)\right|<+\infty$ and equations

$$\xi^2X''(\xi)+\xi X'(\xi)-\left[\lambda_m\xi^2+\mu\right]X(\xi)=0,\ \ \xi>0, \tag{14}$$

$$\eta(1-\eta)\,Y''(\eta)+[1/2-\eta]\,Y'(\eta)+\frac{1}{4}\mu Y(\eta)=0,\ \ \eta>1, \tag{15}$$

where $\mu\in R$ is the parameter of the separation.

Solutions of equation (14) satisfying the condition $X(0)=0$, exist at $\mu>0$ and they (with accuracy to a constant multiplier) have of the form [17]

$$X(\xi)=I_\omega(\sigma_m\xi/c)\,,\ \ m\in N, \tag{16}$$

where $\omega=\sqrt{\mu}$, $I_l(x)$ is the Bessel function of an imaginary argument of order l [17].

(15) is a hypergeometric Gaussian equation [18]. Its general solution is defined by the formula [18]

$$Y(\eta)=c_1\eta^{-\omega/2}F(\omega/2,1/2+\omega/2,1+\omega;1/\eta)+$$
$$+c_2\eta^{\omega/2}F(-\omega/2,1-\omega/2,1-\omega;1/\eta)\,, \tag{17}$$

where c_1, c_2 are arbitrary constants.

$\omega>0$ it since follows from (17) that in order to obtain the function bounded at $\eta\to+\infty$, we need to put $c_2=0$ in the formula, as a result of which, we get

$$Y(\eta)=c_1\eta^{-\omega/2}F(\omega/2,1/2+\omega/2,1+\omega;1/\eta)\,. \tag{18}$$

Consequently, continuous and nontrivial in $\bar{\Delta}_1$ solution of the problem $\{(11),(12)\}$, according to (13), (16) and (18) are defined by the formulas

$$w_m^-(x,y)$$
$$=c_1\eta^{-\omega/2}F(\omega/2,(1+\omega)/2,1+\omega;1/\eta)\,I_\omega(\sigma_m\xi/c)\,, \tag{19}$$

where $c_1\neq 0$, $m\in N$.

Hence, we find

$$\begin{cases}\tau_m^-(x)=\lim\limits_{y\to-0}w_m^-(x,y)=c_1 2^\omega I_\omega\left(\dfrac{\sigma_m x}{c}\right),\ \ x\in[0,1]\,,\\ \nu_m^-(x)=\lim\limits_{y\to-0}\dfrac{\partial}{\partial y}w_m^-(x,y)=\dfrac{2^\omega c_1\omega}{x}I_\omega\left(\dfrac{\sigma_m x}{c}\right),\ x\in(0,1)\,,\end{cases} \tag{20}$$

where $\Gamma(z)$ is the Euler's gamma-function [18].

Now, consider the problem $\{(7),(8)\}$ in case $\lambda=\lambda_m$ in the domain Δ_0, i.e., consider the following problem:

$$w_{xx}+w_{yy}-\lambda_m w=0, \quad (x,y)\in\Delta_0, \tag{21}$$

$$w\left(0,y\right)=0, \quad y\in(0,1). \tag{22}$$

Separating the variables by formula

$$w\left(x,y\right)=Q\left(\rho\right)S\left(\varphi\right), \tag{23}$$

where $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg\left(y/x\right)$, from equation (21) and conditions $w\in C\left(\bar{\Delta}_0\right)$, (22), we obtain the following problems:

$$\rho^2 Q''\left(\rho\right)+\rho Q'\left(\rho\right)-\left[\left(\sigma_m\rho/c\right)^2+\tilde{\mu}\right]Q\left(\rho\right)=0, \ \rho\in(0,1), \tag{24}$$

$$\left|Q\left(0\right)\right|<+\infty, \tag{25}$$

$$S''\left(\varphi\right)+\tilde{\mu}\,S\left(\varphi\right)=0, \ \varphi\in(0,\pi/2), \tag{26}$$

$$S\left(\pi/2\right)=0, \tag{27}$$

where $\tilde{\mu}\in R$ is the separation constant.

We first study the problem $\{(24),(25)\}$. The general solution of equation (24) is defined in the form [17]

$$Q_m\left(\rho\right)=c_3 I_{\tilde{\omega}}\left(\sigma_m\rho/c\right)+c_4 K_{\tilde{\omega}}\left(\sigma_m\rho/c\right), \ \rho\in[0,1], \tag{28}$$

where $\tilde{\omega}=\sqrt{\tilde{\mu}}$, c_3 and c_4 are arbitrary constants, $K_l\left(x\right)$ is a MacDonald function of order l [17].

It follows from (28) that solutions of equation (24), satisfying condition (25), exist $\tilde{\mu}\geq 0$ at and they are defined by equations

$$Q_m\left(\rho\right)=c_3 I_{\tilde{\omega}}\left(\sigma_m\rho/c\right), \ \tilde{\omega}\geq 0, \ m\in N. \tag{29}$$

Now, let us study the problem $\{(26),(27)\}$. The general solution of equation (26) is

$$S\left(\varphi\right)=c_5\cos\left(\tilde{\omega}\varphi\right)+c_6\sin\left(\tilde{\omega}\varphi\right), \tag{30}$$

where c_5 and c_6 are arbitrary constants.

Satisfying the function (30) to the condition (27), we obtain $c_6=k_3\left(\tilde{\omega}\right)c_5$, where $k_3\left(\tilde{\omega}\right)=-ctg\left(\tilde{\omega}\pi/2\right)$. Substituting $c_6=k_3\left(\tilde{\omega}\right)c_5$ into (30) and assuming $c_5=1$ (this does not violate generality), we have

$$S\left(\varphi\right)=\cos\left(\tilde{\omega}\varphi\right)-ctg\left(\tilde{\omega}\pi/2\right)\sin\left(\tilde{\omega}\varphi\right). \tag{31}$$

Based on (23), (29) and (31), we conclude that the continuous and nontrivial in $\bar{\Delta}_0$ solution of the problem $\{(21),(22)\}$, has the form

$$w_m^+\left(x,y\right)=c_3 I_{\tilde{\omega}}\left(\sigma_m\rho/c\right)\left[\cos\left(\tilde{\omega}\varphi\right)-ctg\left(\tilde{\omega}\pi/2\right)\sin\left(\tilde{\omega}\varphi\right)\right], \tag{32}$$

where $c_3 \neq 0,\ m \in N$.

Hence, by direct calculation, one can find

$$\begin{cases} \tau_m^+(x) = \lim\limits_{y\to+0} w_m^+(x,y) = c_3 I_{\tilde{\omega}}(\sigma_m x/c), \ \ x \in [0,1]; \\ \nu_m^+(x) = \lim\limits_{y\to+0} \dfrac{\partial}{\partial y} w_m^+(x,y) = \\ \qquad = -c_3\tilde{\omega} ctg(\tilde{\omega}\pi/2)\, x^{-1} I_{\tilde{\omega}}(\sigma_m x/c), \ x \in (0,1). \end{cases} \tag{33}$$

Then, based on $U(x,y,z) = w(x,y)Z(z)$ and the notation introduced, the following equations follow from the conditions and $U(x,y,z) \in C(\bar{\Omega})$ and (6):

$$\begin{cases} \tau_m^-(x) = \tau_m^+(x), \ x \in [0,1], \\ \nu_m^-(x) = \nu_m^+(x), \ \ x \in (0,1). \end{cases} \tag{34}$$

Substituting (20) and (33) into (34) and assuming $\omega = \tilde{\omega}$, we have a homogeneous system of equations with respect to c_1 and c_3:

$$\begin{cases} 2^\omega c_1 + ctg\dfrac{\omega\pi}{2} c_3 = 0, \\ \quad 2^\omega c_1 - c_3 = 0. \end{cases} \tag{35}$$

From the system (35), we find $ctg\dfrac{\omega\pi}{2} = -1$. Writing out the solutions of this equation and taking into account the condition $\omega > 0$ we find

$$\omega_n = 2n - 1/2, \ \ n \in N. \tag{36}$$

Based on (36), the numbers $\mu_n = \omega_n^2,\ n \in N$ are the eigenvalues of problems $\{(15), \left|\lim\limits_{\eta\to+\infty} Y(\eta)\right| < +\infty\}$ and $\{(26), (27)\}$.

Note that at $\omega = \omega_n$ the function $S(\varphi)$, defined by the equality (31), will be written in the form

$$S_n(\varphi) = \sqrt{2}\sin\left[\left(2n - \frac{1}{2}\right)\varphi + \frac{\pi}{4}\right]. \tag{37}$$

In [19], it was proved that the system of eigenfunctions (37) forms a basis in the space $L_2(0,\pi/2)$.

Taking into account the above proved and equality (19), (32), $\omega = \tilde{\omega} = \omega_n$, we conclude that the functions

$$w_{nm}(x,y)$$

$$
= \begin{cases} c_3\sqrt{2}\sin\left[\left(2n-\frac{1}{2}\right)\varphi+\frac{\pi}{4}\right]\times \\ \qquad\qquad \times I_{2n-\frac{1}{2}}\left(\frac{\sigma_m\rho}{c}\right),\ (x,y)\in\bar{\Delta}_0, \\ \frac{c_3}{2^{\omega_n}}\left(\frac{1}{\eta}\right)^{\frac{\omega_n}{2}} F\left(n-\frac{1}{4}, n+\frac{1}{4}, 2n+\frac{1}{2}; \frac{1}{\eta}\right)\times \\ \qquad\qquad \times I_{2n-\frac{1}{2}}\left(\frac{\sigma_m\xi}{c}\right),\ (x,y)\in\bar{\Delta}_1, \end{cases} \tag{38}
$$

are continuous and nontrivial in $\bar{\Delta}$ solution of the problem $\{(7),(8)\}$.

Then, the functions

$$
U_{nm}(x,y,z) = w_{nm}(x,y)\, Z_m(z), \quad n,m\in N, \tag{39}
$$

where $Z_m(z)$ and $w_{nm}(x,y)$ are the functions defined by equalities (10) and (38) respectively, continuous and nontrivial in Ω solutions of equation (1) satisfying conditions (4)-(5).

3. Singularity of the solution of the problem T

Let $U(x,y,z) = V(\rho,\varphi,z)$ is solution the problem T in the domain Ω_0 and satisfy the condition

$$
V_\varphi(\rho,0,z) = \omega_n V(\rho,0,z), \tag{40}
$$

where ρ,φ,z are the cylindrical coordinates, related to Cartesian coordinates by the equations, $\rho=\sqrt{x^2+y^2}$, $\varphi = arctg\,(y/x)$, $z=z$.

In these coordinates, equations (1) and condition (3) are written in the form

$$
V_{\rho\rho}+\frac{1}{\rho^2}V_{\varphi\varphi}+\frac{1}{\rho}V_\rho+V_{zz}+\frac{2\gamma}{z}V_z=0, \quad (\rho,\varphi,z)\in\tilde{\Omega}, \tag{41}
$$

$$
V(1,\varphi,z) = f(\varphi,z), \varphi\in[0,\pi/2],\ z\in[0,c], \tag{42}
$$

where $\tilde{\Omega} = \{(\rho,\varphi,z) : \rho\in(0,1),\ \varphi\in(0,\pi/2),\ z\in(0,c)\}$, $f(\varphi,z) = F(\cos\varphi,\ \sin\varphi, z)$.

Using $V(\rho,\varphi,z)$ and eigenfunctions (10), (37), let us compose the following function:

$$
\zeta_{nm}(\rho) = d_m \int\limits_0^c \int\limits_0^{\pi/2} V(\rho,\varphi,z)\, S_n(\varphi)\, z^{2\gamma} Z_m(z)\, d\varphi dz, \tag{43}
$$

where $d_m = 2/\left[cJ_{3/2-\gamma}(\sigma_m)\right]^2$, $n,m\in N$.

Based on (43), we introduce the functions

$$
\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = d_m \int\limits_{\varepsilon_2}^{c-\varepsilon_2} \int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1} V(\rho,\varphi,z)\, S_n(\varphi)\, z^{2\gamma} Z_m(z)\, d\varphi dz, \tag{44}
$$

where ε_1 and ε_2 are sufficiently small positive numbers.

Obviously,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)=\zeta_{nm}(\rho).$$

From (44), we find $\left(\frac{\partial^2}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)$:

$$\left(\frac{\partial^2}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)$$

$$=d_m\int\limits_{\varepsilon_1}^{c-\varepsilon_1}\int\limits_{\varepsilon_2}^{\pi/2-\varepsilon_2}\left(\frac{\partial^2}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)V(\rho,\varphi,z)\,S_n(\varphi)\,z^{2\gamma}Z_m(z)\,d\varphi dz.$$

Taking into account equations (41), from the latter, we have

$$\left(\frac{\partial^2}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)$$

$$=-\frac{d_m}{\rho^2}\int\limits_{\varepsilon_2}^{c-\varepsilon_2}\left[\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}V_{\varphi\varphi}S_n(\varphi)\,d\varphi\right]z^{2\gamma}Z_m(z)\,dz$$

$$-d_m\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left[\int\limits_{\varepsilon_2}^{c-\varepsilon_2}\left(V_{zz}+\frac{2\gamma}{z}V_z\right)z^{2\gamma}Z_m(z)\,dz\right]S_n(\varphi)\,d\varphi.$$

Applying the rule integration by parts from the last, we obtain

$$\left(\frac{\partial^2}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)$$

$$=-\frac{d_m}{\rho^2}\int\limits_{\varepsilon_2}^{c-\varepsilon_2}\Bigg\{\left\{\left[V_\varphi S_n(\varphi)-VS'_n(\varphi)\right]\right\}\Big|_{\varphi=\varepsilon_1}^{\varphi=\pi/2-\varepsilon_1}$$

$$\left.-\mu_n\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}V(\rho,\varphi,z)\,S_n(\varphi)d\varphi\right\}z^{2\gamma}Z_m(z)\,dz$$

$$-d_m\int\limits_{\varepsilon_1}^{\pi/2-\varepsilon_1}\Bigg\{\left\{\left[V_z(\rho,\varphi,z)\,Z_m(z)-V(\rho,\varphi,z)\,Z'_m(z)\right]z^{2\gamma}\right\}\Big|_{z=\varepsilon_2}^{z=c-\varepsilon_2}$$

$$\left.-(\sigma_{\gamma m}/c)^2\int\limits_{\varepsilon_2}^{c-\varepsilon_2}V(\rho,\varphi,z)\,z^{2\gamma}Z_m(z)\,dz\right\}S_n(\varphi)\,d\varphi. \qquad (45)$$

Hence, passing to the limit as, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ and considering (2), (4), (5), (27), (40) and boundary conditions of the problems (9), as well as the notation (43), we obtain the equality

$$\zeta''_{nm}(\rho) + \frac{4\beta+1}{\rho}\zeta'_{nm}(\rho) - \left(\lambda_m - \frac{\mu_n}{\rho^2}\right)\zeta_{nm}(\rho) = 0, \;\; \rho \in (0,1).$$

Hence, the function $\zeta_{nm}(\rho)$ satisfies the differential equation (24) for $\mu = \mu_n$.

Moreover, due to the boundary conditions (3), it follows from (43) that the function $\zeta_{nm}(\rho)$ satisfies the following boundary conditions:

$$\zeta_{nm}(1) = f_{nm}, \tag{46}$$

where

$$f_{nm} = d_m \int\limits_0^c \int\limits_0^{\pi/2} f(\varphi, z)\, S_n(\varphi)\, z^{2\gamma} Z_m(z)\, d\varphi dz. \tag{47}$$

Consequently, the function $\zeta_{nm}(\rho)$, defined by equality (43), satisfies equation (24) at $\tilde{\mu} = \mu_n$ and conditions (25), (46). Therefore, by subjecting the general solution (28) of equation (24) to these conditions, we find the coefficients c_3 and c_4: $c_3 = f_{nm}/I_{\omega_n}(\sigma_m/c)$, $c_4 = 0$.

Substituting these values into (28), we unambiguously find the function $\zeta_{nm}(\rho)$

$$\zeta_{nm}(\rho) = I_{\omega_n}(\sigma_m \rho/c) f_{nm}/I_{\omega_n}(\sigma_m/c). \tag{48}$$

Now, we shall prove the following theorem.

THEOREM 3.1. *If a solution to problem T exists if condition (40) is satisfied, then it is unique.*

P r o o f. For this, it is sufficient to prove that the homogeneous problem T, has only a trivial solution. Let $f(\varphi, z) \equiv 0$. Then $f_{nm} = 0$ for all $n, m \in N$. By virtue of this equality, it follows from (48) and (43) that $\int\limits_0^c \int\limits_0^{\pi/2} V(\rho, \varphi, z)\, S_n(\varphi)\, z^{2\gamma} Z_m(z)\, d\varphi dz = 0$. Hence, by virtue of the complenteness of the system of functions (10) with the weight $z^{2\gamma}$ in the space $L_2(0, c)$ and $V(\rho, \varphi, z) \in C\left(\bar{\bar{\Omega}}\right)$ it follows that, $\int\limits_0^{\pi/2} V(\rho, \varphi, z)\, S_n(\varphi)\, d\varphi = 0$, $n \in N$. Given the completeness of the system of functions (37) in the space $L_2(0, \pi/2)$ and $V(\rho, \varphi, z) \in C\left(\bar{\bar{\Omega}}\right)$, it follows from the last equality that $V(\rho, \varphi, z) \equiv 0$ in $\bar{\bar{\Omega}}$.

Using this equality and $U(x,y,z) = V(\rho,\varphi,z)$, it is easy to see that $U(x,+0,z) \equiv 0,\ U_y(x,0,z) \equiv 0,\ x \in [0,1],\ z \in [0,c]$.

Then, by virtue of $U(x,y,z) \in C\left(\overline{\Omega}\right)$, the following equations are true

$$U(x,-0,z) \equiv 0,\ U_y(x,-0,z) \equiv 0,\ x \in [0,1],\ z \in [0,c]. \tag{49}$$

It follows from the results of [20] that the solution of equation

$$U_{xx} - U_{yy} + U_{zz} + \frac{2\gamma}{z} U_z = 0, (x,y,z) \in \Omega_1$$

satisfying conditions (49) is identically zero, i.e., $U(x,y,z) \equiv 0$, $(x,y,z) \in \bar{\Omega}_1$. Theorem 1 is proved. □

4. Construction and justification of the solution to the problem T

Substituting the values $c_3 = f_{nm}/I_{\omega_n}(\sigma_m/c)$ to equality (38), and then the obtained function in (39), we find partial solutions of the problem T in the form of

$$U_{nm}(x,y,z) = \begin{cases} U^{+}_{nm}(x,y,z), & (x,y,z) \in \bar{\Omega}_0,\ n,m \in N, \\ U^{-}_{nm}(x,y,z), & (x,y,z) \in \bar{\Omega}_1,\ n,m \in N, \end{cases}$$

where

$$U^{+}_{nm}(x,y,z) = Z_m(z)\,\zeta_{nm}(\rho)\,S_n(\varphi),\ (x,y,z) \in \bar{\Omega}_0, \tag{50}$$

$$U^{-}_{nm}(x,y,z) = 2^{-\omega_n} Z_m(z)\,X_{nm}(\xi)\,Y_n(\eta),\ (x,y,z) \in \bar{\Omega}_1, \tag{51}$$

$$X_{nm}(\xi) = \frac{I_{2n-1/2}(\sigma_m \xi/c)\, f_{nm}}{I_{2n-1/2}(\sigma_m/c)},\ \xi = \sqrt{x^2 - y^2}, \tag{52}$$

$$Y_n(\eta) = \left(\frac{1}{\eta}\right)^{n-1/4} F\left(n - \frac{1}{4}, n + \frac{1}{4}, 2n + \frac{1}{2}; \frac{1}{\eta}\right),\ \eta = \frac{x^2}{\xi^2}, \tag{53}$$

and $Z_m(z) S_n(\varphi)$ f_{nm} and $\zeta_{nm}(\rho)$ are defined by the equations (10), (37), (47) and (48) respectively.

Theorem 4.1. *If $f(\varphi,z)$ satisfies the following conditions:*

I. $f(\varphi,z) \in C^{4,5}_{\varphi,z}\left(\bar{\Pi}\right)$, where $\Pi = \{(\varphi,z) :\ \varphi \in (0,\pi/2),\ z \in (0,c)\}$;

II. $\left.\frac{\partial^j}{\partial \varphi^j} f(\varphi,z)\right|_{\varphi=0} = 0,\ \left.\frac{\partial^j}{\partial \varphi^j} f(\varphi,z)\right|_{\varphi=\pi/2} = 0,\ j = \overline{0,3}$;

III. $\left.\frac{\partial^j}{\partial z^j} f(\varphi,z)\right|_{z=0} = 0,\ \left.\frac{\partial^j}{\partial z^j} f(\varphi,z)\right|_{z=c} = 0,\ j = \overline{0,4}$.

Then the solution of the problem T exists and is defined by the formula

$$U(x,y,z)=\begin{cases}\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}U_{nm}^{+}(x,y,z)\,, & (x,y,z)\in\bar{\Omega}_0,\\ \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}U_{nm}^{-}(x,y,z)\,, & (x,y,z)\in\bar{\Omega}_1,\end{cases}\tag{54}$$

where $U_{nm}^{+}(x,y,z)$, $U_{nm}^{-}(x,y,z)$ are functions defined by formulas (50) and (51), respectively.

Before proceeding to the proof of this theorem, let us prove some lemmas.

Lemma 1. *If $\gamma\in(0,1/2)$, then the following estimates are valid with respect to the functions $Z_m(z)$, defined by equations (10), for $z\in[0,c]$ and sufficiently large m:*

$$|Z_m(z)|\le c_5 z^{1-2\gamma}(\sigma_m)^{1/2-\gamma},\tag{55}$$

$$\left|z^{2\gamma}Z'_m(z)\right|\le c_6(\sigma_m)^{1/2},\tag{56}$$

$$\left|B_{\gamma-1/2}^{z}Z_m(z)\right|\le c_7 z^{1-2\gamma}(\sigma_m)^{5/2-\gamma},\tag{57}$$

where $c_j,\ j=\overline{5,7}$ are positive constants, $B_q^y\equiv\dfrac{\partial^2}{\partial y^2}+\dfrac{2q+1}{y}\dfrac{\partial}{\partial y}$ is Bessel operator [21].

P r o o f. Let us rewrite the function $Z_m(z)$ in the from

$$Z_m(z)=\frac{(2c)^{\gamma-1/2}}{\Gamma(3/2-\gamma)}z^{1-2\gamma}(\sigma_m)^{1/2-\gamma}\bar{J}_{1/2-\gamma}(\sigma_m z/c)\,,\tag{58}$$

where $\bar{J}_\nu(z)$ is the Bessel-Clifford function [22]:

$$\bar{J}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)=\sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{(\nu+1)_j j!}.$$

The function $\bar{J}_\nu(z)$ is even and infinitely differentiable. Moreover, we have the equality $\bar{J}_\nu(0)=1$ and the inequality $\left|\bar{J}_\nu(z)\right|\le 1$ for all $\nu>-1/2$. Considering this and $1/2-\alpha>0$, from equality (58), we get the estimate (55).

Now, consider the function $z^{2\gamma}Z'_m(z)=\dfrac{\sigma_m}{c}z^{1/2+\gamma}J_{-1/2-\gamma}(\dfrac{\sigma_m z}{c})$. Let us rewrite this function in the form

$$z^{2\gamma}Z'_m(z)=(\sigma_m/c)^{1/2-\gamma}\xi^{1/2+\gamma}J_{-1/2-\gamma}(\xi)\,,\tag{59}$$

where $\xi = \sigma_m z/c$. The function $\xi^{1/2+\gamma} J_{-1/2-\gamma}(\xi)$ is bounded at the point $\xi = 0$ and continuous at $\xi \in [0, +\infty)$. Moreover, by virtue of the asymptotic formula of the Bessel function:

$$J_\nu(\xi) \approx \left(\frac{2}{\pi\xi}\right)^{1/2} \cos\left(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{60}$$

for sufficiently large ξ, we have the estimate $\left|\xi^{1/2+\gamma} J_{-1/2-\gamma}(\xi)\right| < \xi^\gamma \tilde{c}_6$, where $\tilde{c}_6 = const > 0$ is valid. Considering these properties of the function $\xi^{1/2+\gamma} J_{-1/2-\gamma}(\xi)$, it follows from (60) that for sufficiently large ξ the inequality $\left|z^{2\gamma} Z'_m(z)\right| \le \tilde{c}_6(\sigma_m/c)^{1/2-\gamma}\xi^\gamma = \tilde{c}_6(\sigma_m/c)^{1/2} z^\gamma \le c_6(\sigma_m)^{1/2}$, i.e., the estimate (56) is valid.

It is known that the function $Z_m(z)$ satisfies equation from (9) at $\lambda_m = (\sigma_m/c)^2$. It follows that $B^z_{\gamma-1/2} Z_m(z) = -(\sigma_m/c)^2 Z_m(z)$. Then, by virtue of evaluation (55), evaluation (57) is valid. Lemma 1 is proven. □

Lemma 2. [23] *For sufficiently large $m \in N$, the following estimate is valid*

$$\left|J_{3/2-\gamma}(\sigma_m)\right| \ge c_8(\sigma_m)^{-1/2}, \tag{61}$$

where c_8 is the positive constant.

Lemma 3. *Let the conditions of Theorem 2 be satisfied. Then, for the coefficients f_{nm}, defined by equality (47), the following estimate is valid:*

$$|f_{nm}| \le c_9 \omega_n^{-4} (\sigma_m)^{-4,5}, \tag{62}$$

where c_9 is some positive constants.

Proof. Let us represent the coefficient f_{nm} in the form

$$f_{nm} = d_m \int_0^c F_n(z)\, z^{1/2+\gamma} J_{1/2-\gamma}(\sigma_m z/c)\, dz, \tag{63}$$

where $F_n(z) = \int\limits_0^{\pi/2} f(\varphi, z) \sin\left(\omega_n \varphi + \frac{\pi}{4}\right) d\varphi$.

First, consider the function $F_n(z)$ and for it applying the rule of integration by parts four times, we obtain

$$F_n(z) = -\frac{1}{\omega_n} f(\varphi, z) \cos(\omega_n\varphi + \pi/4)\Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$+\frac{1}{\omega_n^2} f_\varphi(\varphi, z) \sin(\omega_n\varphi + \pi/4)\Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$+\frac{1}{\omega_n^3} f_{\varphi\varphi}(\varphi, z) \cos(\omega_n\varphi + \pi/4)\Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$-\frac{1}{\omega_n^4} f_{\varphi\varphi\varphi}(\varphi, z)\sin(\omega_n\varphi + \pi/4)\Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$+\frac{1}{\omega_n^4}\int\limits_0^{\pi/2}\sin\left(\omega_n\varphi + \frac{\pi}{4}\right)\frac{\partial^4}{\partial\varphi^4} f(\varphi, z)\,d\varphi.$$

By virtue of the statements of Theorem 2, the non-integral terms in the last term are zero. Hence,

$$F_n(z) = \frac{1}{\omega_n^4}\int\limits_0^{\pi/2}\sin\left(\omega_n\varphi + \frac{\pi}{4}\right)\frac{\partial^4}{\partial\varphi^4} f(\varphi, z)\,d\varphi. \tag{64}$$

Based on the conditions of Theorem 2, it is true $\frac{\partial^4}{\partial\varphi^4} f(\varphi, z) \in C\left(\bar{\Pi}\right).$ Taking this into account and $|\sin(\omega_n\varphi + \pi/4)| \leq 1$, we conclude that the integral in (64) exists and $F_n(z) \in C[0, c].$

Now consider the coefficients f_{nm}, defined by equality (63).

Using the equality

$$z^{1/2+\gamma} J_{1/2-\gamma}(\sigma_m z/c) = -\frac{c}{\sigma_m}\frac{d}{dz}\left[z^{1/2+\gamma} J_{-1/2-\gamma}(\sigma_m z/c)\right], \tag{65}$$

coefficient f_{nm} we will write in the form

$$f_{nm} = -\frac{cd_m}{\sigma_m}\int\limits_0^c \frac{d}{dz}\left[z^{1/2+\gamma} J_{-1/2-\gamma}(\sigma_m z/c)\right] F_n(z)\,dz.$$

Applying the rule of integration by parts, we obtain

$$f_{nm} = -d_m(c/\sigma_m)\, z^{1/2+\gamma} J_{-1/2-\gamma}(\sigma_m z/c)\, F_n(z)\Big|_{z=0}^{z=c}$$

$$+d_m(c/\sigma_m)\int\limits_0^c z^{1/2+\gamma} J_{-1/2-\gamma}(\sigma_m z/c)\, F'_n(z)\,dz. \tag{66}$$

By virtue of the equality

$$z^{1/2+\gamma} J_{-1/2-\gamma}\left(\frac{\sigma_m z}{c}\right) = \frac{z^{2\gamma} c}{\sigma_m}\frac{d}{dz}\left[z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z/c)\right], \tag{67}$$

we rewrite equation (66) in the form

$$f_{nm} = -d_m(c/\sigma_m)\, z^{1/2+\gamma} J_{-1/2-\gamma}(\sigma_m z/c)\, F_n(z)\Big|_{z=0}^{z=c}$$

$$+d_m(c/\sigma_m)^2\int\limits_0^c \frac{d}{dz}\left[z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z/c)\right] z^{2\gamma} F'_n(z)\,dz.$$

Hence, applying the rule of integration by parts again, we have

$$f_{nm} = -d_m \left(c/\sigma_m\right) z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+d_m \left(c/\sigma_m\right)^2 z^{1/2+\gamma} J_{1/2-\gamma} \left(\sigma_m z/c\right) F'_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$-d_m (c/\sigma_m)^2 \int\limits_0^c z^{1/2-\gamma} J_{1/2-\gamma} \left(\sigma_m z/c\right) \left(z^{2\gamma} F'_n \left(z\right)\right)' dz.$$

Taking into account the equalities $\left(z^{2\gamma} F'_n \left(z\right)\right)' = z^{2\gamma} B^z_{\gamma-1/2} F_n \left(z\right)$ and (65), let us rewrite the last equality in the form

$$f_{nm} = -d_m \left(c/\sigma_m\right) z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+d_m \left(c/\sigma_m\right)^2 z^{1/2+\gamma} J_{1/2-\gamma} \left(\sigma_m z/c\right) F'_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+d_m (c/\sigma_m)^3 \int\limits_0^c \frac{d}{dz} \left[z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right)\right] B^z_{\gamma-1/2} F_n \left(z\right) dz. \tag{68}$$

Taking into account formulas (65) and (67), we apply the rule of integration by parts three more times to the integral in (68). As a result, equality (68) takes the form

$$f_{nm} = d_m \Big\{ - \left(c/\sigma_m\right) z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+ \left(c/\sigma_m\right)^2 z^{1/2+\gamma} J_{1/2-\gamma} \left(\sigma_m z/c\right) F'_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+ \left(c/\sigma_m\right)^3 z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) B^z_{\gamma-1/2} F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$- \left(c/\sigma_m\right)^4 z^{1/2+\gamma} J_{1/2-\gamma} \left(\sigma_m z/c\right) \frac{d}{dz} B^z_{\gamma-1/2} F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$- \left(c/\sigma_m\right)^5 z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) \left[B^z_{\gamma-1/2}\right]^2 F_n \left(z\right)\Big|_{z=0}^{z=c}$$
$$+ (c/\sigma_m)^5 \int\limits_0^c z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) \frac{d}{dz} \left[B^z_{\gamma-1/2}\right]^2 F_n \left(z\right) dz \Big\}. \tag{69}$$

Since the integral in (64) converges uniformly with respect to z, all derivatives and operators acting on z functions $F_n \left(z\right)$, passes to functions $f \left(\varphi, z\right)$. By virtue of, $z^{1/2+\gamma} J_{-1/2-\gamma} \left(\frac{\sigma_m z}{c}\right) \in C \left[0, c\right]$, $J_{1/2-\gamma} \left(\sigma_m\right) = 0$ and the conditions of Theorem 2, the non-integral terms in (69) are zero. Hence,

$$f_{nm} = d_m (c/\sigma_m)^5 \int\limits_0^c z^{1/2+\gamma} J_{-1/2-\gamma} \left(\sigma_m z/c\right) \frac{d}{dz} \left[B^z_{\gamma-1/2}\right]^2 F_n \left(z\right) dz.$$

If we use equalities, $z^{1/2+\gamma}J_{-1/2-\gamma}(\sigma_m z/c) = \dfrac{c}{\sigma_m} z^{2\gamma}Z'_m(z)$, then the last equality can be written as

$$f_{nm} = (\sigma_m/c)^{-6} \int\limits_0^c z^{2\gamma}Z'_m(z)\,\frac{d}{dz}\left[B^z_{\gamma-1/2}\right]^2 F_n(z)\,dz. \qquad (70)$$

Using the operator decomposition $B^z_{\gamma-1/2}$, it is easy to see that

$$\frac{\partial}{\partial z}\left[B^z_{\gamma-1/2}\right]^2 f(\varphi,z)$$

$$= \frac{\partial^5}{\partial z^5} f(\varphi,z) + \frac{4\gamma}{z}\frac{\partial^4}{\partial z^4} f(\varphi,z) + \frac{4\gamma^2-8\gamma}{z^2}\frac{\partial^3}{\partial z^3} f(\varphi,z)$$

$$-\frac{12\gamma^2-12\gamma}{z^3}\frac{\partial^2}{\partial z^2} f(\varphi,z) + \frac{12\gamma^2-12\gamma}{z^4}\frac{\partial}{\partial z} f(\varphi,z).$$

Hence, based on the conditions of Theorem 2, it follows that

$$\frac{d}{dz}\left[B^z_{\gamma-1/2}\right]^2 f(\varphi,z) \in C\left(\bar{\Pi}\right).$$

Taking into account this and $z^{2\gamma}Z'_m(z) \in C[0,c]$, we conclude that the integral in (70) exists.

Substituting the function $F_n(z)$, defined by equality (64) into (70), we have

$$f_{nm} = \frac{c^6 d_m}{\omega_n^4 \sigma_m^6} \int\limits_0^c \int\limits_0^{\pi/2} \sin\left(\omega_n\varphi + \frac{\pi}{4}\right) z^{2\gamma}Z'_m(z)$$

$$\times \frac{\partial}{\partial z}\left[B^z_{\gamma-1/2}\right]^2 f_{\varphi\varphi\varphi\varphi}(\varphi,z)\,d\varphi dz. \qquad (71)$$

By virtue of the condition of Theorem 2, it is true that

$$\frac{\partial}{\partial z}\left[B^z_{\gamma-1/2}\right]^2 f(\varphi,z) \in C\left(\bar{\Pi}\right), \quad f_{\varphi\varphi\varphi\varphi}(\varphi,z) \in C\left(\bar{\Pi}\right).$$

Then

$$\frac{\partial}{\partial z}\left[B^z_{\gamma-1/2}\right]^2 f(\varphi,z) \in C\left(\bar{\Pi}\right), \quad f_{\varphi\varphi\varphi\varphi}(\varphi,z) \in C\left(\bar{\Pi}\right).$$

Taking this into account, and

$$\sin\left(\omega_n\varphi + \frac{\pi}{4}\right) z^{1/2+\gamma}J_{-1/2-\gamma}\left(\frac{\sigma_m z}{c}\right) \in C\left(\bar{\Pi}\right),$$

we conclude that the integrand is continuous in $\bar{\Pi}$, and the repeated integral in (71) exists.

Based on the estimates (61), we obtain $|d_{nm}| \le c_{10}\sigma_m$, where $c_{10} = const > 0$. Given this and estimates (56), from (71), we obtain estimates (62). Lemma 3 is proved.

Lemma 4. *For any $n, m \in N$ for functions $\zeta_{nm}(\rho)$, defined by equality (48), it is valid to evaluate at $\rho \in [0,1]$:*

$$|\zeta_{nm}(\rho)| \le c_{11}\omega_n^{-4}\sigma_m^{-4,5}, \tag{72}$$

and at $\rho \in (0,1)$

$$\left|\zeta''_{nm}(\rho) + \frac{1}{\rho}\zeta'_{nm}(\rho)\right| \le c_{12}\left(\sigma_m^2 + \omega_n^2/\rho^2\right)\omega_n^{-4}\sigma_m^{-4,5}, \tag{73}$$

where c_{11}, c_{12} are positive constants.

Proof. It is easy to see that $I_{\omega_n}(\sigma_m\rho/c)$ is an increasing function. By virtue of $\omega_n > 0$ this function has zero at the point $\rho = 0$ and its maximum is the point $\rho = 1$. If we take this into account, then from (48) by virtue of (62), it follows the estimate (72).

Moreover, the function $\zeta_{nm}(\rho)$, defined by equality (48) satisfies the differential equation (24) at $\tilde{\mu} = \mu_n$. Therefore, the equality

$$\zeta''_{nm}(\rho) + (1/\rho)\,\zeta'_{nm}(\rho) = \left(\lambda_m + \mu_n/\rho^2\right)\zeta_{nm}(\rho)\,, \rho \in (0,1)\,. \tag{74}$$

By virtue of evaluation (72), evaluation (73) follows from (74). Lemma 4 has been proved.

Similarly, one can prove the following lemmas.

Lemma 5. *For any $n, m \in N$ functions $X_{nm}(\xi)$, defined by equality (52), estimates (72) and (73) are valid.*

Lemma 6. *For any $n \in N$ for functions $S_n(\varphi)$, defined by equality (37), the estimates are valid*

$$|S_n(\varphi)| \le \sqrt{2}, \quad |S'_n(\varphi)| \le \sqrt{2}\omega_n at \varphi \in [0, \pi/2]\,, \tag{75}$$

$$|S''_n(\varphi)| \le \sqrt{2}\omega_n^2 at \varphi \in (0, \pi/2). \tag{76}$$

The fairness of the estimates (75), (76) follows easily from the property of trigonometric functions.

Lemma 7. *For any $n \in N$ for functions $Y_n(\eta)$, defined by the equality (53), the estimates are valid*

$$|Y_n(\eta)| \le c_{13}, \;\; \eta \ge 1, \quad |Y'_n(\eta)| \le c_{14}\omega_n, \;\; \eta > 1, \tag{77}$$

$$|Y''_n(\eta)| \le c_{15}\omega_n^2, \;\; \eta > 1, \tag{78}$$

where $c_{13}, c_{14}, c_{15} = const > 0$.

Proof. By virtue of $0 < 1/\eta \le 1$, the function $Y_n(\eta)$ is bounded. Using the well-known formula [18]

$$\frac{d}{dx}\left[x^a F(a,b,c;x)\right] = a x^{a-1} F(a+1,b,c;x)\,, \tag{79}$$

from (53), we obtain

$$Y'_n(\eta) = -\frac{\omega_n}{2}\left(\frac{1}{\eta}\right)^{\omega_n/2+1} F\left(\frac{\omega_n}{2}+1, \frac{1+\omega_n}{2}, 1+\omega_n; \frac{1}{\eta}\right). \tag{80}$$

From this equality, by virtue of $0 < 1/\eta < 1$, the second estimate (77) follows.

Once again, using formula (79) from (80), we obtain

$$Y''_n(\eta) = \omega_n/2\,(\omega_n/2+1)\,(1/\eta)^{\omega_n/2+2}\times$$

$$\times F\,(\omega_n/2+2, (1+\omega_n)/2, 1+\omega_n; 1/\eta)\,.$$

From the last equality, by virtue of $0 < 1/\eta < 1$, follows the estimate (78). Lemma 7 has been proved.

Proof of Theorem 2. According to (50) and (51), all terms of series (54) satisfy conditions (4)-(6). Then to prove the theorem it is enough to prove the uniform convergence of series (54) and series, $z^{2\gamma}V_z, V_\varphi$ in $\bar{\Omega}_0$, as well as series, $V_{\rho\rho} + \frac{1}{\rho}V_\rho$, $B^z_{\gamma-1/2}V$, $V_{\varphi\varphi}$, $U_{\xi\xi} + \frac{1}{\xi}U_\xi$, U_η and $U_{\eta\eta}$ in any compacta $K \subset \Omega_0 \cup \Omega_1$.

According to [17], for sufficiently large m for m – one positive root of the equation $J_{1/2-\gamma}(x) = 0$, there is a relation

$$\sigma_m \approx \pi m. \tag{81}$$

For eigenvalues ω_n, approximate equations are valid

$$\omega_n \approx n. \tag{82}$$

According to the estimates (55)-(57), (72), (73), (75), (76), (81) and (82), the series (54), $z^{2\gamma}V_z$, V_φ and in the region $\bar{\Omega}_0$ are estimated by the numerical series, respectively

$$c_{16}\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{m=1}^{\infty}\frac{1}{m^{4+\gamma}},\ c_{17}\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{m=1}^{\infty}\frac{1}{m^4},\ c_{18}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{1}{m^{4+\gamma}}, \tag{83}$$

and the series, $V_{\rho\rho} + \frac{1}{\rho}V_\rho$, $B^z_{\gamma-1/2}V$ and $V_{\varphi\varphi}$ are estimated respectively by numerical series

$$c_{19}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{m^{2+\gamma}},\ c_{20}\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{m=1}^{\infty}\frac{1}{m^{2+\gamma}},\ c_{21}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{m=1}^{\infty}\frac{1}{m^{4+\gamma}}, \tag{84}$$

where c_j, $j = \overline{16, 21}$ are the positive constants.

Since both multipliers of the number series in (83), (84) converge, the series (54) and the series, $z^{2\gamma}V_z$, V_φ converge absolutely and uniformly to $\bar{\Omega}_0$ and the series, $V_{\rho\rho} + \frac{1}{\rho}V_\rho$, $B^z_{\gamma-1/2}V$ and $V_{\varphi\varphi}$ converge on every compact $K \subset \Omega_0$.

Now consider the series (54) in the region $\bar{\Omega}_1$. Given $\omega_n > 0$ it follows that

$$2^{-\omega_n} < c_{22}, \;\; c_{22} = const > 0. \tag{85}$$

According to estimates (55)-(57), (72), (73), (77), (78) and (85), the series (54) and $z^{2\gamma}U_z$ in the region $\bar{\Omega}_1$ are estimated in absolute value by the following products of numerical series, respectively

$$c_{23}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-4-\gamma}, \;\; c_{24}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-4} \tag{86}$$

and the rows, $B^z_{\gamma-1/2}U$, $U_{\xi\xi}+\dfrac{1}{\xi}U_\xi$, U_η and $U_{\eta\eta}$ are numerical rows, respectively

$$c_{25}\sum_{n=1}^{\infty} n^{-4}\sum_{m=1}^{\infty} m^{-2-\gamma}, \;\; c_{26}\sum_{n=1}^{\infty} n^{-2}\sum_{m=1}^{\infty} m^{-2-\gamma}, \tag{87}$$

$$c_{27}\sum_{n=1}^{\infty} n^{-3}\sum_{m=1}^{\infty} m^{-4-\gamma}, \;\; c_{28}\sum_{n=1}^{\infty} n^{-2}\sum_{m=1}^{\infty} m^{-4-\gamma}, \tag{88}$$

where $c_j, \; j = \overline{23, 28}$ are positive constants. Both multipliers of the numerical series in (86)-(88) converge, then the series (54) and $z^{2\gamma}U_z$ converge absolutely and uniformly in to $\bar{\Omega}_1$ and the series, $B^z_{\gamma-1/2}U\left(x,y,z\right)$, $U_{\xi\xi}\left(x,y,z\right) + \dfrac{1}{\xi}U_\xi\left(x,y,z\right)$, $U_\eta\left(x,y,z\right)$ and $U_{\eta\eta}\left(x,y,z\right)$ converge on every compacta of $K \subset \Omega_1$. Therefore, the function $U\left(x,y,z\right)$, defined by row (54), satisfies all the conditions of the problem T. Theorem 2 is proved.

5. **Conclusion**

In this work, in a mixed domain, for which the elliptic part consists of a quarter cylinder and the hyperbolic part of a triangular straight prism, the Tricomi problem is studied for a mixed type equation with a singular coefficient. The method of spectral analysis has been used to prove the unique solvability of the problem posed. The solutions to the considered problem in the areas of hyperbolicity and ellipticity of the equation are constructed in the form of a double series. When justifying the uniform convergence of the constructed series, asymptotic estimates of the Bessel functions of the real and imaginary argument, as well as the properties of hypergeometric Gauss functions, were used. On their basis, estimates were obtained for each member of the series, which made it possible to prove the convergence of the resulting series and its derivatives up to the second order inclusive, as well as the existence theorem in the class of regular solutions.

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