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NONLOCAL CONTROLLABILITY OF MILD SOLUTIONS FOR EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY IN FRÉCHET SPACES

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Abstract

In this paper, we establish the nonlocal controllability of mild solutions of partial functional evolution equations with state-dependent delays in Fréchet spaces. We give sufficient conditions to obtain the nonlocal controllability of mild solutions by using Avramescu's nonlinear alternative to for the sum of compact and contraction operators in Fréchet spaces, combined with semigroup theory.

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1. Introduction

We demonstrate in this paper the controllability of mild solutions defined on the semi-infinite real interval $J := [0, +\infty)$, for a class of first order semilinear functional differential evolution equations with infinite state-dependent delay and with nonlocal conditions in a real Banach space $(E, |\cdot|)$. In Section 3, we consider the following nonlocal semilinear functional differential evolution equation

$$y'(t) - A(t)y(t) = Cu(t) + f(t, y_{\rho(t,y_t)}), \quad a.e.\ t \in J, \tag{1}$$

$$y(t) = \phi(t) - h_t(y), \quad t \in (-\infty, 0], \tag{2}$$

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where $\mathcal{B}$ is an abstract phase space which will be defined later; $\phi \in \mathcal{B}$, $f : J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ and $h_t : \mathcal{B} \to E$ are given functions; the control function $u(\cdot)$ is given in $L^2(J, E)$ is the Banach space of admissible control function; C is a bounded linear operator from E into E and $\{A(t)\}_{t \in J}$ is a family of linear closed (not necessarily bounded) operators from E into E which generates an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $s \leq t$. For any continuous function y and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$, for $\theta \leq 0$. Here $y_t(\cdot)$ represents the history of the state from time $t \leq 0$ up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$. Then in Section 4, we illustrate by an example the previous abstract theory obtained.

Functional and partial functional differential equations have been used to model the evolution of physical, biological and economic systems, where the response of the system depends not only on the current state but also on the system's past. In recent decades, many authors have extensively studied the existence and uniqueness of mild, strong and classical solutions to semilinear functional differential equations using semigroup theory, fixed point arguments, degree theory and non compactness measures. For example, we mentioned the books of Ahmed [4], Pazy [26] and Wu [28]. When the delay is infinite, the concept of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A common choice is a seminorm space satisfying the appropriate axioms, which is introduced by Hale and Kato in [21]. The problem of controllability of linear and nonlinear systems represented by ODEs in finite-dimensional spaces has been extensively studied. Some authors have extended the concept of controllability to infinite-dimensional systems with unbounded operators in Banach spaces. Quinn and Carmichael [27] showed that controllability problems can be transformed into fixed point problems. Using fixed-point parameters, Benchohra *et al.* examined many classes of functional differential equations and inclusions and presented some controllability results in [7]. Baghli *et al.* considered the existence, uniqueness, and controllability of mild solutions to various evolution problems with finite and infinite delay in [2], [10]-[15]. However, in recent years, complex cases where the delay depends on an unknown function have been proposed in modeling. These equations are often referred to as state delay equations. We refer readers to the work of Abada *et al.* [1] and Baghli *et al.* in [5, 6, 9, 16]. More recently, Baghli and Mebarki provided results for the existence of mild solutions to the class of neutral-type integral-differential evolution inclusions constraints with infinite state-dependent delay in [25]. Byszewski introduced the concept of nonlocal constraints in [18, 19] to extend classical constraint-based problems. Nonlocal constraints represent their usefulness in describing certain physical phenomena. Nonlocal constraints are implemented in physics due to their better efficiency compared to classical initial constraints. Furthermore, due to the accuracy of non-local constraints, they are largely involved in boundary value problems. In recent years, several papers have been devoted to the existence of solutions to differential equations with nonlocal conditions as in [17, 24].

Our aim in this paper is to extend Baghli *et al.* controllability results obtained specially in [5] earlier for our nonlocal problem (1)−(2). We use Avramescu's nonlinear alternative method [8] for the sum of compact operators and contraction maps in Fréchet spaces, combined with semigroup theory [4, 26], to provide sufficient conditions for the existence of mildly controllable solutions.

2. Preliminaries

In this section, we introduce notations, definitions and theorems which are used throughout this paper.

Let $C(J,E)$ be the continuous functions space from J into E and $B(E)$ be the all bounded linear operators space from E into E, with the norm: $\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}$.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by: $\|y\|_{L^1} = \int_0^{+\infty} |y(t)|dt$.

Let X be a Fréchet space with a semi-norms family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the semi-norms family $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X.$$

In what follows, we assume that $\{A(t)\}_{t\ge 0}$ is a closed densely defined linear unbounded operators family on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.1. A family $\{U(t,s)\}_{(t,s)\in J\times J}$ of bounded linear operators $U(t,s) : J \times J \to E$ for $s \le t$ is called an evolution system if the following properties are satisfied:

(1) $U(t,t) = I$ where I is the identity operator in E,
(2) $U(t,s)U(s,\tau) = U(t,\tau)$ for $\tau \le s \le t$,
(3) $U(t,s) \in B(E)$, where for every $s \le t$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)y$ is continuous.

In this paper we use the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [21] and follow the terminology used by Hino, Murakami and Naito in [23]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfying the following axioms:

(A_1) If $y : (-\infty, b) \to E,\ b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant $\mathcal{D}$ such that

$$|y(t)| \leq \mathcal{D}\|y_t\|_{\mathcal{B}};$$

(iii) There exist two functions $K(\cdot),\ M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ independent of $y(t)$ with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup_{0 \leq s \leq t} |y(s)| + M(t)\|y_0\|_{\mathcal{B}}.$$

Denote $K_b = \sup_{t \in [0,b]} K(t)$ and $M_b = \sup_{t \in [0,b]} M(t)$.

(A_2) For the function $y(\cdot)$ in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1.

1. (ii) is equivalent to $|\phi(0)| \leq \mathcal{D}\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can check $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From equivalence, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$. This implies necessarily that $\phi(0) = \psi(0)$.

Here is an example of phase spaces. For more details we refer the reader to the book by Hino *et al.* [23].

Example 2.1. Let BC denote the bounded continuous functions space defined from $\mathbb{R}^-$ to E; BUC denote the bounded uniformly continuous functions space defined from $\mathbb{R}^-$ to E;
$C^\infty := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\}$;
$C^0 := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\}$, endowed with the uniform norm $\|\phi\| = \sup_{\theta \leq 0} |\phi(\theta)|$. Then, we have that the spaces BUC, C^∞ and C^0 satisfies assumptions $(A_1) - (A_3)$. However, BC satisfy axioms $(A_1), (A_3)$ not axiom (A_2). □

Set $\mathcal{R}(\rho^-) = \{\rho(s, \Phi) : (s, \Phi) \in J \times \mathcal{B},\ \rho(s, \Phi) \leq 0\}$. We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:b

(H_Φ) The function $t \to \Phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and for every $t \in \mathcal{R}(\rho^-)$, there exists a continuous and bounded function $\mathcal{L}^\Phi : \mathcal{R}(\rho^-) \to (0, +\infty)$ such that $\|\Phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\Phi(t)\|\Phi\|_{\mathcal{B}}$.

Remark 2.2. Continuous and bounded functions verified frequently the condition (H_Φ) (see [23]).

Lemma 2.1. [22] *If $y : (-\infty, b] \to E$ is a function such that $y_0 = \Phi \in \mathcal{B}$, then for each $s \in \mathcal{R}(\rho^-) \cup J$*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + \mathcal{L}^\Phi)\|\Phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|;\ \theta \in [0, \max\{0, s\}]\},$$

where $\mathcal{L}^\Phi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\Phi(t)$.

Proposition 2.1. [5] *From (H_Φ), (A_1) and Lemma 2.1, for all $t \in [0, n]$ and $n \in \mathbb{N}$ we have*

$$\|y_{\rho(t, y_t)}\|_{\mathcal{B}} \leq K_n|y(t)| + \left(M_n + \mathcal{L}^\Phi\right)\|\Phi\|_{\mathcal{B}}.$$

Definition 2.2. A function $f : J \times \mathcal{B} \to E$ is said to be an L^1_{loc}-Carathéodory function if it satisfies:

(i) for each $t \in J$ the function $f(t, \cdot) : \mathcal{B} \to E$ is continuous;
(ii) for each $y \in \mathcal{B}$ the function $f(\cdot, y) : J \to E$ is measurable;
(iii) for every positive integer q there exists $\vartheta_q \in L^1_{loc}(J, \mathbb{R}^+)$ such that $|f(t, y)| \leq \vartheta_q(t)$ for all $\|y\|_{\mathcal{B}} \leq q$ and a.e. $t \in J$.

Definition 2.3. A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that:

$$\|f(x) - f(y)\|_n \leq k_n\|x - y\|_n \quad \text{for all } x, y \in X.$$

Now we introduce the nonlinear alternative used in this paper given by Avramescu in Fréchet spaces which is an extension of the Burton and Kirk alternative given in Banach spaces, we refer to [8] and the references therein.

Theorem 2.1. (Nonlinear Alternative of Avramescu)

Let X be a Fréchet space and let $A, B : X \to X$ be two operators satisfying: A is a compact operator and B is a contraction. Then either one of the following statements holds:

(C1) *The operator $A + B$ has a fixed point;*
(C2) *The set $\left\{x \in X,\ x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0, 1)$.*

3. Semilinear Evolution Equations

In this section, we give a nonlocal controllability results for the problem (1)−(2). Before stating and proving this result, we introduce the definition of mild solutions for the nonlocal problem (1)−(2) and we define its controllability sense.

Definition 3.1. We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of (1)−(2) if $y(t) = \phi(t) - h_t(y)$ for all $t \in (-\infty, 0]$ and y satisfies the following integral equation

$$y(t) = U(t,0)[\phi(0) - h_0(y)] + \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds \quad \text{for each } t \in J. \tag{3}$$

Definition 3.2. The evolution problem (1)−(2) is said to be non locally controllable if for every initial function $\phi \in \mathcal{B}$, $y^\star \in E$ and $n \in \mathbb{N}$, there is some control $u \in L^2([0,n], E)$ such that the mild solution $y(\cdot)$ of (1)−(2) satisfies the terminal condition

$$y(n) + h_n(y) = y^\star. \tag{4}$$

We will need to introduce the following hypotheses which are assumed thereafter:

- $(H1)$ $U(t,s)$ is compact for $t - s > 0$ and there exists a constant $\widehat{M} \geq 1$ such that $\|U(t,s)\|_{B(E)} \leq \widehat{M}$ for every $s \leq t$.
- $(H2)$ There exists a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0, +\infty)$ and such that:
$$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \quad \text{for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$
- $(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J, \mathbb{R}^+)$ such that:
$$|f(t,u) - f(t,v)| \leq l_R(t)\|u - v\|_{\mathcal{B}}$$
for all $u, v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq R$ and $\|v\|_{\mathcal{B}} \leq R$.
- $(H4)$ For each $n \in \mathbb{N}$, the linear operator $W : L^2([0,n], E) \to E$ is defined by $Wu = \int_0^n U(n,s)Cu(s)ds$, has a pseudo invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2([0,n], E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that: $\|C\|_{B(E)} \leq \widetilde{M}$ and $\|\widetilde{W}^{-1}\| \leq \widetilde{M}_1$.
- $(H5)$ For each $n \in \mathbb{N}$, there exists a constant $\sigma_n > 0$ such that
$$|h_t(u) - h_t(v)| \leq \sigma_n \|u - v\|_{\mathcal{B}}$$
for all $u, v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq n$ and $\|v\|_{\mathcal{B}} \leq n$.
- $(H6)$ There exists $\widehat{\sigma}_n > 0$ such that $|h_t(u)| \leq \widehat{\sigma}_n$ for each $t \in J$ and $u \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq n$.

Remark 3.1. For the construction of $\widetilde{W}^{-1}$, see the paper of Quinn and Carmichael [27].

Corollary 3.1. *From* (H_Φ) *and Proposition 2.1, the function* $y:(-\infty,b]\rightarrow E$ *such that* $y(t)=\phi(t)-h_t(y)$ *for* $t\leq 0$ *when* h_t *is satisfying* $(H6)$*, then for each* $t\in[0,n]$ *and* $n\in\mathbb{N}$ *we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}}\leq K_n|y(t)|+\left(M_n+\mathcal{L}_h^{\phi}\right)(\|\phi\|_{\mathcal{B}}+\widehat{\sigma}_n),$$

for $\rho\in\mathcal{R}(\rho^-)$*, and* $\mathcal{L}_h^{\phi}=\sup\limits_{t\in\mathcal{R}(\rho^-)}\mathcal{L}^{\phi(t)-h_t(\cdot)}$.

Consider the following space

$$B_{+\infty}=\{y:\mathbb{R}\rightarrow E: y|_{[0,T]} \text{ continuous for } T>0 \text{ and } y_0\in\mathcal{B}\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau>1$. For every $n\in\mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by: $\|y\|_n:=\sup\limits_{t\in[0,n]}e^{-\tau L_n^*(t)}|y(t)|$ where $L_n^*(t)=\int_0^t\bar{l}_n(s)ds$, $\bar{l}_n(t)=K_n\widehat{M}l_n(t)$ and l_n is the function from $(H3)$. Then $B_{+\infty}$ is a Fréchet space with those family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Theorem 3.1. *Assume that* (H_Φ) *and* $(H1)$*-*$(H6)$ *hold. Then the partial functional evolution equation with infinite state-dependant delay* $(1)-(2)$ *is non locally controllable on* $\mathbb{R}$.

P r o o f. We transform the problem $(1)-(2)$ into a fixed-point problem. For that, let us consider the operator $N:B_{+\infty}\rightarrow B_{+\infty}$ defined by

$$N(y)(t)=\begin{cases}\phi(t)-h_t(y), & \text{if } t\leq 0;\\ U(t,0)[\phi(0)-h_0(y)]+\displaystyle\int_0^t U(t,s)Cu_y(s)ds & \\ +\displaystyle\int_0^t U(t,s)f(s,y_{\rho(s,y_s)})ds, & \text{if } t\in J.\end{cases}$$

□

First, let us introduce the following proposition.

Proposition 3.1. *From the inequalities* (3) *and* (4) *and the hypotheses* $(H1)$, $(H2)$, $(H4)$ *and* $(H6)$, *for all* $t \in [0, n]$ *and* $n \in \mathbb{N}$ *we have*

$$|u_y(t)| \leq \widetilde{M}_1 \Bigg[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n + \widehat{M}\int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}})d\tau \Bigg]. \quad (5)$$

P r o o f. Using assumption $(H4)$, for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \widetilde{W}^{-1}\Bigg[y^\star - h_n(y) - U(n,0)\,[\phi(0) - h_0(y)] - \int_0^n U(n,s) f(s, y_{\rho(s,y_s)})ds \Bigg](t).$$

By the hypotheses $(H1)$, $(H6)$ and using Remark 2.1, we get

$$\begin{aligned} |u_y(t)| &\leq \|\widetilde{W}^{-1}\| \Bigg[|y^\star| + |h_n(y)| + \|U(t,0)\|_{B(E)}\left[|\phi(0)| + |h_0(y)|\right] \\ &\quad + \int_0^n \|U(n,\tau)\|_{B(E)} |f(\tau, y_{\rho(\tau,y_\tau)})| d\tau \Bigg] \\ &\leq \widetilde{M}_1 \Bigg[|y^\star| + \widehat{\sigma}_n + \widehat{M}\,[\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n] + \widehat{M}\int_0^n |f(\tau, y_{\rho(\tau,y_\tau)})| d\tau \Bigg]. \end{aligned}$$

Applying $(H2)$, we get

$$|u_y(t)| \leq \widetilde{M}_1 \Bigg[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n + \widehat{M}\int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}})d\tau \Bigg].$$

□

Using the previous proposition, we shall show that the operator N has a fixed point $y(\cdot)$ which is the mild solution of the nonlocal evolution equation (1)–(2).

P r o o f. For $\phi \in \mathcal{B}$, we define the function $x(\cdot) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leq 0; \\ U(t,0)[\phi(0) - h_0(y)], & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi - h_0(y)$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y(\cdot)$ satisfies (3), we can decompose it as $y(t) = z(t) + x(t)$ for $t \in J$, which implies $y_t = z_t + x_t$, for every $t \in J$. The function $z(\cdot)$ satisfies $z_0 = 0$ and for $t \in J$, we get

$$\begin{aligned} z(t) = &\int_0^t U(t,s)Cu_{z+x}(s)ds \\ &+ \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds. \end{aligned}$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}$. We define for $t \in J$ the operators $F,\ G : B^0_{+\infty} \to B^0_{+\infty}$ by $F(z)(t) = \int_0^t U(t,s)Cu_{z+x}(s)ds$ and $G(z)(t) = \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds$.

Clearly the operator N has a fixed point is equivalent to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in several steps.

Step 1: F is continuous. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_k \to z$ in $B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H4)$, we get for every $t \in [0, n]$

$$\begin{aligned} &|F(z_k)(t) - F(z)(t)| \\ &\leq \int_0^t \|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z_k+x}(s) - u_{z+x}(s)|ds \\ &\leq \widehat{M}\widetilde{M}\int_0^t |u_{z_k+x}(s) - u_{z+x}(s)|ds. \end{aligned}$$

Using the hypotheses $(H1)$, $(H4)$, $(H5)$ and assumption (A_1), we get

$$\begin{aligned} |u_{z_k+x}(s) - u_{z+x}(s)| \leq &\|\widetilde{W}^{-1}\|\Big[|h_n(z_k+x) - h_n(z+x)| \\ &+ \|U(s,0)\|_{B(E)}|h_0(z_k+x) - h_0(z+x)| \\ &+ \int_0^n \|U(n,\tau)\|_{B(E)}|f(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) \\ &\qquad - f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})|d\tau\Big] \\ \leq &\widetilde{M}_1\Big[\sigma_n(\widehat{M}+1)K_n|z_k(s) - z(s)| \\ &+ \widehat{M}\int_0^n |f(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)}) \\ &\qquad - f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})|d\tau\Big]. \end{aligned}$$

Then

$$|F(z_k)(t) - F(z)(t)| \le \widehat{M}\widetilde{M}\widetilde{M}_1 \int_0^t \sigma_n(\widehat{M}+1)K_n|z_k(s) - z(s)|ds$$
$$+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 \int_0^t \int_0^n |f(\tau, z_{k\rho(\tau,z_{k\tau}+x_\tau)} + x_{\rho(\tau,z_{k\tau}+x_\tau)})$$
$$- f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})|d\tau ds$$
$$\le \sigma_n K_n \widehat{M}\widetilde{M}\widetilde{M}_1(\widehat{M}+1)\int_0^t |z_k(s) - z(s)|ds$$
$$+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n \int_0^n |f(s, z_{k\rho(s,z_{ks}+x_s)} + x_{\rho(s,z_{ks}+x_s)})$$
$$- f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.$$

Since f is Caratheodory, by the Lebesgue dominated convergence theorem, we obtain $|F(z_k)(t) - F(z)(t)| \to 0$ as $k \to +\infty$. Thus F is continuous.

Step 2: F maps bounded sets of $B^0_{+\infty}$ into bounded sets. Indeed, it is enough to show that for any $d > 0$, there exists a positive constant ℓ such that for each $z \in \mathcal{B}_d$ $\mathcal{B}_d = \{z \in B^0_{+\infty} : \|z\|_n \le d\}$ one has $\|F(z)\|_n \le \ell$. Let $z \in B_d$. By the hypotheses $(H1)$, $(H4)$ and the inequality (5), we have for each $t \in [0, n]$

$$|F(z)(t)| \le \int_0^t \|U(t,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|ds$$
$$\le \widehat{M}\widetilde{M}\int_0^t \widetilde{M}_1\Bigg[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n$$
$$+\widehat{M}\int_0^n p(\tau)\psi(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}})d\tau\Bigg] ds$$
$$\le \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n\right]$$
$$+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})ds.$$

Using Corollary 3.1 and the fact that $x_0 = \phi - h_0(z+x)$, we get

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \le \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}$$
$$\le K_n|z(s)| + (M_n + \mathcal{L}^\phi_h)\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}^\phi_h)\|x_0\|_{\mathcal{B}}$$
$$\le K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + K_n\|U(s,0)\|_{B(E)}|h_0(z+x)|$$
$$+ (M_n + \mathcal{L}^\phi_h)\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right)$$
$$\le K_n|z(s)| + K_n\widehat{M}\left(\mathcal{D}\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right) + (M_n + \mathcal{L}^\phi_h)\left(\|\phi\|_{\mathcal{B}} + \widehat{\sigma}_n\right)$$
$$\le K_n|z(s)| + (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}^\phi_h)\|\phi\|_{\mathcal{B}} + (K_n\widehat{M} + M_n + \mathcal{L}^\phi_h)\widehat{\sigma}_n.$$

Set $c_n := (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi})\|\phi\|_{\mathcal{B}} + (K_n\widehat{M} + M_n + \mathcal{L}_h^{\phi})\widehat{\sigma}_n$ to get

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_n|z(s)| + c_n. \tag{6}$$

Since $z \in B_d$, then we have

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_n d + c_n := \delta_n. \tag{7}$$

We get, using the nondecreasing character of ψ, for each $t \in [0,n]$

$$\begin{aligned}|F(z)(t)| &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n\right] \\ &+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi\left(K_n|z(s)| + c_n\right)ds \\ &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n\right] \\ &+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\psi(\delta_n)\|p\|_{L^1} := \ell_n.\end{aligned}$$

Thus there exists a positive number ℓ_n such that $\|F(z)\|_n \leq \ell_n$. Hence $F(B_d) \subset B_{\ell_n}$.

Step 3: F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1,\ \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| &\leq \int_0^{\tau_1}\Big|[U(\tau_2,s) - U(\tau_1,s)]Cu_{z+x}(s)\Big|ds \\ &+ \int_{\tau_1}^{\tau_2}|U(\tau_2,s)Cu_{z+x}(s)|\,ds \\ &\leq \int_0^{\tau_1}\|U(\tau_2,s) - U(\tau_1,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|ds \\ &+ \int_{\tau_1}^{\tau_2}\|U(\tau_2,s)\|_{B(E)}\|C\|_{B(E)}|u_{z+x}(s)|ds.\end{aligned}$$

By the inequalities (5) and (7) and using the nondecreasing character of ψ, we get

$$\begin{aligned}|u_{z+x}(t)| \leq \widetilde{M}_1\Big[&|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n \\ &+ \widehat{M}\psi(\delta_n)\|p\|_{L^1}\Big] := \omega_n.\end{aligned} \tag{8}$$

Then

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| &\leq \widetilde{M}\omega_n\int_0^{\tau_1}\|U(\tau_2,s) - U(\tau_1,s)\|_{B(E)}ds \\ &+ \widetilde{M}\omega_n\int_{\tau_1}^{\tau_2}\|U(\tau_2,s)\|_{B(E)}ds.\end{aligned}$$

Noting that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \to 0$ independently of $z \in B_d$. The right-hand side of the above inequality tends to zero as

$\tau_2 - \tau_1 \to 0$. Since $U(t,s)$ is a strongly continuous and compact operator for $t > s$ this is implies the continuity in the uniform operator topology (see [3, 26]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_d$ we define

$$\begin{aligned} F_\epsilon(z)(t) &= \int_0^{t-\epsilon} U(t,s) C u_{z+x}(s) ds \\ &= U(t, t-\epsilon) \int_0^{t-\epsilon} U(t-\epsilon, s) C u_{z+x}(s) ds. \end{aligned}$$

Note that the set $\left\{ \int_0^{t-\epsilon} U(t-\epsilon, s) C u_{z+x}(s) ds : z \in B_d \right\}$ is bounded. Since $U(t,s)$ is a compact operator, $Z_\epsilon(t) = \{F_\epsilon(z)(t) : z \in B_d\}$ is a precompact set in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover using (8), we have

$$\begin{aligned} |F(z)(t) - F_\epsilon(z)(t)| &\le \int_{t-\epsilon}^t \|U(t,s)\|_{B(E)} \|C\|_{B(E)} |u_{z+x}(s)| ds \\ &\le \widetilde{M} \omega_n \int_{t-\epsilon}^t \|U(t,s)\|_{B(E)} ds. \end{aligned}$$

Then $|F(z)(t) - F_\epsilon(z)(t)| \to 0 \quad as\ \epsilon \to 0$. Therefore there are arbitrary closed pre-compact sets to the set $\{F(z)(t) : z \in B_d\}$. Hence the set $\{F(z)(t) : z \in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.

Step 4: G is a contraction. Indeed, consider z, $\overline{z} \in B^0_{+\infty}$. By $(H1)$ and $(H3)$ for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned} |G(z)(t) - G(\overline{z})(t)| &\le \int_0^t \|U(t,s)\|_{B(E)} \\ &\times \left| f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) \right| ds \\ &\le \int_0^t \widehat{M} l_n(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}\ ds. \end{aligned}$$

Using inequality (6), we obtain

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \widehat{M} K_n l_n(s) |z(s) - \overline{z}(s)| ds \\
&\leq \int_0^t \left[\overline{l}_n(s) e^{\tau L_n^*(s)}\right] e^{-\tau L_n^*(s)} |z(s) - \overline{z}(s)| ds \\
&\leq \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau}\right]' ds \|z - \overline{z}\|_n \\
&\leq \frac{1}{\tau} e^{\tau L_n^*(t)} \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore, $\|G(z) - G(\overline{z})\|_n \leq \frac{1}{\tau}\|z - \overline{z}\|_n$. So, the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5: To apply Theorem 2.1, we must check (C2): i.e. it remains to show that the following set is bounded

$$\Gamma = \left\{ z \in B^0_{+\infty} : z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right) \text{ for some } 0 < \lambda < 1 \right\}.$$

Let $z \in \Gamma$. By $(H1)$, $(H2)$, $(H4)$ and the inequality (5), we have for each $t \in [0, n]$

$$\begin{aligned}
|z(t)| &\leq \lambda \int_0^t \|U(t,s)\|_{B(E)} \|C\|_{B(E)} |u_{z+x}(s)| ds \\
&+ \lambda \int_0^t \|U(t,s)\|_{B(E)} \left| f\left(s, \frac{z_{\rho(s, \frac{z_s}{\lambda} + x_s)}}{\lambda} + x_{\rho(s, \frac{z_s}{\lambda} + x_s)}\right)\right| ds \\
&\leq \lambda \widehat{M} \widetilde{M} \int_0^t \widetilde{M}_1 \Big[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n \\
&\qquad + \widehat{M} \int_0^n p(\tau)\psi(\|z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}\|_{\mathcal{B}}) d\tau \Big] ds \\
&+ \lambda \widehat{M} \int_0^t p(s)\psi\left(\left\| \frac{z_{\rho(s, \frac{z_s}{\lambda} + x_s)}}{\lambda} + x_{\rho(s, \frac{z_s}{\lambda} + x_s)} \right\|_{\mathcal{B}} \right) ds \\
&\leq \lambda \widehat{M} \widetilde{M} \widetilde{M}_1 n \left[|y^\star| + \widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + (\widehat{M}+1)\widehat{\sigma}_n \right] \\
&+ \lambda \widehat{M}^2 \widetilde{M} \widetilde{M}_1 n \int_0^n p(s)\psi(\|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}}) ds \\
&+ \lambda \widehat{M} \int_0^t p(s)\psi\left(\left\| \frac{z_{\rho(s, \frac{z_s}{\lambda} + x_s)}}{\lambda} + x_{\rho(s, \frac{z_s}{\lambda} + x_s)} \right\|_{\mathcal{B}} \right) ds.
\end{aligned}$$

Using Corollary 3.1 and inequality (6), we obtain

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} \leq \frac{1}{\lambda}\|z_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}+\|x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}$$
$$\leq \frac{K_n|z(s)|}{\lambda}+(M_n+\mathcal{L}_h^{\phi})\|z_0\|_{\mathcal{B}}+K_n|x(s)|+(M_n+\mathcal{L}_h^{\phi})\|x_0\|_{\mathcal{B}}$$
$$\leq \frac{K_n|z(s)|}{\lambda}+K_n\|U(s,0)\|_{B(E)}|\phi(0)|+K_n\|U(s,0)\|_{B(E)}|h_0(z+x)|$$
$$+(M_n+\mathcal{L}_h^{\phi})\left(\|\phi\|_{\mathcal{B}}+\widehat{\sigma}_n\right)$$
$$\leq \frac{K_n|z(s)|}{\lambda}+(K_n\widehat{M}\mathcal{D}+M_n+\mathcal{L}_h^{\phi})\|\phi\|_{\mathcal{B}}+(K_n\widehat{M}+M_n+\mathcal{L}_h^{\phi})\widehat{\sigma}_n.$$

Then, we get

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} \leq \frac{K_n|z(s)|}{\lambda}+c_n. \tag{9}$$

By inequality (6) and the previous one and the nondecreasing character of ψ, we obtain

$$|z(t)| \leq \lambda\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star|+\widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)\widehat{\sigma}_n\right]$$
$$+\lambda\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi\left(K_n|z(s)|+c_n\right)ds$$
$$+\lambda\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n|z(s)|}{\lambda}+c_n\right)ds.$$

Consider the function $\widetilde{u}(t) := \sup_{\theta\in[0,t]}|z(\theta)|$. Then by the nondecreasing character of ψ, we get for $t\in[0,n]$

$$\frac{K_n\widetilde{u}(t)}{\lambda}+c_n \leq c_n+K_n\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star|+\widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)\widehat{\sigma}_n\right]$$
$$+K_n\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(K_n\widetilde{u}(s)+c_n)ds$$
$$+K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds.$$

Set $\alpha_n := c_n+K_n\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|y^\star|+\widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}}+(\widehat{M}+1)\widehat{\sigma}_n\right]$.

By the nondecreasing character of ψ and for $\lambda<1$, we obtain

$$\frac{K_n\widetilde{u}(t)}{\lambda}+c_n \leq \alpha_n+K_n\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds$$
$$+K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\left\{\frac{K_n\widetilde{u}(s)}{\lambda} + c_n : 0 \le s \le t\right\}, \quad t \in J.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = \frac{K_n\widetilde{u}(t^\star)}{\lambda} + c_n$. If $t^\star \in [0,n]$, by the previous inequality, we have for $t \in [0,n]$

$$\begin{aligned}\mu(t) &\le \alpha_n + K_n\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\psi(\mu(s))ds \\ &+ K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))ds \\ &\le \alpha_n + K_n\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1)\int_0^n p(s)\psi(\mu(s))ds.\end{aligned}$$

Consequently,

$$\frac{\|z\|_n}{\alpha_n + K_n\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1)\psi(\|z\|_n)\|p\|_{L^1}} \le 1.$$

Then this shows that the set Γ is bounded, i.e. the statement (C2) in Theorem 2.1 does not hold. Then the Avramescu nonlinear alternative [8] implies that (C1) holds: i.e. the operator $F+G$ has a fixed-point $z^\star$. Then, there exists at least $y^\star(t) = z^\star(t) + x(t),\ t \in \mathbb{R}$ which is a fixed point of the operator N, which is a mild solution of the nonlocal problem (1)−(2). Thus the evolution system (1)−(2) is non locally controllable on $\mathbb{R}$. □

4. **Example**

We give in this section an example to illustrate the previous results. Consider the following control problem given by the partial functional differential equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + a_0(t,\xi)v(t,\xi) + d(\xi)u(t) \\ + \displaystyle\int_{-\infty}^0 a_1(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\eta)|v(t,\eta)|^2 d\eta\right), \xi\right] ds, \\ \qquad\qquad\qquad\qquad t \ge 0,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \qquad\qquad t \ge 0, \\ v(\theta,\xi) + \displaystyle\sum_{i=1}^p c_i v(\theta + t_i, \xi) = v_0(\theta,\xi), \qquad \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{10}$$

where $a_0 : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is a given function such that $a_0(\cdot,\xi)$ is continuous and $a_0(t,\cdot)$ is uniformly Hölder continuous in t (see [20]); $a_1 : \mathbb{R}^- \to \mathbb{R}$; $\rho_1 : \mathbb{R}^+ \to \mathbb{R}$; $\rho_2 : \mathbb{R} \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$ and $v_0 : \mathbb{R}^- \times [0,\pi] \to \mathbb{R}$ are continuous functions. $c_i,\ i = 1,\cdots,p$, are given constants and $0 < t_1 < \cdots < t_p < +\infty$.

Let $E = L^2([0,\pi],\mathbb{R})$, $u(\cdot) : \mathbb{R}^+ \to E$ is a given control and $d : [0,\pi] \to E$ is a continuous function. Consider the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with domain

$$D(A) := \{w \in E : w'' \in E, w(0) = w(\pi) = 0\}.$$

Thus A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2,\ n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, z_n)z_n \ x \in E, t \geq 0.$$

It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$. On the domain $D(A)$, we define the operator $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(\cdot)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}$, $\xi \in [0,\pi]$, it follows that the system $u'(t) = A(t)u(t) \quad t \geq s,\ u(s) = x \in E$, has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)\exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that $\|U(t,s)\| \leq e^{-(1+\delta_0)(t-s)}$ for every $s \leq t$.

Set $\mathcal{B} = BUC(\mathbb{R}^-, E)$ the space of bounded uniformly continuous functions defined from $\mathbb{R}^-$ to E endowed with the uniform norm $\|\phi\| = \sup_{\theta\in\mathbb{R}^-} |\phi(\theta)|$.

THEOREM 4.1. *Let $\phi \in \mathcal{B}$. Assume that condition (H_Φ) holds and the functions $d : [0,\pi] \to E$, $\rho_1 : \mathbb{R}^+ \to \mathbb{R}$, $\rho_2 : \mathbb{R} \to \mathbb{R}$, $a_1 : \mathbb{R}^- \to \mathbb{R}$, $a_2 : [0,\pi] \to \mathbb{R}$ and $v_0 : \mathbb{R}^- \times [0,\pi] \to \mathbb{R}$ are continuous. Then the partial differential equation* (10) *is non locally controllable on* $\mathbb{R}$.

P r o o f. From the assumptions, we have the following well defined functions for each $\xi \in [0,\pi]$:

for $t \in \mathbb{R} : y(t)(\xi) = v(t,\xi)$;
for $t \geq 0 : f(t,\psi)(\xi) = \int_{-\infty}^0 a_1(s)\psi(s,\xi)ds$, $Cu(t)(\xi) = d(\xi)u(t)$,
for $u \in \mathbb{R},\ d(\xi) \in E$, and for $C \in B(\mathbb{R}, E)$;
for $t \leq 0 : \rho(t,\psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\eta)|\psi(0,\xi)|^2 d\eta\right)$,
$h_t(v)(\xi) = \sum_{j=1}^p c_j v(t+t_j,\xi)$, and $\phi(t)(\xi) = v_0(t,\xi)$

which permit to transform system (10) into the abstract system (1)−(2). Moreover, the function f is bounded linear operator. Then, the nonlocal controllability of mild solutions can be deduced from a direct application of Theorem 3.1 and the conclusion of our theorem hold. □

From Remark 2.2, we have the following result.

COROLLARY 4.1. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then the system* (10) *is non locally controllable on* $\mathbb{R}$.

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