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ACTING OF COVARIANT FUNCTORS ON MAPPINGS

Boburmirzo M. Toshbuvayev

Fergana State University

Fergana - 100150, UZBEKISTAN

Abstract

In this paper, exponential space and its properties, the topological properties of exponential spaces are studied. It is known that the exponential space in category of compact space generates covariant functor. It is checked the impact of this functor into some types of mappings.

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Key Words and Phrases: almost-open mapping, pseudo-open mapping, covariant functor, τ-closed topological space

1. Introduction

Throughout this paper, exponential space and its properties, the topological properties of exponential spaces are studied. It is known that the exponential space in category of compact space generates covariant functor. It is checked the impact of this functor into some types of mappings.

Let us recall that a cover of a set X is a family $\{A_s\}_{s\in S}$ of subset of X such that $\bigcup_{s\in S} A_s = X$, and that - if X is a topological space- $\{A_s\}_{s\in S}$ is an open (closed) cover of X if all sets A_S are open (closed). We say that a cover $\beta = \{B_t\}_{t\in T}$ is a refinement of another cover $A = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $B_t \subset A_s$; in this situation we say also that β refines A. A cover $A' = \left\{A_{s'}\right\}_{s\in S'}$ of X is a subcover of another cover $A = \{A_s\}_{s\in S}$ of X if $S' \subset S$ and $A'_s = A_s$ for every $s \in S'$. In particular, any subcover is a refinement.

A topological space X is called a compact space if X is a Hausdorff space and every open cover of X has a finite subcover, i.e., if for every open cover

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$\{U_s\}_{s\in S}$ of the space X there exists a finite set $\{S_1, S_2, ..., S_k\} \subset S$ such that $X = U_{S_1} \cup U_{S_2} \cup, ..., \cup U_{S_k}$, [3].

Definition 1.1. Let (X, τ) and (Y, τ') be two topological space; a mapping f of X to Y is called continuous if $f^{-1}(U) \in \tau$ for any $U \in \tau'$, i.e., if the inverse image of any open subset of Y is open in X. The fact that f is a continuous of X to Y will be often written in symbols as $f : X \to Y$, [1].

Definition 1.2. A mapping f is called an almost-open, if for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y for each neighborhood U of x, [1].

Definition 1.3. A mapping f is called pseudo-open if for each $y \in Y$ and each neighborhood U of $x \in f^{-1}(y)$ in X, $f(U)$ is a neighborhood of y in Y.

Proposition 1.1. *Let $f : X \to Y$ be a mapping. If X is first countable (especially, if X is metric), then the following are equivalent* ([1])*:*

1. *f is an almost-open mapping;*
2. *f is an almost weak-open mapping;*
3. *f is an almost sn-open mapping;*
4. *f is a 1-sequence-covering, quotient mapping.*

Definition 1.4. A covariant functor $F : Comp \to Comp$ acting in the category of compact is called normal, if it

1) preserves the weight;
2) preserves singletons and empty set;
3) monomorphic (preserves embeddings);
4) epimorphic (preserves surjections);
5) preserves intersections of closed subsets;
6) preserves inverse images;
7) is continuous with respect to inverse limits.

2. **Acting of covariant functors on mappings**

A covariant functor $F : Comp \to Comp$ is said to be weakly normal, if it satisfies all the properties from Definition 1 except the inverse image preserving property.

Let $F : Comp \to Comp$ be an arbitrary normal functor and let $X \in Tych$. Put $F_\beta(X) = \{x \in F(\beta X) : supp(x) \subseteq X\}$. For a morphism $f : X \to Y$ in the category $Tych$ put $F\beta(f) : F(\beta f) \mid F_\beta(X)$, where $\beta f : \beta(X) \to \beta(Y)$ is the Stone-Czech extension of the mapping f. Using normality of the functor

F one can show that $F_\beta(f)$ is well-defined, i.e. $F(\beta f)F_\beta(X) \subseteq F_\beta(Y)$. It is easy to see that F_β is a covariant functor in the category $Tych$.

Let X be a topological T_1-space. The set of all non-empty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, ..., U_n\rangle = \left\{ F: \ F \in \exp(X), \ F \subset \bigcup_{i=1}^{n} U_i, \ F \bigcap U_i \neq \emptyset, \ i = 1, 2, ..., n \right\},$$

where $U_1, ..., U_n$ are open subsets of X, generates a base of the topology on the set $\exp(X)$. This topology is called the Vietoris topology. The set $\exp(X)$ with the Vietoris topology is called exponential space or the hyperspace of a space X, [3].

Corollary 2.1. *The mapping* $\exp_n f$ *is almost-open for every almost-open mapping* $f: X \to Y$ *(see Definition 2).*

P r o o f. Let $f: X \to Y$ be a almost-open mapping. For an arbitrary $y \in Y$ there exist such element $x_i \in X$ and the image $f(U_{x_i})$ of an arbitrary neighborhood U_{x_i} is a neighborhood of y. Therefore, for an arbitrary $y \in Y$ there exist $f^{-1}(y)$ and $y \in Int(f(U_{x_i}))$, $i = \overline{1, n}$.

Let us choose an arbitrary $C \in \exp_n X$. Here $C = \{y_1, y_2, ..., y_n\}$

$$(\exp_n f)^{-1}(C) = (\exp_n f)^{-1}(y_1, y_2, ..., y_n). \tag{1}$$

Since $\exp_n$ is a normal functor, then (1) is equal to $\exp_n(f^{-1}(y_1, y_2, ..., y_n))$.

It follows that there is such a set $C' = \{x_1, x_2, ..., x_n\} \in \exp_n X$, and given $x_i \in f^{-1}(y_i)$ $i = \overline{1, n}$, so that the relation

$$C' = \{x_1, x_2, ..., x_n\} \subset (\exp_n f)^{-1}(\{y_1, y_2, ..., y_n\})$$

holds.

Let it be $C' \in O\langle U_1, U_2, ..., U_n\rangle$. It is possible to say $x_1 \in U_1, x_2 \in U_2, ..., x_n \in U_n$ without loss of generality.

$f(x_i) \in Int(f(U_i))$ $i = \overline{1, n}$, $Intf(U_i) = V_i$ shows that $C \in O\langle V_1, V_2, ..., V_n\rangle$.

$\exp f(O\langle U_1, U_2, ..., U_n\rangle) = O\langle f(U_1), f(U_2), ..., f(U_n)\rangle$, and since $Intf(U_i) \subset f(U_i), \forall V_i \subset f(U_i), V_i \in O\langle V_1, V_2, ..., V_n\rangle$,

$$O\langle V_1, V_2, ..., V_n\rangle \subset \exp_n f(O\langle U_1, U_2, ..., U_n\rangle)$$

it follows that the relationship holds. □

Corollary 2.2. *The mapping* $\exp_n f$ *is pseudo-open for every pseudo-open mapping* $f: X \to Y$.

P r o o f. Let $f : X \to Y$ is a pseudo-open mapping. For an arbitrary $y \in Y$ there exists such elements $x_i \in X$ and the image $f(U_{x_i})$ of an arbitrary neighborhood U_{x_i} is a neighborhood of y. Therefore, for an arbitrary $y \in Y$ there exist $f^{-1}(y)$ and $y \in Int(f(U_{x_i}))$, $i = \overline{1,n}$.

Let us choose an arbitrary $C \in \exp_n X$. Here $C = \{y_1, y_2, ..., y_n\}$,

$$(\exp_n f)^{-1}(C) = (\exp_n f)^{-1}(y_1, y_2, ..., y_n). \quad (2)$$

Since $\exp_n$ is a normal functor, then (2) is equal to $\exp_n(f^{-1}(y_1, y_2, ..., y_n))$.

It follows that there is such a set $C'_k = \left\{x_1^k, x_2^k, ..., x_n^k\right\} \in \exp_n X$, and given $x_i \in f^{-1}(y_i)$ $i = \overline{1,n}$, it follows that the relation

$$C'_k = \left\{x_1^k, x_2^k, ..., x_n^k\right\} \subset (\exp_n f)^{-1}(\{y_1, y_2, ..., y_n\})$$

holds.

Let it be $C' \in O\langle U_1, U_2, ..., U_n\rangle$. It is possible to say $x_1^k \in U_1, x_2^k \in U_2, ..., x_n^k \in U_n$ without changing the generality. $f(x_i^k) \in Int(f(U_i))$ $i = \overline{1,n}$, $Int f(U_i) = V_i$ shows that $C \in O\langle V_1, V_2, ..., V_n\rangle$. $\exp cf(O\langle U_1, U_2, ..., U_n\rangle) = O\langle f(U_1), f(U_2), ..., f(U_n)\rangle$ equality, and since $Int f(U_i) \subset f(U_i), \forall V_i \subset f(U_i), V_i \in O\langle V_1, V_2, ..., V_n\rangle$,

$$O\langle V_1, V_2, ..., V_n\rangle \subset \exp_n f(O\langle U_1, U_2, ..., U_n\rangle)$$

the relationship follows. □

3. On τ-closed subsets of hyperspaces. On the functor of closed sets of finitely many components.

In [3] it is proven that the union of compact subspace in hyperspace is closed in initial topological space.

DEFINITION 3.1. A subset A of a topological space X is called τ-closed if for some $B \subset A$, $|B| \leq \tau$ we have $[B] \subset A$, [6].

Recall that a topological space X is called τ-bounded (see [6]), if the closure in X of every subset of cardinality at most τ is compact.

COROLLARY 3.1. *Let X be a topological space. Then the set $\Gamma = \{F \in \exp X : F \cap A \neq \emptyset\}$ is τ-closed, if $A \subset X$ is τ-closed in X,* [5].

COROLLARY 3.2. *Let X be an infinite regular space, then $\cup\beta$ is τ-closed in X for every τ-bounded subspace β of the hyperspace $\exp X$,* [6].

For $X \in Comp$ let us consider the spaces $\exp_n(X)$ and $C_n(X)$. Put $f_X(F) = F$ for every $F \in \exp_n(X)$. It is clear that $f_X(F) \in C_n(X)$, i.e. the map $f_X : \exp_n(X) \to C_n(X)$ is an embedding. Commutability of the following diagram is obvious:

$$\begin{array}{ccc} \exp_n(X) & \xrightarrow{f_X} & C_n(X) \\ {\scriptstyle \exp_n(f)}\downarrow & & \downarrow{\scriptstyle C_n(f)} \\ \exp_n(Y) & \xrightarrow{f_Y} & C_n(Y) \end{array},$$

where $f : X \to Y$ is a continuous mapping and $X, Y \in Comp$.

From the above discussion we directly obtain the following

Corollary 3.3. *The functor* $\exp_n : Comp \to Comp$ *is a subfunctor of* $C_n : Comp \to Comp$. *Moreover, the functor* $C_n : Comp \to Comp$ *is a subfunctor of* $\exp : Comp \to Comp$, [5].

References

[1] X. Ge, Notes on almost open mappings, *Matematički Vesnik*, **233**, No 60 (2008), 181-186.

[2] R.B. Beshimov, D.N. Georgiou, N.K. Mamadaliev, On τ-bounded spaces and hyperspaces, *Filomat*, **36**, No 1 (2021), 187-193.

[3] V.V. Fedorchuk, V.V. Filippov, *General Topology*, Moscow (2006).

[4] O. Okunev, A.R. Paramo, Functional tightness, R-quotient mappings and products, *Topology and its Applications*, **228** (2017), 236-242.

[5] D.N. Georgiou, N.K. Mamadaliev, R.M. Zhuraev, A note on functional tightness and minitightness of space of the G-permutation degree, *Comment. Math. Univ. Carolinae*, **64**, No 1 (2023), 97-108.

[6] R.B. Beshimov, N.K. Mamadaliyev, On functional tightness and local density of the space of probability measures, *Topology and its Applications*, **329** (2023).

[7] R.B. Beshimov, F.G. Mukhamadiev, N.K. Mamadaliev, The local density and the local weak density of hyperspaces, *Int. J. Geom.*, **4**, No 1 (2015), 42-49.