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INTEGRATION OF THE LOADED NEGATIVE ORDER KORTEWEG-DE VRIES EQUATION IN THE CLASS OF PERIODIC FUNCTIONS

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Abstract

In this paper, we consider the loaded Korteweg-de Vries equation of negative order in the class of periodic functions corresponding to the eigenvalues of the corresponding spectral problem. It is shown that the considered equation can be integrated by the method of the inverse spectral problem. The evolution of the spectral data of the Sturm-Liouville operator with a periodic potential associated with the solution of the considered equation is determined. The obtained results make it possible to apply the inverse problem method for solving the loaded Korteweg-de Vries equation of negative order in the class of periodic functions corresponding to the eigenvalues of the corresponding spectral problem.

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1. Introduction

The Korteweg-de Vries (KdV) equation is one of the representatives of the class of completely integrable nonlinear partial differential equations, which is of great practical importance. The complete integrability of this equation by the inverse problem method, in the class of rapidly decreasing functions, was first established in [1]. The works [2, 3, 4, 5, 6, 7, 8] are devoted to the investigation of the KdV equation in the class of finite-zone periodic and quasi-periodic functions. In [9], the KdV equation with a self-consistent source was considered in the class of rapidly decreasing functions, and the KdV equation with a self-consistent source in the class of periodic functions was studied in [10].

In the work [11] it was integrated the Korteweg-de Vries equation with a loaded term in the class of periodic functions. The works [12, 13] are devoted to the studies of the nonlinear Schrödinger equation and the modified Korteweg–de Vries equation with a loaded term in the class of periodic functions.

The (G'/G) - expansion method was used to integrate the loaded Korteweg-de Vries (KdV) equation and the loaded modified Korteweg-de Vries (mKdV) equation in [14, 15, 16].

Most of the studies concerning the study of integrable equations with a self-consistent source are related to non-linear evolutionary equations of positive order.

Works [17, 18] are devoted to the study of the KdV equation of negative order. In particular, J.M. Verosky [17], while studying symmetries and negative powers of a recursive operator, obtained the following KdV equation of negative order:

$$\begin{cases} q_t = p_x \\ p_{xxx} + 4qp_x + 2q_x p = 0. \end{cases} \tag{1}$$

S.Y. Lou [18] presented additional symmetries based on the invertibility of the recursive operator of the KdV system and, in particular, derived the KdV equation of negative order in the following form

$$q_t = 2pp_x, \quad p_{xx} + qp = 0 \;\Leftrightarrow\; \left(\frac{p_{xx}}{p}\right)_t + 2pp_x = 0. \tag{2}$$

The study of integrable hierarchies of negative order plays a significant role in the theory of cusp solitons [19, 20]. In [21] it was studied the hierarchy of the KdV equation of negative order, in particular, equations (1) and (2).

In [22, 23, 24, 25, 26] it was investigated the Hamiltonian structure, an infinite set of conservation laws, N-soliton, quasi-periodic wave solutions for the KdV equation of negative order.

In [34, 35], the KdV equation of negative order with a self-consistent integral source was studied in the class of periodic functions.

In this paper, the method of the inverse spectral problem is applied to the integration of the loaded Korteweg-de Vries equation of negative order in the class of periodic functions.

Consider the following loaded Korteweg-de Vries equation of negative order

$$\begin{cases} q_t = -2pp_x + \gamma(t) \cdot q\,|_{x=0} \cdot q_x \\ pq - p_{xx} = 0 \end{cases} ,t > 0, \quad x \in R^1, \tag{3}$$

with the conditions

$$q(x,t)|_{t=0} = q_0(x), \tag{4}$$
$$p(x,t)|_{x=0} = p_0(t),$$

where $q_0(x)$,$p_0(t)$ and $\gamma(t) \in C[0,\ \infty)$ are given real continuous functions, besides $q_0(x)$ - π-periodic function. It is required to find the real functions $q(x,t)$ and $p^2(x,t)$, which are π - periodic with respect to the variable$x$:

$$p^2(x+\pi,t) \equiv p^2(x,t), q(x+\pi,t) \equiv q(x,t), t \geq 0, \quad x \in R^1, \tag{5}$$

and satisfied the smooth conditions:

$$\begin{matrix} q(x,t) \in C_x^1(t>0) \cap C_t^1(t>0) \cap C(t \geq 0), \\ p(x,t) \in C_x^2(t>0) \cap C(t \geq 0). \end{matrix} \tag{6}$$

The purpose of this work is to provide a procedure for constructing a solution to problem (3)-(6), within the framework of the inverse spectral problem for the Sturm-Liouville operator with a periodic coefficient.

2. Basic facts about the direct and inverse spectral problem for the Sturm-Liouville operator with periodic coefficient

In this section, for the sake of completeness, we present some basic information concerning the inverse spectral problem for the Sturm-Liouville operator with a periodic potential (see [27, 28, 29, 30, 31, 32, 33]).

Consider the following Sturm-Liouville operator on the line

$$Ly \equiv -y'' + q(x)y = \lambda\, y, x \in R, \tag{7}$$

where $q(x)$ - real continuous π - periodic function.

Denote by $c(x,\lambda)$ and $s(x,\lambda)$ solutions of (7) satisfied initial conditions $c(0,\lambda) = 1$, $c'(0,\lambda) = 0$ and $s(0,\lambda) = 0$, $s'(0,\lambda) = 1$. The function $\Delta(\lambda) = c(\pi,\ \lambda) + s'(\pi,\ \lambda)$ is called Lyapunov's function or Hill's discriminant.

The spectrum of the operator (7) is purely continuous and coincides with the following set

$$E = \{\lambda \in R^1 : \ -2 \leq \Delta(\lambda) \leq 2\ \}$$
$$= [\lambda_0, \lambda_1] \bigcup [\lambda_2,\ \lambda_3] \bigcup ... \bigcup [\lambda_{2n}, \lambda_{2n+1}] \bigcup ... .$$

The intervals $(-\infty,\ \lambda_0)$, $(\lambda_{2n-1},\ \lambda_{2n})$, $n \geq 1$ are called gaps. Here λ_0, λ_{4k-1}, λ_{4k} - are eigenvalues of periodic problem ($y(0) = y(\pi)$, $y'(0) = y'(\pi)$), and

λ_{4k+1}, λ_{4k+2} - are eigenvalues of antiperiodic problem ($y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$) for equation (7).

Let ξ_n, $n \geq 1$ be the roots of equation $s(\pi, \lambda) = 0$. Note that, ξ_n, $n \geq 1$ coinside with eigenvalues of the Dirichlet problem ($y(0) = y(\pi) = 0$) for the equation (7), in addition the inclusions $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$, $n \geq 1$ hold. The numbers ξ_n, $n \geq 1$ with the signs $\sigma_n = sign\{s'(\pi, \xi_n) - c(\pi, \xi_n)\}$, $n \geq 1$ are called the spectral parameters of the problem (7). The spectral parameters ξ_n, σ_n, $n \geq 1$ with boundaries λ_n, $n \geq 0$ of the spectrum are called the spectral data of the operator (7). Reconstruction of the coefficient $q(x)$ from spectral data is called the inverse spectral problem for the operator (7).

The spectrum of the Sturm-Liouville operator with coefficient $q(x+\tau)$ does not depend on the real parameter τ, and the spectral parameters depend on τ: $\xi_n(\tau)$, $\sigma_n(\tau)$, $n \geq 1$. The spectral parameters satisfy the following Dubrovin system of equations

$$\frac{d\xi_n}{d\tau} = 2(-1)^{n-1}\sigma_n(\tau)\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}$$
$$\times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1. \tag{8}$$

The Dubrovin system of equations and the following trace formulas

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)),$$

give a method for solving the inverse problem.

3. **Evolution of spectral parameters**

The main result of this paper is the following theorem.

Theorem. *Let $q(x,t)$ - is the solution of the problem* (3)-(6). *Then the spectrum of the operator* (7) *does not depend on parameter t, and the spectral parameters $\xi_n(t)$, $n = 1, 2, \ldots$, satisfy the analog of the system of Dubrovin equations:*

$$\dot{\xi}_n = 2(-1)^{n+1}\sigma_n(t)\left\{\frac{1}{2\xi_n}p^2(0,t) + \gamma(t)q(0,t)\right\}$$
$$\times \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \tag{9}$$

where $n \geq 1$, the sign of $\sigma_n(t)$ changes to the opposite for each collision of a point $\xi_n(t)$ with the boundaries of its gap $[\lambda_{2n-1},\ \lambda_{2n}]$. Moreover, the following initial conditions are satisfied

$$\xi_n(t)|_{t=0} = \xi_n^0, \quad \sigma_n(t)|_{t=0} = \sigma_n^0\ , n \geq 1,$$

where ξ_n^0, σ_n^0, $n \geq 1$ - are spectral parameter of the Sturm-Liouville operator with coefficient $q_0(x)$.

P r o o f. In [34] it was shown, that if $q(x,t)$ is a solution of system

$$\begin{cases} q_t = -2pp_x + G(x,t) \\ pq - p_{xx} = 0, \end{cases} \tag{10}$$

then the following equalities hold

$$\dot{\xi}_n = -\frac{1}{2\xi_n}\left((y_n^{'})^2(\pi,t) - (y_n^{'})^2(0,t)\right)p^2(0,t) + \int_0^\pi y_n^2(x,t)G(x,t)dx, \tag{11}$$

where $y_n(x,t)$, $n = 1,\ 2,\ ...$ are orthonormal eigenfunctions of the Dirichlet problem $(y(0) = 0,\ \ y(\pi) = 0)$ for equation (7) corresponding to the eigenvalues $\xi_n(t)$, $n = 1,\ 2,\ ...$

Assuming

$$G(x,t) = \gamma(t)q(0,t)q_x(x,t),$$

we get

$$\int_0^\pi G \cdot y_n^2 dx = \left(-(y_n^{'})^2(\pi,t) - (y_n^{'})^2(0,t)\right)\gamma(t)q(0,t). \tag{12}$$

Substituting the expression (12) into (11) we obtain

$$\dot{\xi}_n = [(y_n^{'})^2(\pi,t) - y_n^{'})^2(0,t)] \times \left\{-\frac{1}{2\xi_n}p^2(0,t) - \gamma(t)q(0,t)\right\}. \tag{13}$$

Using the equalities

$$y_n(x,t) = \frac{1}{c_n(t)}s(x,\xi_n(t),t),$$

$$c_n^2(t) \equiv \int_0^\pi s^2(x,\xi_n(t),t)dx = s'(\pi,\xi_n(t),t)\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda},$$

we have

$$(y_n^{'})^2(\pi,t) - (y_n^{'})^2(0,t) = \frac{1}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}}\left(s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}\right).$$

By virtue of $s'(\pi,\xi_n,t) - \frac{1}{s'(\pi,\xi_n,t)} = \sigma_n(t)\sqrt{\Delta^2(\xi_n(t)) - 4}$ we get $(y_n^{'})^2(\pi,t) - (y_n^{'})^2(0,t) = \frac{\sigma_n(t)\sqrt{\Delta^2(\xi_n(t))-4}}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}}$.

Here $\sigma_n(t) = sign\{s'(\pi,\xi_n(t),t) - c(\pi,\xi_n(t),t)\}$.

It follows from the expansions

$$\Delta^2(\lambda) - 4 = 4\pi^2(\lambda_0 - \lambda)\prod_{k=1}^{\infty}\frac{(\lambda_{2k-1} - \lambda)(\lambda_{2k} - \lambda)}{k^4},$$

$$s(\pi, \lambda, t) = \pi\prod_{k=1}^{\infty}\frac{\xi_k(t) - \lambda}{k^2},$$

that

$$(y_n^{'})^2(\pi, t) - (y_n^{'})^2(0, t) = 2(-1)^n\sigma_n(t)\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}$$

$$\times\sqrt{(\xi_n - \lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}. \quad (14)$$

Due to (13) and (14) we obtain (9).

Now we prove the independence on t of the eigenvalues λ_n, $n = 0,\ 1,\ 2,\ ...$ of the periodic and antiperiodic problems for the Sturm-Liouville equation (7). According to [34]

$$\dot{\lambda}_n(t) = \int_0^{\pi} G(x, t)v_n^2(x, t)dx,$$

where $v_n(x, t)$ - is normalized eigenfunction of a periodic or antiperiodic problem for the Sturm-Liouville equation (7). Taking into account the form of the function $G(x, t)$, and acting as before, we get $\dot{\lambda}_n(t) = 0$. The theorem is proven. □

Result 1. If we consider $q(x+\tau, t)$ instead of $q(x, t)$, then the eigenvalues of the periodic and antiperiodic problem do not depend on the parameters $\tau$and t, while the eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ and t: $\xi_n = \xi_n(\tau, t)$, $\sigma_n = \sigma_n(\tau, t) = \pm 1$, $n \geq 1$. In this case, the system (9) has the form

$$\frac{\partial\xi_n}{\partial t} = 2(-1)^{n+1}\sigma_n(\tau, t)\left\{\frac{1}{2\xi_n}p^2(\tau, t) + \gamma(t)q(0, t)\right\}$$

$$\times\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\sqrt{(\xi_n - \lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad (15)$$

where $n \geq 1$.

Taking into account the trace formulas, we get

$$q(\tau,t)=\lambda_0+\sum_{k=1}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k(\tau,t)), \tag{16}$$

$$p^2(\tau,t)=2\sum_{k=1}^{\infty}\int_0^{\tau}\frac{\partial\xi_k(s,t)}{\partial t}ds$$
$$+\gamma(t)q(0,t)q(\tau,t)-\gamma(t)q^2(0,t)+p_0^2(t). \tag{17}$$

Result 2. This theorem provides a method for solving problem (3)-(6). To do this, first find the spectral data $\lambda_n,\ \xi_n^0(\tau),\ \sigma_n^0(\tau),\ \ n\ge 1$, of the Sturm-Liouville operator corresponding to the potential $q_0(x+\tau)$. Then, solving for the $\tau=0$ the Cauchy problem

$$\xi_n(\tau,t)|_{t=0}=\xi_n^0(\tau),\quad \sigma_n(\tau,t)|_{t=0}=\sigma_n^0(\tau)\ ,n\ge 1 \tag{18}$$

for the Dubrovin system of equations (15), we find $\xi_n(0,t)$ and $\sigma_n(0,t)$, $n\ge 1$. Based on these data, we find $q(0,t)$. After, substitute the found expression for $q(0,t)$ into equation (15), and solving the Cauchy problem for an arbitrary value τ, we find $\xi_n(\tau,t)$, $n\ge 1$. By the trace formula (16) we determine $q(x,t)$ and then from the formula (17) we determine $p^2(x,t)$.

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