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RESULTS ON EXISTENCE AND UNIQUENESS OF SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALE VIA GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The non-absolutely convergent integral of some functions of dynamic equations failed the requirement of the derivative of the function existing everywhere, and so the loss uniqueness of the indefinite integral is the setback in the theory of timescale calculus. In this work, we addressed this challenge in the context of the generalized ordinary differential equation. The well-established relationship between the dynamic equations on time-scale and the generalized ordinary differential equations is used to develop theorems and proofs on the existence and uniqueness of solutions of the dynamic equations on time-scale. The notion is demonstrated with examples and the outcomes obtained validate the concept's applicability.

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1. **Introduction**

The introduction of the theory of dynamic equation on time scale in 1988 by Hilger as found in [3] addresses one of the major challenges of mathematical sciences by unifying the seemingly disparate fields of discrete dynamical systems (difference equations) and continuous dynamical systems (differential equations) into one comprehensive mathematical model equation known as dynamic equation. Dynamic equations on time scale are defined on connected, discrete or combination of both sets. It provides a generalization of differential and difference analysis. Extensive research on dynamic equation on time scale are found in [2], [4], [5].

The theory of Lebesgue integral fails to apply effectively to mathematical space that admits more structure than just measure. The discovery of the Kurzweil integral inevitably led to the problem of finding the integral that integrates all derivatives (including the Lebesgue integral). This also results in multiple definitions of the integrals that are referred to as non-absolutely convergent

Denjoy [7] and Luzin [17] were the earliest mathematicians to have presented the fundamental description of the non-absolutely convergent integrals. They defined the function $F : I \to R$ as the antiderivative (or indefinite integral) of the function $f : I \to R$ if $F'(x) = f(x)$ for $I = [a, b]$ a bounded interval. Thus the increment of F over I is the definite integral of f over I. However, the constraint of everywhere existence of F' might be exceedingly limiting, and so ignoring the set of measure zero is a good idea. This suggests that $F'(x) = f(x)$ is not necessarily everywhere, but rather for some points that do not belong to an exceptional set. As a result, the indefinite integral loses its uniqueness, which can only be addressed by making additional assumptions on $F(x)$, such as absolute continuity or its generalization. Thus [6] developed the concept of generalized ordinary differential equation.

We consider the dynamic equation on time scale of the form

$$x^{\Delta}(t) = f(x(t), t), \quad t \in T, \tag{1}$$

that is non-absolutely convergent, and so lost the uniqueness of the indefinite integral of the equation due to the extremely restrictive condition of everywhere existence of $x^{\Delta}(t)$. However, the introduction of the Kurzweil integral, which is studied as the generalized ordinary differential equations (GODE) is employed to address the lapses. This is made possible by the established correspondence between the dynamic equations on time-scale and generalized ordinary differential equations as in Slavik [5], using constructed local flow of topological dynamics that satisfy the technical conditions of absolute continuity and generalization conditions.

We consider a compact interval $[a, b] \in R$ with a finite set of points $\Lambda = \{a = t_0 \le \alpha_1 \le t_1 \le \alpha_2 \le t_2 \le ... \le t_{1-i} \le \alpha_i \le t_i = b\}$ such that $a =$

$t_0 < t_1 < ...t_{i+1} = b$ and define a tagged division of the compact interval as a finite collection of point-interval pairs $P(\alpha, \Lambda) = (\alpha_i, [ti-1, t_i])$ such that $[ti-1, t_i]$ is non-overlapping. A gauge on $[a, b]$ is the function $\delta : [a, b] \in (0, \infty)$ and a tagged division $P(\alpha_i, [ti-1, t_i])$ is δ-fine, if for every $i = 1, 2, ...[t_{i-1}, t_i] \subset (\alpha_i - \delta(\alpha_i), \alpha_i + \delta(\alpha_i))$. Using the definition of the topological norm $\|.\|_B$ in the Banach space B_c, $G([a, b], B_c)$, we denote $A(x, t) : B_c \times [a, b] \to R^n$ the set of all regulated functions on $[a, b] \subset R$ which is in the Banach space when equipped with the supremum norm $\|A\|_\infty = \sup\{\|A(t, x)\|\}$. The regulated function is of bounded variation on $[a, b]$ if $var_b^a A(x, t) < \infty$, where

$$var_b^a A(x, t) = \sup\{\sum_{i=1}^{n(p)} \|A(x(t_i), t_i) - A(x(t_{i-1})t_{i-1})\|_W\}.$$

Definition 1.1. Given any $\epsilon > 0$ and a gauge $\delta : [a, b] \to (0, \infty)$ such that there exists a unique element $I \in B_c$ on the interval $[a, b]$ satisfying

$$\|I - S(dA, P)\| < \epsilon,$$

for all $\delta - fine$ partition $P([a, b], \lambda)$, where

$$S(dA, P) = \sum_{i=1}^{n} A(x(t_i), t_i) - A(x(t_{i-1})t_{i-1}),$$

is the integral sum that corresponds to the function $A(x(t), t)$ and the partition P, then

$$I = \int_a^b DA(x(t), t), \tag{2}$$

is known as the Kurzweil integral.

The generalized ordinary differential equation is the differential equation that emerges from the fundamental Kurzweil integral equation (2). The generalized ordinary differential equation's qualitative characteristics are extensively studied in [6], [10], [16], [9], [11], [12].

2. Preliminary results

We consider the generalized ordinary differential equation resulting from (2) as

$$\frac{dx}{dt} = DA(x, t), \tag{3}$$

for $A : B_c \times [a,b] \to R^n$. The real value function $x : [a,b] \to R^n$ is a solution of (3) on the interval $[a,b] \subset R$ for all $t \in [a,b]$ if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DA(x(t),t) \tag{4}$$

holds for every $s_1, s_2 \in [a,b]$.

Lemma 2.1. (Proof in [14])
If $x : [a,b] \to R^n$ is a solution of (4), then

$$\lim_{s\to v}[x(s) - A(x(v),s) + A(x(v),v)] = x(v), \quad s \in [a,b].$$

Lemma 2.2. (Proof in [14])
Assume that a function $U : [a,b] \times [a,b] \to R^n$ is given for which the integral $\int_a^b DU(\tau,t)$ exists. If $V : [a,b]\times[a,b] \to R^n$ is such that the integral $\int_a^b DU(\tau,t)$ exists, and if there is a guage $\delta \in [a,b]$ such that

$$|t-\tau|\,\|U(\tau,t) - U(\tau,\tau)\| \leq [t-\tau][V(\tau,t) - V(\tau,\tau)],$$

for every $t \in [\tau - \delta(\tau), \tau + \delta(\tau)]$, then the inequality

$$\left\| \int_a^b DU(\tau,t) \right\| \leq \int_a^b DV(\tau,t)$$

holds.

Remark 2.1. Consider $A : W \to R^n$ for which $W = B_c \times [a,b]$, such that $\int_{t_1}^{t_2} DA(x,t)$ exists for $t_1, t_2 \in [a,b]$. Assume there exist a nondecreasing function $h : [a,b] \to R$, such that $\int_{t_1}^{t_2} dh(t) = h(t_2) - h(t_1)$ and a continuous increasing function $w : [0,\infty) \to R$ with $w(0) = 0$, then consequent of Lemma 2.2, the following conditions hold for all $t \in [a,b]$:

i $\|A(x,t_2) - A(x,t_1)\| \leq |h(t_2) - h(t_1)|$.
ii $\|A(x,t_2) - A(x,t_1) - A(y,t_1) + A(y,t_2)\|$
$\leq w(\|x-y\|)|h(t_2) - h(t_1)|$.

Let $f \in BV([a,b], B_c)$ be a regulated function, and $g : [a,b] \to R$ a strictly increasing function on $[a,b]$, then for every $\epsilon > 0$ there exists a gauge $\delta \in [a,b]$ such that

$$\|S(f,dg,P) - 1\| < \epsilon,$$

for all δ-fine partitions P of $[a,b]$, where

$$S(f,dg,P) = \sum_{i=1}^{n(p)} f(s)[g(\alpha_i) - g(\alpha_{i-1})], \tag{5}$$

and

$$I = \int_a^b f(s)dg(s), \tag{6}$$

is known as the Kurzwiel-Stieltjes integral.

Let T be a time scale, $a, b \in T, a < b$ and $I_T = [a, b]_T$ a dense set. Consider a strictly increasing real-valued function g on I_T, then for any partition $P \subset I_T$ we have

$$\begin{aligned} g(P) &= \{g(a) = g(t_0), g(t_1), ...g(t_{n-1}), g(t_n) \\ &= g(b)\} \subset g(I_T) = [g(a), g(b)]_R, \end{aligned}$$

and $\Delta g_i = g(t_i) - g(t_{i-1})$, such that $\sum_{i=1}^n \Delta g_i = g(b) - g(a)$.

Let $f \in BV([a, b], B_c)$, for every $t \in [t_i - 1, t_i]_T \subset I_T$ and $P = \{t_0, t_1, ..., t_n\}$ $\subset I_T$, such that $m_{\Delta i} = \inf_{t \in [t_{i-1}, t_i]} f(t)$ and $M_{\Delta i} = \sup_{t \in [t_{i-1}, t_i]} f(t)$, then $U_\Delta(P, f, g)$ $= \sum_{i=1}^n M\Delta g_i$ and $L_\Delta(P, f, g) = \sum_{i=1}^n m\Delta g_i$ are the upper and lower Darboux integral respectively. Also, for $g : [a, b]_T \to R$ be an increasing function on a dense set $[a, b]_T$, we defined the real number $t^* = \sup\{s \in T, s < t\}$, so that $g(t) = t^* = t$, and $U_\Delta(P, f, g) = \sum_{i=1}^n M\Delta t$ and $L_\Delta(P, f, g) = \sum_{i=1}^n m\Delta t$, are the Riemann sums of upper and lower Delta (Δ) integral respectively.

Suppose $f \in BV([a, b], B_c)$ and $m \leq f(t) \leq M, t \in I$, we have that

$$m(g(b)) - g(a)) \leq L_\Delta(P, f, g) \leq U_\Delta(P, f, g) \leq M(g(b) - g(a)), \tag{7}$$

so that $\inf U_\Delta(P, f, g) = \int_a^{\overline{b}} f(t)\Delta g$ and $\sup L_\Delta(P, f, g) = \int_{\overline{a}}^b f(t)\Delta g$. If

$$\int_a^{\overline{b}} f(t)\Delta g = \int_{\overline{a}}^b f(t)\Delta g, \tag{8}$$

then f is Riemann-Stieltjes integrable with respect to g on $[a, b]_T$ and

$$I_T = \int_a^b f(t)\Delta g(t) \tag{9}$$

is known as the Riemann-Stieltjes integral. In particular, for $g(t) = t$, (9) has the form

$$I_T = \int_a^b f(t)\Delta t, \tag{10}$$

which is known as Riemann-Delta integral.

THEOREM 2.1. (**Equivalent Theorem**) *Let $t \in [a, b]_T$ be a dense point and $g : [a, b]_T \to R$ be strictly increasing function on $[a, b]_T$. Let $f \in BV(W, R^n)$, such that f_T and g_T are restriction of f and g to T. Assume that*

$t \in T$ is right dense and $t \leq \sup T$ such that there exist $t^* = \sup\{s \in T, s < t\}$, then $g(t) = t^* = t$ and

$$\int_a^b f(t)dg(t) = \int_a^b f_T(t)\Delta(t). \tag{11}$$

P r o o f. Given that $f \in BV(W, R^n)$ is a regulated function and $g : [a, b] \to R$ a strictly increasing function on $[a, b]$, then the Kurzweil-Stieltjes integral of f with respect to the strictly increasing function g on $[a, b]$ is well defined. That is given any partition $P \in P([a, b]_T)$ such that $P = \{a = t_0 < t_1 < ... < t_n = b\}$. Then by (5),

$$\int_a^b f(t)dg(t) \leq \sum_{i=0}^{n} M_{\Delta i}\Delta g_i(t) = U_\Delta(f_i, \Delta g_i, P), \tag{12}$$

where $M_{\Delta i} = \sup_{t\in[a,b]_T} f(t)$. If we take the infimum of the right hand side of (12) over all partition of $P([a, b]_T, f(t))$, we have

$$\inf_{t\in[a,b]_T} U_\Delta(f_i(t), \Delta g_i(t), P) = \int_a^{\overline{b}} f_T(t)\Delta g_T(t).$$

Hence,

$$\int_a^b f(t)dg(t) \leq \int_a^{\overline{b}} f_T(t)\Delta g_T(t). \tag{13}$$

Again,

$$\int_a^b f(t)dg(t) \geq \sum_{i=0}^{n} m_{\Delta i}\Delta g_i(t) = L_\Delta(f_i, \Delta g_i, P), \tag{14}$$

where $m_{\Delta i} = \inf_{t\in[a,b]_T} f(t)$. If we take the supremum of the right hand side of (14) over all partition of $P([a, b]_T, f(t))$, we have

$$\sup_{t\in[a,b]_T} L_\Delta(f_i(t), \Delta g_i(t), P) = \int_{\overline{a}}^b f_T(t)\Delta g_T(t).$$

Hence,

$$\int_a^b f(t)dg(t) \geq \int_{\overline{a}}^b f_T(t)\Delta g_T(t). \tag{15}$$

Combining (13) and (15), we have

$$\int_{\overline{a}}^b f_T(t)\Delta g_T(t) \leq \int_a^b f(t)dg(t) \leq \int_a^{\overline{b}} f_T(t)\Delta g_T(t),$$

and by (8) we have

$$\int_a^b f(t)dg(t) = \int_a^b f_T(t)\Delta g_T(t). \tag{16}$$

In particular, given that $t \in T$ is a dense point such that $t \leq \sup T$ and there exists $t^* = \sup\{s \in T, s < t\}$, so that $g(t) = t^* = t$ and

$$\int_a^b f(t)dg(t) = \int_a^b f_T(t)\Delta t,$$

hence the theorem is proved. □

Consider an initial value problem of the dynamic equation on $[a,b]_T$,

$$x^{\Delta}(t) = f(x,t), \quad x(t) = x(t_0), \quad t \in [a,b]_T. \tag{17}$$

The real value function $x : [a,b]_T \to R^n$ is a solution of (17) for all $t \in [a,b]$ if

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x,s)\Delta s, \quad t, t_0 \in [a,b]. \tag{18}$$

Remark 2.2. Let $f : B_c \times T \to R^n$ be a Lebesgue integrable, rd-continuous function on $[a,b]_T$. The Caratheodory assumptions we make on the integral of the function of (18) are as follows:

A1 if $f(x,t)$ is rd-continuous function,for $x : T \to R$ being continuous, then $t \to \int_a^t f(x,s)\Delta s$ is rd-continuous.

A2 there exists a Lebesgue integrable function $m_o : T \to R$ such that

$$\| \int_{t_0}^{t_1} f(x,s)\Delta s\| \leq \int_{t_0}^{t_1} m_0(s)\Delta s$$

for $t_0, t_1 \in T$, $x \in G(T, R^n)$.

A3 there exists a Lebesgue integrable function $m_1 : T \to R$ such that

$$\| \int_{t_0}^{t_1} f(x-y,s)\Delta s\| \leq \int_{t_0}^{t_1} m_1(s)\|x-y\|\Delta s$$

for $t_0, t_1 \in T$, $x, y \in B_c$.

By the implication of Theorem 2.2 and equation (11), for $f : B_c \times [a,b] \to R^n$ belonging to the class of Caratheodory function, measurable with references to a regulated function $g : [a,b] \to R$, then the Kurzweil-Stieltjes integral

$$x(t) = x(t_0) - \int_a^t f(x,s)dg(s), \tag{19}$$

also satisfies Remark 2.2 everywhere on $[a,b] \subset R$.

Proposition 2.1. *Let $f \in BV(B_c \times [a,b], R^n)$ be a Caratheodory function and $g : [a,b] \to R$ a regulated non-decreasing function. Then, the Kurzweil integrated function $A : B_c \times [a,b] \to R^n$ defined as*

$$A(x,t) = \int_{t_0}^{t} f(x,s)dg(s), \tag{20}$$

satisfies Remark 2.1 if we can find a non-decreasing function $h(t) : [a,b] \to R$ *that satisfies*

$$h(t) = \int_{t_0}^{t_1} (m_0(s) + m_1(s))dg(s). \tag{21}$$

P r o o f. Let the regulated function $g : [a,b] \to R$ be given such that $g = g_+ - g_-$, and $\int_R f dg = \int_R f dg_+ - \int_R f dg_-$, so that by (20) we can have $A^+(x,t) = \int_{t_0}^{t} f(x,s)dg_+(s)$ and $A^-(x,t) = \int_{t_0}^{t} f(x,s)dg_-(s)$ respectively. By Remark 2.2, we have

$$\begin{aligned} \|A^-(x,t_1) - A^-(x,t_0)\| &= \left\| \int_{t_0}^{t_1} f(x,s)dg_-(s) \right\| \\ &= \left\| \int_{t_0}^{t_1} f(x,s)\Delta^-(s) \right\| \\ &\leq \left\| \int_{t_0}^{t_1} m_0(s)ds^- \right\|, \end{aligned}$$

and

$$\begin{aligned} \|A^+(x,t_1) - A^+(x,t_0)\| &= \left\| \int_{t_0}^{t_1} f(x,s)dg_+(s) \right\| \\ &= \left\| \int_{t_0}^{t_1} f(x,s)\Delta^+(s) \right\| \\ &\leq \left\| \int_{t_0}^{t_1} m_0(s)ds^+ \right\|, \end{aligned}$$

so that

$$\begin{aligned} &\|A(x,t_1) - A(x,t_0)\| \\ &= \|A^+(x,t_1) - A^+(x,t_0) + A^-(x,t_0) - A^-(x,t_1)\| \\ &\leq \int_{t_0}^{t_1} m_0(s)ds^+ + \int_{t_0}^{t_1} m_0(s)ds^- \\ &= \int_{t_0}^{t_1} m_0(s)\Delta s. \end{aligned}$$

We define

$$h_1(t) = \int_{t_0}^{t} (m_0(s))\Delta s, \quad t, t_0 \in [a,b], \tag{22}$$

for $h_1 : [a,b] \to R$ is a non-decreasing function, and m_0 a nonnegative function on $[a,b]$, so that

$$\|A(x,t_1) - A(x,t_0)\| \leq \int_{t_0}^{t_1} m_0(s)\Delta s \leq |h_1(t_1) - h_1(t_0)|.$$

Also

$$\begin{aligned}
&\|A^+(x,t_1) - A^+(x,t_0) + A^+(y,t_0) - A^+(y,t_1)\| \\
&= \left\| \int_{t_0}^{t_1} f(x,s)dg_+(s) - \int_{t_0}^{t_1} f(y,s)dg_+(s) \right\| \\
&= \left\| \int_{t_0}^{t_1} f(x-y,s)\Delta^+ s \right\| \\
&\leq w\|x-y\| \int_{t_0}^{t_1} m_1(s)\Delta s^+
\end{aligned}$$

and

$$\begin{aligned}
&\|A^-(x,t_1) - A^-(x,t_0) + A^-(y,t_0) - A^-(y,t_1)\| \\
&= \left\| \int_{t_0}^{t_1} f(x,s)dg_-(s) - \int_{t_0}^{t_1} f(y,s)dg_-(s) \right\| \\
&= \left\| \int_{t_0}^{t_1} f(x-y,s)\Delta^- s \right\| \\
&\leq w\|x-y\| \int_{t_0}^{t_1} m_1(s)\Delta s^-,
\end{aligned}$$

so that,

$$\begin{aligned}
&\|A(x,t_1) - A(x,t_0) + A(y,t_0) - A(y,t_1)\| \\
&\leq w\|x-y\| \int_{t_0}^{t_1} m_1(s)\Delta^+ s + w\|x-y\| \int_{t_0}^{t_1} m_1(s)\Delta^- s \\
&= w\|x-y\| \int_{t_0}^{t_1} m_1(s)\Delta s.
\end{aligned}$$

We define

$$h_2(t) = \int_{t_0}^{t} m_1(s)\Delta s, \quad t, t_0 \in [a,b] \tag{23}$$

for $h_2 : [a,b] \to R$ is a non-decreasing function, and m_1 a nonnegative function on $[a,b]$, so that

$$\|A(x,t_1) - A(x,t_0) + A(y,t_0) - A(y,t_1)\| = w\|x-y\||h_2(t_0) - h_1(t_0)|.$$

Then for $h(t) = h_1(t) + h_2(t)$, the proposition is proved. □

LEMMA 2.3. (Proof in [5])
Let $x : [a,b] \to R^n$ be a regulated function on $[a,b]$, then any defined step

function $\varphi : [a, b] \to R^n$ *uniformly approximate* $x \in B_c$ *on* $[a, b]$*, such that for* $\epsilon > 0$, $\|x(t) - \varphi(t)\| < \epsilon$.

Definition 2.1. Let $A : B_c \times [a, b] \to R^n$ be a regulated function on $[a, b]$, for every step function $x(t) : [a, b] \to R^n$ on a partition $a = t_0 < t_1 < ...t_n = b$ such that $x(t) = c_i$ for $t \in (t_{i-1,t_i})$, then

$$\begin{aligned}\int_a^b DA(x(t), t) &= \sum_{i=0}^{n}((A(c_i, t_i^-) - A(c_i, t_{i-1}^+)\\ &\quad + (A(x(t_{i-1}), t_{i-1}^+) - A(x(t_{i-1}), t_{i-1}))\\ &\quad + (A(x(t_i), t_i) - A(x(t_i), t_i^-))).\end{aligned}$$

Lemma 2.4. *Let the regulated function* $f : B_c \times T \to R^n$ *be Kurzweil-Stieltjes integrable with a measure function* $g : [a, b] \to R$*, which is of bounded variation on* $[a, b]$*. Given any step function* $x : [a, b] \to R^n$ *such that for* $t \to A(x, t)$ *is regulated and (20) holds, then*

$$\int_a^b DA(x, t) = \int_a^b f(x, t)dg(t). \tag{24}$$

P r o o f. By Definition 2.1 for $t \to A(x, t)$ being a regulated function on $[a, b]$ and $x(t) : [a, b] \to R^n$ a step function on a partition $a = t_0 < t_1 < ... < t_n = b$, then for any $\epsilon > 0$ and $x(t) = c_i$ for $t \in (t_{i-1}, t_i)$, we have

$$\begin{aligned}&\int_a^b DA(x(t), t)\\ &= \lim_{\epsilon \to 0^+} \sum_{i=0}^{n}((A(x(t_{i-1}), t_{i-1} + \epsilon) - A(x(t_{i-1}), t_{i-1})))\\ &+ \lim_{\epsilon \to 0^+} \sum_{i=0}^{n}((A(c_i, t_i - \epsilon) - A(c_i, t_{i-1} + \epsilon)\\ &+ \lim_{\epsilon \to 0^+} \sum_{i=0}^{n}(A(x(t_i), t_i) - A(x(t_i), t_i - \epsilon)).\end{aligned} \tag{25}$$

Also for $t \to f(x, t)$ being a regulated function, we have

$$\int_a^b f(x, t)dg(t) = \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} f(x(s), s)dg(s)$$

$$\begin{aligned} &= \lim_{\epsilon\to 0^-}\sum_{i=0}^{n}\int_{t_{i-1}}^{t_{i-1}+\epsilon} f(x(s),s)dg(s) \\ &\quad + \lim_{\epsilon\to 0^-}\sum_{i=0}^{n}\int_{t_{i-1}+\epsilon}^{t_i-\epsilon} f(x(s),s)dg(s) \\ &\quad + \lim_{\epsilon\to 0^-}\sum_{i=0}^{n}\int_{t_i-\epsilon}^{t_i} f(x(s),s)dg(s). \end{aligned} \tag{26}$$

Comparing (25) and (26) term by term, its observed that

$$\begin{aligned} &\lim_{\epsilon\to 0^+}\sum_{i=0}^{n}((A(x(t_{i-1}),t_{i-1}+\epsilon)-A(x(t_{i-1}),t_{i-1}))) \\ &= \lim_{\epsilon\to 0^+}\int_{t_{i-1}}^{t_{i-1}+\epsilon} f(x(s),s)dg(s) \\ &= f(x(t_{i-1}).t_{i-1})\Delta^+ g(t_{i-1}), \\ &\lim_{\epsilon\to 0^+}\sum_{i=0}^{n}((A(c_i,t_i-\epsilon)-A(c_i,t_{i-1}+\epsilon))) \\ &= \lim_{\epsilon\to 0^+}\int_{t_{i-1}+\epsilon}^{t_{i-\epsilon}} f(x(s),s)dg(s) \\ &= f(x(t_{i-1}),t_{i-1})\Delta^+ g(t_{i-1}). \end{aligned}$$

Hence, (24) holds and the theorem is proved. □

Theorem 2.2. (**Correspondence theorem** [5]) *Let $f : B_c \times [a,b]_T \to R^n$ satisfies Remark 2.2 and $A(x,t) : W \to R^n$ satisfies Remark 2.1. If we have a dense point $t^* = sup\{s \in T, s < t\}$ and $g(t) = t^* = t$, then $x(t^*) : T \to R^n$ is a solution of*

$$x^{\Delta}(t) = f(x,t), \quad t \in T \tag{27}$$

and also a solution of the generalized differential equation

$$\frac{dx}{dt} = DA(x,t), \tag{28}$$

where

$$A(x,t) = \int_{t_0}^{t} f(x,s)dg(s).$$

Also, every solution $y : T \to R^n$ of (28) can be expressed as $y = x$, where $x : T \to R^n$ is a solution of (27).

P r o o f. Let $v \in T$. If $x : [a,b]_T \to R^n$ is a solution of (27), then

$$x(s) = x(v) + \int_v^s f(x,t)\Delta t, \quad s \in T.$$

It follows that by (11) for $g(t^*) = t$, we have

$$x(s) = x(v) + \int_v^s f(x,t)dg(t).$$

Let $s, v \in [a,b] \subset T$ be a compact interval, such that $A(x,t)$ is regulated, since f satisfies the Remark 2.2, then by Lemma 2.4, we have that

$$x(s) = x(v) + \int_v^s DA(x,t),$$

which implies that x is also a solution of the generalized ordinary differential equation.

Conversely, let $y : [a,b] \to R^n$ be a solution of (28). Then

$$y(s) = y(a) + \int_a^s DA(y(\tau),t), \quad s \in [a,b].$$

Let $[\alpha,\beta]_T$ be a time scale interval such that $a, s \in [\alpha,\beta]_T$. Then by Lemma 2.1

$$\begin{aligned} y(\tau) &= \lim_{u\to\tau^+}(y(u) - A(y(\tau),u) + A(y(\tau),\tau)) \\ &= \lim_{u\to\tau^+}(y(u) - \int_\tau^u f(y(\tau),s)dg(s)) \\ &= \lim_{u\to\tau^+}(y(u) - \int_\tau^u f(y(\tau),\tau)\Delta g(\tau)) \\ &= \lim_{u\to\tau^+} y(u) \text{ for } \ \text{all } \tau \in [\alpha,\beta], \end{aligned}$$

and therefore $\lim_{u\to\tau} y(u)$ exists. Similarly, for every $\tau^- \in (\alpha,\beta]$ we have

$$\begin{aligned} y(\tau) &= \lim_{u\to\tau^-}(y(u) - A(y(\tau),u) + A(y(\tau),\tau)) \\ &= \lim_{u\to\tau^-}(y(u) - \int_\tau^u f(y(\tau),s)dg(s)) \\ &= \lim_{u\to\tau^-}(y(u) - \int_\tau^u f(y(\tau),\tau)\Delta g(\tau)) = \lim_{u\to\tau^-} y(u). \end{aligned}$$

Since y is regulated and therefore bounded on $[\alpha,\beta]$ there exists a bounded set $W \subset B_c \times [a,b]$ such that $y(t) \in W$ for every $t \in [\alpha,\beta]$. The function f satisfies Remark 2.1 on W and by Preposition 2.1, the function A satisfies Remark 2.1 on W. Then by Lemma 2.4, we obtain

$$y(s) = y(a) + \int_a^s f(y(t),t)dg(t), \quad s \in T. \tag{29}$$

Thus $y = x$ where $x : T \to R^n$ is the restriction of y to T. □

3. Main Results

THEOREM 3.1. *Let $f : B_c \times [a,b]_T \to R^n$ and m_0, m_1 be Lebesgue measurable functions, satisfying Remark 2.2. Assume $A : W \to R^n$ satisfies Remark 2.1 for some continuous increasing function $w : [0,w) \to R$ with $w(0) = 0$. Then there exists a solution $x : [a,b]_T \to B_c$ in the open domain $W = [(x,t) : \|x(t) - x(t_0)\| \leq |h(t_1) - h(t_0)|, |t - t_0| < \epsilon]$ which satisfies (17).*

P r o o f. Suppose the set $y(t) \in B_b \subset B_c$ is the best approximation to the point $x \in B_c$ such that $\|y(t) - \tilde{x}\| \leq \epsilon$, for $x(t_0) = \tilde{x}$ and $f(y(t),t)$ being a regulated function on $[t_0 - \Delta, t_0 + \Delta]$. Then for all $y(t_1), y(t_2) \in B_b$ and any $\alpha \in [0,1]$ we have that

$$\begin{aligned}\|\alpha(y(t_2)) + (1-\alpha)(y(t_1))\| &= \|\alpha y(t_2) + y(t_1) - \alpha y(t_1)\| \\ &\leq \|\alpha(y(t_2) - y(t_1))\| + \|y(t_1)\| \\ &\leq |\alpha\|h(t_2) - h(t_1)\| + c,\end{aligned}$$

which is in B_b for any arbitrary $c > 0$. Hence B_b is a convex set.

We show that B_b is a closed set.

Let $y_k \in B_b$ for $k = N$, be a sequence converging to $y \in B_b$ in $[t_0 - \Delta, t_0 + \Delta]$ such that $\lim_{k\to 0} \|y_k(t) - y(t) = 0$, for all $t \in [t_0 - \Delta, t_0 + \Delta]$. Then we have

$$\begin{aligned}&\|y(t) - \tilde{x}\| = \left\| \int_{t_0}^{t} f(y(s))\Delta s \right\| = \left\| \int_{t_0}^{t} DA(y(s),s) \right\| \\ &= \|(A(y(t),t) - A(y_k(t),t) + A(y_k(t),t) - A(y(t),t_0))\| \\ &\leq w(\|y_k - y\|)|h(t_2) - h(t_1)| \\ &\leq w(\|\epsilon\|)|h(t_2) - h(t_1)|.\end{aligned}$$

Therefore for any arbitrary ϵ, $\|y(t) - \tilde{x}\| \in B_b$ and $B_b \subset B_c$ is closed.

Let the map $T : B_b \to B_b$ be defined such that $Ty = x$, where

$$Ty(t) = \tilde{x} + \int_{t_0}^{t} DA(y(s),s), \quad s \in [t_0 - \epsilon, to + \epsilon].$$

Then, using Lemma 2.1 and for $t^+ \in [t_0, t_0 + \Delta]$, we have

$$\begin{aligned}\|Ty(t) - \tilde{x}\| &= \left\| \int_{t_0}^{t^+} f(y(s),s)\Delta s \right\| = \left\| \int_{t_0}^{t^+} DA(y(s),s) \right\| \\ &= A(y(t^+),t^+) - A(y(t_0),t_0)\| \\ &\leq |h(t^+) - h(t_0)|. \end{aligned} \tag{30}$$

Also for $t^- \in [t_0 - \Delta, t_0]$ we have

$$\begin{aligned} \|Ty(t) - \tilde{x}\| &= \left\| \int_{t^-}^{t_0} f(y(s), s)\Delta s \right\| = \left\| \int_{t^-}^{t_0} DA(y(s)) \right\| \\ &= \|A(y(t_0), t_0) - A(y(t^-), t^-)\| \\ &\leq |h(t_0) - h(t^-). \end{aligned} \tag{31}$$

Combining (30), (31), and by Remark 2.1, we obtain

$$\|Ty(s) - \tilde{x}\| \leq |h(t^+) - h(t^-)\|$$

and since $f(y(t), t) \in B_b$ is rd-continuous, $T : B_b \to B_b$ map into itself.

Consider $u, v \in B_b$, such that by the definition of a bounded variation, we have

$$\begin{aligned} &\|Tu(t) - Tv(t)\|_{BV} \\ &= \|Tu(t_0 - \Delta) - Tv(t_0 + \Delta) + var_{t-\Delta}^{t+\Delta}(Tu - Tv) \\ &\leq \|Tu(t_0) - Tv(t_0)\| + 2var_{t+\Delta}^{t-\Delta}(Tu - Tv) \\ &= 2var_{t-\Delta}^{t+\Delta}(Tu - Tv). \end{aligned}$$

By Lemma 2.2 and Remark 2.1, and for $t^+ \in [t_0, t_0 + \Delta]$ we have

$$\begin{aligned} \|Tu(t) - Tv(t)\| &= \|Tu(t^+) - Tv(t^+) - Tv(t_0) + Tu(t_0)\| \\ &= \left\| \int_{t_0}^{t^+} D[A(u(s), s) - A(v(s), s)] \right\| \\ &= \int_{t_0}^{t^+} f(s, u(s) - v(s))\Delta s \\ &\leq w\|u(s) - v(s)\| \int_{t_0}^{t^+} m_1(s)\Delta^+ s \\ &\leq w\|u(s) - v(s)\||h(t^+) - h(t_0)|. \end{aligned} \tag{32}$$

Also, for $t^- \in [t_0 - \Delta, t_0]$

$$\begin{aligned} \|Tu(t) - Tv(t)\| &= \|Tu(t^-) - Tv(t^-) - Tv(t_0) + Tu(t_0)\| \\ &= \left\| \int_{t^-}^{t_0} D[A(u(s), s) - A(v(s), s)] \right\| \\ &= \int_{t^-}^{t_0} f(s, u(s) - v(s))\Delta s \\ &\leq w\|u(s) - v(s)\| \int_{t^-}^{t_0} m_1(s)\Delta^- s \\ &\leq w\|u(s) - v(s)\||h(t_0) - h(t^-)|. \end{aligned} \tag{33}$$

Combining (32) and (33) we have for $t \in [t_0 - \Delta, t_0 + \Delta]$,

$$\begin{aligned} \|Tu(t) - Tv(t)\| &= \|Tu(t^+) - Tv(t^+) - Tv(t^-) + Tv(t^-) \\ &\leq w\|u - v\||h(t^+) - h(t^-)|, \end{aligned}$$

and thus

$$var_{t_0-\Delta}^{t_0+\Delta}(Tu(t) - Tv(t)) \leq w\|u - v\||h(t^+) - h(t^-)|. \tag{34}$$

Assume $u, u_k \in B_b$, k$\in$ N is such that $\lim_{k\to\infty} \|u_k(t) - u(t)\| = 0$ for $t \in [t_0 - \Delta, t_0 + \Delta]$, then

$$\lim_{k\to\infty} \|u_k - u\||h(t^+) - h(t^-)| = 0$$

and

$$\lim_{k\to\infty} var_{t_0-\Delta}^{t_0+\Delta}(Tu_k(t) - Tu(t)) = \lim_{k\to\infty} (Tu_k(t) - u(t)) = 0.$$

Hence T is a continuous map.

Let $y_k \in B_b, k \in N$ be an arbitrary sequence in B_b. Since y_k is uniformly bounded and consists of equally bounded functions, then by Helly's selection theorem y_k contains a pointwise convergent subsequence denoted by y_k, such that $\lim_{k\to\infty} y_k(s) = y(s)$ for every $s \in [t_0 - \Delta^-, t_0 + \Delta^+]$. Then, for

$$x(t) = T(y(t)) = \tilde{x} + \int_{t_0}^{t} DA(x, s)ds,$$

we have

$$\begin{aligned} \|Ty_k(s) - \tilde{x}\| &= \left\| \int_{t_0}^{t} DA(x_k, s)ds \right\| = \|A(x_k, t) - A(x_k, t_0)\| \\ &= \int_{t_0}^{t} f(x_k, s)\Delta s\| \leq \left\| \int_{t_0}^{t} m_0(s)\Delta s \right\| \leq |h(t) - h(t_0)| \end{aligned}$$

and

$$\lim_{k\to\infty} Ty_k(t) = x(t).$$

This shows that every sequence in B_b contains a convergent subsequence on $[t_0 - \Delta, t_0 + \Delta]$ and then B_b is sequentially compact. Then there exists at least one fixed point $x = Ty$ such that

$$x = \tilde{x} + \int_{t_0}^{t} DA(x(s), s), \quad s \in [t_0 - \epsilon, t_0 + \epsilon]$$

and by Theorem 3.4 we conclude that there exists a solution $x \in B_c$ satisfying (17). □

Theorem 3.2. *Consider the functions $A : W \to R^n$ and $f : B_c \times [a, b]_T \to R$ satisfying Remarks 2.1 and 2.2 respectively. Let there exist a continuous increasing function $w : [0, \infty) \to R$ with $w(0) = 0, w(r) > 0$ for $r > 0$, such* *that for any* $\epsilon > 0$ *we have* $\alpha = \frac{\epsilon(u-v)}{w(r)} > 0$ *for every* $u, v \in [a,b]$. *Then the solution* $x : [a,b] \to R^n$ *of (17) is locally unique on* $[a,b]$.

P r o o f. Assume that $y, x \in B_c$ are solutions of (17) such that $y(t_0) = x(t_0) = \tilde{x}$, then

$$\begin{aligned}
\|x(t) - y(t)\| &= \| \int_{t_0}^{t} [DA(y(s), s) - A(x(s), s)] \| \\
&= \| \int_{t_0}^{t} f(y(s) - x(s), s) dgs \| \\
&= \| \int_{t_0}^{t} f(y(s) - x(s), s) \Delta s \| \\
&\leq \int_{t_0}^{t} m_1(s) w \|y(s) - x(s)\| \Delta s \\
&\leq \int_{t_0}^{t_0+} m_1(s) w \|y(s) - x(s)\| \Delta s \\
&+ \int_{t_0+}^{t} m_1(s) w \|y(s) - x(s)\| \Delta s.
\end{aligned}$$

But

$$\begin{aligned}
\int_{t_0}^{t_0+} w\|y(s) - x(s)\| m_1(s) \Delta s &\leq w\|\tilde{y} - \tilde{x}\| |h(t_{0+}) - h(t_0)| \\
&= w(0) = 0.
\end{aligned}$$

Assume $\lim_{t \to 0^+} |h(t) - h(t_{0+})| = 0$, then there exists $\alpha > 0$ such that $var_{t_{0+}}^{t} h(s) < \alpha < \infty$, so that

$$\begin{aligned}
\|y(t) - x(t)\| &\leq \int_{t_0+}^{t_0} w\|y(s) - x(s)\| m_1(s) \Delta s \\
&\leq w\|y - x\| |h(t) - h(t_0+)| \\
&\leq w(\|y - x\|)\alpha \\
&= \epsilon(t - t_{0+}).
\end{aligned}$$

Since $\epsilon(t - t_{0+}) \to$ for $t \to t_{0+}$ then $\|y(t) - x(t) = 0$ which implies that $y(t) = x(t)$ and the solution is unique. □

4. **Illustrations**

EXAMPLE 4.1. Consider the function $f : T \to R$ defined by

$$f(t) = \begin{cases} 1, & \text{if } \ t \in Q \\ 0, & \text{if } \ t \notin Q \end{cases}$$

on $[0,1]_T \subset T$. Let be partition $P = \{0 = t_0 < t_1 < ... < t_n = 1\} \subset I_T$. Then the function $f(t)$ is Delta (Δ) integrable on $[0,1]$ if

$$\int_{\overline{0}}^{1} f(t)\Delta t = \int_{0}^{\overline{1}} f(t)\Delta t = \int_{0}^{1} f(t)\Delta t.$$

The Riemann Sums of the Upper and lower Delta integral are

$$U_\Delta(f,P) = \sum_{i=1}^{n} M\Delta t = \sum_{i=1}^{n} 1.(t_i - t_{i-1}) = 1$$

and

$$L_\Delta(f,P) = \sum_{i=1}^{n} m\Delta t = \sum_{i=1}^{n} 0.(t_i - t_{i-1}) = 0,$$

respectively.

Also,

$$I_{L_\Delta(f,P)} = \int_{\overline{0}}^{1} f(t)\Delta t = \sup(L_\Delta(f,P)) = 0$$

and

$$I_{U(f,P)} = \int_{0}^{\overline{1}} f(t)\Delta t = \inf(L_\Delta(f,P)) = 1$$

are the lower and upper Delta integral, respectively.

Since $I_{L_\Delta(f,P)} \neq I_{U(f,P)}$, then $f(t)$ is not Delta integrable on $[0,1]_T$.

However applying the concept of Kurzwiel integral, we can refine the partition P into an over lapping interval as

$$P = 0 = x_0 \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq ... \leq x_{i-1} \leq \alpha_i \leq x_i = 1.$$

Let $(\alpha, P) = (\alpha_i, [t_{i-1}, t_i])$ be a tag such that $\alpha_i = \frac{1}{2}(t_i + t_{i-1}) \in [t_{i-1}, t_i]$ and

$$[t_{i-1}, t_i] \subset (\alpha_i - \frac{\epsilon}{2^{i+1}}, \alpha_i + \frac{\epsilon}{2^{i+1}},$$

if the $t_i s'$ are the rationales, the gauge $\delta(\alpha_i) = \frac{\epsilon}{2^{i+1}}$. and if $t_i s'$ are irrationals(in between the rationales) the gauge $\delta(\alpha_i) = 1$. There can only be two subintervals with α_i as tag and the length of this subintervals is $\frac{\epsilon}{2^i}$. Hence, the contribution to the Kurzweil sum $(S(f,P))$, over the partition P, from subintervals with α_i as tag is $\frac{\epsilon}{2^i}$ and the contribution of the tags at irrational points to the Kurzweil sum $(S(f,P))$, over P, is 0. Then

$$0 \leq S(f,P) \leq \sum_{i=0}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Since ϵ is arbitrary, we therefore conclude that $f(t)$ is Kurzweil integrable on $[0,1]_T$ and

$$\int_0^1 f(t)dg(t) = 0.$$

EXAMPLE 4.2. Consider the function $f : T \to R$ defined by

$$f(t) = \begin{cases} 1, & \text{if } \quad -1 \le t < 0, \\ -1 & \text{if } \quad 0 \le t \le 1, \end{cases}$$

on $[-1, 1]_T \subset T$. We define the partition $P = \{-1 = t_0 < t_1 < ... < t_n = 1\}$, such that the Riemann Sums of the Upper and lower Delta integral are

$$U_\Delta(P, f) = \sum_{i=1}^{n} M\Delta t = \sum_{i=1}^{n} (x_i - x_{i-1}) \,.\, 1 = 1$$

and

$$L(P, f) = \sum_{i=1}^{n} m\Delta t = \sum_{i=1}^{n} (x_i - x_{i-1}) \,.\, (-1) = -1,$$

respectively.

Also the Upper and lower Delta integrals are

$$I_{U_\Delta(P,f)} = \overline{\int_{-1}^{1}} f(t)\Delta t = \inf(U_\Delta(P, f)) = 1$$

and

$$I_{L(P,f)} = \underline{\int_{-1}^{1}} f(t)\Delta t = \sup(L_\Delta(P, f)) = -1,$$

respectively.

Since $I_{U_\Delta(P,f)} \neq I_{L(P,f)}$, then $f(t)$ is not Delta integrable on $[-1, 1]_T$.

Defining a δ-fine partition on P such that

$$P = \{-1 = t_0 \le \alpha_1 \le t_1 \le \alpha_2 \le ... \le \alpha_n \le t_n = 1\},$$

such that $P = \{-1, -\frac{\epsilon}{2}, \frac{\epsilon}{2}, 1\}$, and choosing $\alpha_i = \frac{1}{2}(t_i + t_{i-1}) \in [t_{i-1}, t_i]$ for $i = 1, 2, ...n$, we have

$$\begin{aligned} S(f, P) &= \sum_{i=1}^{n} f(\alpha_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \frac{1}{2}(t_i + t_{i-1})(t_i - t_{i-1}) \\ &= \frac{1}{2} \sum_{i=1}^{n} (t_n^2 - t_0^2) \end{aligned}$$

$$= \frac{1}{2}\left(\left((-\frac{\epsilon}{2})^2 - (-1)^2\right) + \left(\,(\frac{\epsilon}{2})^2 - (-\frac{\epsilon}{2})^2\right) + \left((1)^2 - (\frac{\epsilon}{2})^2\right)\right) = 0.$$

And so $f(t)$ is Kurzweil integrable on $[-1, 1]_T$.

5. Conclusion

The setback of non-absolutely convergent integral of some functions of the dynamic equations on time scale, which makes the everywhere existence of $f^{\Delta}(t)$ very restrictive, and so the loss of the uniqueness of the indefinite integral was addressed by the use of Kurzweil integral. Theorems on the existence and uniqueness of solution of dynamic equations were formulated and proved via the generalized ordinary differential equation. Examples were used for illustration, and the results obtained confirmed the suitability of the approach.

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