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# ON PERIODIC PROBLEMS FOR THE NONLOCAL POISSON EQUATION IN THE CIRCLE

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**Abstract:** This work is devoted to the study of the methods of solving the boundary value problem with transformed arguments. In the studied problems, the arguments are transformed by mapping the type of involution. Moreover, these mappings are used both in the equation and in the boundary conditions. The equation under consideration is a nonlocal analogue of the Poisson equation. The boundary conditions are specified as a relation between the value of the desired function in the upper semicircle and its value in the lower semicircle. Two types of boundary conditions are considered. They generalize the known periodic and antiperiodic conditions for circular regions. When solving the main problems for the classical Poisson equation, auxiliary problems are obtained. Using well-known assertions for these auxiliary problems, theorems on the existence and uniqueness of solutions are proved. Exact conditions for the solvability of the studied problems are found.

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**Key Words:** involution, Poisson equation, nonlocal equation, periodic conditions, Dirichlet problem, Neumann problem

## 1. Introduction

Let $\Omega = \left\{x \in R^2 : |x| = 1\right\}$ be a unit circle, $\partial\Omega$ - a circumference. Let us introduce the notation $\partial\Omega_+ = \{x \in \partial\Omega : x_1 \geq 0\}, \partial\Omega_- = \{x \in \partial\Omega : x_1 \leq 0\}, I =$

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$\{x \in \partial\Omega : x_1 = 0\}$.

For any $x = (x_1, x_2) \in R^2$ consider the mappings:

$$S_1x = (-x_1, x_2), S_2x = (x_1, -x_2), S_3x = (-x_1, -x_2).$$

It is obvious that $S_j^2x = x, S_1S_2 = S_2S_1 \equiv S_3, S_1S_3 = S_3S_1 \equiv S_2, S_2S_3 = S_3S_2 \equiv S_1$.

Let $a_0, a_1, a_2, a_3$ be some real numbers. We introduce the operator

$$Lu(x) \equiv -a_0\Delta u(x) - a_1\Delta u(S_1x) - a_2\Delta u(S_2x) - a_3\Delta u(S_3x).$$

Denote $x^* = S_jx$, where $j$ takes one of the values $j = 1$ or $j = 3$.

Consider the following problem in the domain $\Omega$:

**Problem P.** Find the function $u(x) \in C^2(\Omega) \cup C^1(\bar{\Omega})$ satisfying the conditions

$$Lu(x) = f(x),\ x \in \Omega, \tag{1}$$

$$u(x) + (-1)^k u(x^*) = g_0(x),\ x \in \partial\Omega_+, \tag{2}$$

$$\frac{\partial u(x)}{\partial \nu} - (-1)^k \frac{\partial u(x^*)}{\partial \nu} = g_1(x),\ x \in \partial\Omega_+, \tag{3}$$

where $k = 1$ or $k = 2$, $g_0(x)$ and $g_1(x)$ are given functions.

Note that if $x \in I$, then the point $x^*$ also belongs to the set $I$. Therefore, the boundary conditions (2) and (3) imply the following matching conditions for $x \in I$

$$g_0(x) = (-1)^k g_0(x^*),$$

$$\frac{\partial g_0(x)}{\partial x_j} = (-1)^k \frac{\partial g_0(x^*)}{\partial x_j},\ k, j = 1, 2, x \in I, \tag{4}$$

$$g_0(x) = (-1)^k g_0(x^*),\ k = 1, 2, x \in I. \tag{5}$$

Further, we will consider these conditions to be satisfied.

Boundary value problems with involutively transformed arguments for the Laplace equation were first studied in [9]. In this paper, generalizations of Dirichlet, Neumann, and Robin boundary value problems were studied in the two-dimensional case. Problems P in the case of the classical Poisson equation, i.e. when $a_0 = 1$ and $a_j = 0, j = 1, 2, 3$ were studied in [13],[14]. Later, some generalizations of these problems with conditions of the Dirichlet, Neumann, and Robin types, as well as of the Samarskii-Ionkin type, were studied in [6],[7],[10],[11],[15],[19]. Further, in [3],[4] for a nonlocal Laplace operator with involutively transformed arguments in rectangular domains, problems of the Cauchy and Dirichlet types were studied. In a more general case, the main boundary value problems for nonlocal analogues of the Poisson equation and spectral questions for the nonlocal Laplace operator were studied in [1],[5],[8],[16],[18].

## 2. Problem P for the case $k = 1$

Let $k = 1$. Consider the function

$$v(x) = a_0 u(x) + a_1 u(S_1 x) + a_2 u(S_2 x) + a_3 u(S_3 x). \tag{6}$$

Replacing the point $x$ by $S_j x, j = 1, 2, 3$ in equality (6), we obtain

$$v(S_1 x) = a_1 u(x) + a_0 u(S_1 x) + a_3 u(S_2 x) + a_2 u(S_3 x), \tag{7}$$

$$v(S_2 x) = a_2 u(x) + a_3 u(S_1 x) + a_0 u(S_2 x) + a_1 u(S_3 x), \tag{8}$$

$$v(S_3 x) = a_3 u(x) + a_2 u(S_1 x) + a_1 u(S_2 x) + a_0 u(S_3 x). \tag{9}$$

From equation (1) for the function $v(x)$ we obtain $-\Delta v(x) = f(x), x \in \Omega$. Note that if $x \in \partial\Omega_+$, then $S_1 x, S_3 x \in \partial\Omega_-$ and $S_2 x \in \partial\Omega_+$. Therefore if $x^* = S_1 x$, then from condition (2) for points $y = S_2 x \in \partial\Omega_+$ and $y^* = S_1 (S_2 x) = S_3 x \in \partial\Omega_-$ we get

$$u(y) - u(y^*) = g_0(y^*) \Leftrightarrow u(S_2 x) - u(S_1(S_2 x)) = g_0(S_2 x), x \in \partial\Omega_+.$$

And if $x^* = S_3 x$, then for points $y = S_2 x \in \partial\Omega+$ and $y^* = S_2 (S_1 x) = S_3 x \in \partial\Omega_-$ we get

$$u(y) - u(y^*) = g_0(y^*) \Leftrightarrow u(S_2 x) - u(S_2(S_1 x)) = g_0(S_2 x), x \in \partial\Omega_+.$$

Thus, if $x \in \partial\Omega_+$, then

$$u(x) - u(S_1 x) + u(S_2 x) - u(S_3 x) = g_0(x) + g_0(S_2 x), x \in \partial\Omega_+.$$

In a similar way, it can be shown that condition (3) implies

$$\frac{\partial u(x)}{\partial\nu} + \frac{\partial u(S_1 x)}{\partial\nu} + \frac{\partial u(S_2 x)}{\partial\nu} + \frac{\partial u(S_3 x)}{\partial\nu} = g_1(x) + g_2(S_1 x), x \in \partial\Omega_+.$$

Hence, for the function $v(x)$ for $x \in \partial\Omega_+$ we obtain

$$v(x) - v(S_1 x) = a_0 u(x) + a_1 u(S_1 x) + a_2 u(S_2 x) + a_3 u(S_3 x)$$

$$-a_1u(x) - a_0u(S_1x) - a_3u(S_2x) - a_2u(S_3x)$$
$$= (a_0 - a_1)\left[u(x) - u(S_1x)\right] + (a_2 - a_3)\left[u(S_2x) - u(S_3x)\right]$$
$$= (a_0 - a_1)\, g_0(x) + (a_2 - a_3)\, g_0(S_2x),$$
$$\frac{\partial v(x)}{\partial \nu} + \frac{\partial v(S_1x)}{\partial \nu} = (a_0 + a_1)g_1(x) + (a_2 + a_3)g_1(S_2x),\ x \in \partial\Omega_+.$$

Thus, if $u(x)$ is a solution to problem (1), then the function $v(x)$ satisfies the conditions of the following problem

$$-\Delta v(x) = f(x), x \in \Omega, \tag{10}$$

$$v(x) - v(S_1x) = h_0(x), x \in \partial\Omega_+, \tag{11}$$

$$\frac{\partial v(x)}{\partial \nu} + \frac{\partial v(Sx)}{\partial \nu} = h_1(x), x \in \partial\Omega_+, \tag{12}$$

where

$$h_0(x) = (a_0 - a_1)\, g_0(x) + (a_2 - a_3)\, g_0(S_2x),$$
$$h_1(x) = (a_0 + a_1)\, g_1(x) + (a_2 + a_3)\, g_1(S_2x). \tag{13}$$

Let $G_D(x, y)$ and $G_N(x, y)$ be Green's functions of the classical Dirichlet and Neumann problems for the Poisson equation. Note that in the case of a ball, the explicit form of the function $G_D(x, y)$ is given in textbooks on the equations of mathematical physics (see, for example,[2], p.40), and the function $G_N(x, y)$ was constructed in [12],[17].

The following assertion was proved in [13].

**Lemma 1.** *If $f(x) \in C^{\delta}\left(\bar{\Omega}\right), h_0(x) \in C^{1+\delta}\left(\partial\Omega_+\right)$ and $h_1(x) \in C^{\delta}\left(\partial\Omega_+\right)$, $0 < \delta < 1$ and for functions $h_0(x)$ and $h_1(x)$ the matching conditions of type* (4),(6) *are satisfied, then the solution of problem* (10)-(12) *exists, is unique and is represented in the form*

$$v(x) = \int\limits_{\Omega} G_1(x, y)f(y)dy - \int\limits_{\partial\Omega_+} \frac{\partial G_1(x, y)}{\partial \nu_y} h_0(y)dS_y$$
$$+ \int\limits_{\partial\Omega_+} G_1(x, y)h_1(y)dS_y, \tag{14}$$

*where*

$$G_1(x, y) = \frac{G_D(x, y) + G_D\left(x, y^*\right) + G_N(x, y) - G_N\left(x, y^*\right)}{2}.$$

Further, we use the notation

$$V = (v(x), v(S_1x), v(S_2x), v(S_3x))^T,$$

$$U = (u(x), u(S_1x), u(S_2x), u(S_3x))^T.$$

Then system (6)-(9) can be represented in the matrix form

$$V = AU,$$

where matrix $A$ is written as

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}.$$

Eigenvectors of matrix $A$ have the form

$$M_1 = (1,1,1,1)^T, M_2 = (1,1,-1,-1)^T,$$

$$M_3 = (1,-1,1,-1)^T, M_4 = (1,-1,-1,1)^T,$$

and their corresponding eigenvalues are represented as

$$\varepsilon_1 = a_0 + a_1 + a_2 + a_3, \varepsilon_1 = a_0 + a_1 - a_2 - a_3,$$

$$\varepsilon_3 = a_0 - a_1 + a_2 - a_3, \varepsilon_4 = a_0 - a_1 - a_2 + a_3.$$

The determinant of matrix $A$ is calculated by the formula

$$|A| = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \varepsilon_4.$$

Hence, if $\varepsilon_j \neq 0, j = 1,2,3,4$, then the matrix $A$ is invertible. The structure of the inverse matrix $B = A^{-1}$ will be the same as that of the matrix $A$ (see, for example, [5]), i.e. it has the form

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ b_1 & b_0 & b_3 & b_2 \\ b_2 & b_3 & b_0 & b_1 \\ b_3 & b_2 & b_1 & b_0 \end{pmatrix}.$$

Let us formulate the main statement for Problem P in the case $k = 1$.

**Theorem 2.** *Let $k = 1$, coefficients $a_j \neq 0, j = 0,1,2,3$ are such that the conditions $\varepsilon_j \neq 0, j = 1,2,3,4$ are satisfied. If $f(x) \in C^{\delta}\left(\bar{\Omega}\right), g_0(x) \in C^{1+\delta}\left(\partial\Omega_+\right)$ and $g_1(x) \in C^{\delta}\left(\partial\Omega_+\right), 0 < \delta < 1$, then a solution to Problem P exists, is unique, and can be represented as*

$$
\begin{aligned}
u(x) = &\int\limits_{\Omega} G_{P_1}(x,y) f(y) dy \\
&- \int\limits_{\partial\Omega_+} \left[ (a_0 - a_1) \frac{\partial G_{P_1}(x,y)}{\partial \nu_y} + (a_2 - a_3) \frac{\partial G_{P_1}(x, S_2 y)}{\partial \nu_y} \right] g_0(y) dS_y \\
&+ \int\limits_{\partial\Omega_+} \left[ (a_0 + a_1) G_{P_1}(x,y) + (a_2 + a_3) G_{P_1}(x, S_2 y) \right] g_1(y) dS_y, \quad (15)
\end{aligned}
$$

*where $G_{P_1}(x,y)$ is the Green's function of Problem P, which is defined by the equality*

$$G_{P_1}(x,y) = b_0 G_1(x,y) + b_1 G_1(S_1 x, y) + b_2 G_1(S_2 x, y) + b_3 G_1(S_3 x, y).$$

*Proof.* Let the function $v(x)$ be a solution to problem (10)-(12) with functions $h_0(x)$ and $h_1(x)$ from equality (13). Consider the function

$$u(x) = b_0 v(x) + b_1 v(S_1 x) + b_2 v(S_2 x) + b_3 v(S_3 x). \quad (16)$$

Let us show that this function satisfies all the conditions of Problem P. Indeed, applying the operator $-\Delta$ to function (16), we obtain

$$-\Delta u(x) = b_0 f(x) + b_1 f(S_1 x) + b_2 f(S_2 x) + b_3 f(S_3 x),$$

$$
\begin{aligned}
-\Delta u(S_1 x) &= b_1 f(x) + b_0 f(S_1 x) + b_3 f(S_2 x) + b_2 f(S_3 x), \\
-\Delta u(S_2 x) &= b_2 f(x) + b_3 f(S_1 x) + b_0 f(S_2 x) + b_1 f(S_3 x), \\
-\Delta u(S_3 x) &= b_3 f(x) + b_2 f(S_1 x) + b_1 f(S_2 x) + b_0 f(S_3 x).
\end{aligned}
$$

Hence, taking into account that $AB = E$ we get

$$
\begin{aligned}
Lu(x) = a_0 &\left[ b_0 f(x) + b_1 f(S_1 x) + b_2 f(S_2 x) + b_3 f(S_3 x) \right] \\
&+ a_1 \left[ b_0 f(S_1 x) + b_1 f(x) + b_2 f(S_3 x) + b_3 f(S_2 x) \right] \\
&+ a_2 \left[ b_0 f(S_2 x) + b_1 f(S_3 x) + b_2 f(x) + b_3 f(S_1 x) \right] \\
&+ a_3 \left[ b_0 f(S_3 x) + b_1 f(S_2 x) + b_2 f(S_1 x) + b_3 f(x) \right]
\end{aligned}
$$

$$= (a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)\, f(x)$$
$$+ (a_0b_1 + a_1b_0 + a_2b_3 + a_3b_2)\, f(S_1x)$$
$$+ (a_0b_2 + a_1b_3 + a_2b_0 + a_3b_1)\, f(S_2x)$$
$$+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\, f(S_3x) = f(x).$$

Let us check the fulfillment of the boundary conditions in the case $x^* = S_1x$. If $x \in \partial\Omega_+$, we get

$$u(x) - u(x^*) = b_0v(x) + b_1v(S_1x) + b_2v(S_2x) + b_3v(S_3x)$$
$$-b_0v(x^*) - b_1v(S_1x^*) - b_2v(S_2x^*) - b_3v(S_3x^*)$$
$$= b_0\left[v(x) - v(x^*)\right] + b_1\left[v(S_1x) - v(S_1x^*)\right]$$
$$+b_2\left[v(S_2x) - v(S_2x^*)\right] + b_3\left[v(S_3x) - v(S_3x^*)\right]$$
$$= b_0\left[v(x) - v(x^*)\right] - b_1\left[v(x) - v(S_1x)\right]$$
$$+b_2\left[v(S_2x) - v(S_3x)\right] - b_3\left[v(S_2x) - v(S_3x)\right]$$
$$= (b_0 - b_1)\, h_0(x) + (b_2 - b_3)\, h_0(S_2x)$$
$$= (b_0 - b_1)\left[(a_0 - a_1)\, g_0(x) + (a_2 - a_3)\, g_0(S_2x)\right]$$
$$+ (b_2 - b_3)\left[(a_0 - a_1)\, g_0(S_2x) + (a_2 - a_3)\, g_0(x)\right]$$
$$= [a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3]g_0(x) - [a_1b_0 + a_0b_1 + a_2b_3 + a_3b_2]g_0(x)$$
$$+ \left[a_2b_0 + a_3b_1 + a_0b_2 + a_1b_3\right] g_0(S_2x)$$
$$- \left[a_2b_1 + a_3b_0 + a_1b_2 + a_0b_3\right] g_0(S_2x) = g_0(x).$$

Similarly, we can show the fulfillment of the second boundary condition

$$\frac{\partial u(x)}{\partial \nu} + \frac{\partial u(x^*)}{\partial \nu} = \frac{\partial}{\partial \nu}\left[b_0v(x) + b_1v(S_1x) + b_2v(S_2x) + b_3v(S_3x)\right]$$
$$-\frac{\partial}{\partial \nu}\left[b_0v(x^*) + b_1v(S_1x^*) + b_2v(S_2x^*) + b_3v(S_3x^*)\right] = g_1(x).$$

The case when $x^* = S_3x$ is checked in a similar way. Thus, the function $u(x)$ from (16) satisfies equation (1) and conditions (2) and (3). Further, using the representation of the function $v(x)$ from (14) we obtain representation (15) for $u(x)$. The theorem is proved. □

## 3. Problem P in the case $k = 2$

In the case $k = 2$ for Problem P the following assertion is valid.

**Theorem 3.** *Let $k = 2$ and conditions $\varepsilon_j \neq 0, j = 1, 2, 3, 4$, $f(x) \in C^\delta\left(\bar{\Omega}\right), g_0(x) \in C^{1+\delta}\left(\partial\Omega_+\right)$, $g_1(x) \in C^\delta\left(\partial\Omega_+\right), 0 < \delta < 1$, are satisfied. Then, for the existence of a solution to Problem P, it is necessary and sufficient that the condition*

$$\int\limits_{\Omega} f(x)dx - \gamma \int\limits_{\partial\Omega_+} g_1(x)dS_x = 0, \tag{17}$$

*where $\gamma = (a_0 - a_1) + (a_2 - a_3)$, be satisfied. If a solution to the problem exists, then it is unique up to a constant term and is represented as*

$$\begin{aligned} u(x) = Const + \int\limits_{\Omega} G_{P_2}(x, y)f(y)dy \\ - \int\limits_{\partial\Omega_+} \left[(a_0 + a_1)\frac{\partial G_{P_2}(x, y)}{\partial\nu_y} + (a_2 + a_3)\frac{\partial G_{P_2}\left(x, S_2 y\right)}{\partial\nu_y}\right] g_0(y)dS_y \\ + \int\limits_{\partial\Omega_+} \left[(a_0 - a_1)G_{P_2}(x, y) + (a_2 - a_3)G_{P_2}\left(x, S_2 y\right)\right] g_1(y)dS_y, \end{aligned} \tag{18}$$

*where $G_{P_2}(x, y)$ is Green's function of Problem P for the case $k = 2$, which is defined by the equality*

$$G_{P_2}(x, y) = b_0 G_2(x, y) + b_1 G_2(S_1 x, y) + b_2 G_2(S_2 x, y) + b_3 G_2(S_3 x, y),$$

$$G_2(x, y) = \frac{G_D(x, y) - G_D\left(x, y^*\right) + G_N(x, y) + G_N\left(x, y^*\right)}{2} + Const.$$

*Proof.* Let $k = 2$ and a solution to Problem P exists. Let us denote this solution $u(x)$ and as in the case $k = 1$ introduce the function $v(x) = a_0 u(x) + a_1 u(S_1 x) + a_2 u(S_2 x) + a_2 u(S_3 x)$. In this case, for the function $v(x)$ we get the following problem

$$-\Delta v(x) = f(x), x \in \Omega, \tag{19}$$

$$v(x) + v(S_1 x) = h_0(x), x \in \partial\Omega_+,$$

$$\frac{\partial v(x)}{\partial\nu} - \frac{\partial v(S_1 x)}{\partial\nu} = h_1(x), x \in \partial\Omega_+, \tag{20}$$

where

$$h_0(x) = (a_0 + a_1)\, g_0(x) + (a_2 + a_3)\, g_0(S_2 x),$$
$$h_1(x) = (a_0 - a_1)\, g_1(x) + (a_2 - a_3)\, g_2(S_2 x).$$

The following assertion was proved in [13].

**Lemma 4.** *Let* $f(x) \in C^{\delta}\left(\bar{\Omega}\right), h_0(x) \in C^{1+\delta}\left(\partial\Omega_+\right), h_1(x) \in C^{\delta}\left(\partial\Omega_+\right), 0 < \delta < 1$, *and functions* $h_0(x)$ *and* $h_1(x)$ *satisfy matching conditions of type* (4),(5). *Then, for existence of a solution to problem* (19)-(20) *it is necessary and sufficient that the condition*

$$\int\limits_{\Omega} f(x)dx - \int\limits_{\partial\Omega_+} h_1(x)dS_x = 0. \tag{21}$$

*is satisfied. If a solution to the problem exists, it is unique up to a constant term and is represented in the form*

$$v(x) = \int\limits_{\Omega} G_2(x,y)f(y)dy - \int\limits_{\partial\Omega_+} \frac{\partial G_2(x,y)}{\partial \nu_y} h_0(y)dS_y + \int\limits_{\partial\Omega_+} G_2(x,y)h_1(y)dS_y, \tag{22}$$

*where*

$$G_2(x,y) = \frac{G_D(x,y) - G_D\left(x,y^*\right) + G_N(x,y) + G_N\left(x,y^*\right)}{2} + Const.$$

Thus, if a solution to Problem P exists, then condition (21) must be satisfied. Taking into account the representations of functions $h_1(x)$ for the integral over the hemisphere $\partial\Omega_+$, we get

$$\int\limits_{\partial\Omega_+} h_1(x)dS_x = (a_0 - a_1)\int\limits_{\partial\Omega_+} g_1(x)dS_x + (a_2 - a_3)\int\limits_{\partial\Omega_+} g_2(S_2 x)dS_x$$
$$= (a_0 - a_1)\int\limits_{\partial\Omega_+} g_1(x)dS_x + (a_2 - a_3)\int\limits_{\partial\Omega_+} g_1(S_2 x)dS_x$$
$$= \left[(a_0 - a_1) + (a_2 - a_3)\right]\int\limits_{\partial\Omega_+} g_1(x)dS_x.$$

Hence, condition (21) can be rewritten in the form (17). The converse statement is also valid, namely, if condition (17) is satisfied for the functions $f(x)$ and $g_1(x)$, then condition (21) is obviously satisfied for the functions $f(x)$ and $h_1(x) = (a_0 - a_1)\, g_1(x) + (a_2 - a_3)\, g_1(S_2 x)$. If condition (21) is fulfilled, the solution to problem (19) - (20) exists and is represented in the form (22). Substituting the value of the function $v(x)$ into the right side of equality (16), as in the case $k = 1$, we can show that the resulting function $u(x)$ satisfies all the conditions of Problem P. Finally, if we use equality (22), we obtain representation (18) for the function $u(x)$. The theorem is proved. □


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