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PRESERVER OF SOME SPECTRAL DOMAINS OF PRODUCT OPERATORS

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Abstract: Let $B(X)$ be the algebra of all bounded linear operators on infinite-dimensional complex Banach space X. For $T \in B(X)$ and $i \in \{1, 2, 3\}$, let $\sigma_i(T)$ denote any one of the semi-Fredholm domain, the Fredholm domain and the Weyl domain in the spectrum. We prove that if two maps φ_1 and φ_2 from $B(X)$ onto $B(X)$ satisfy

$$\sigma_i(\varphi_1(T)\varphi_2(S)) = \sigma_i(TS)$$

for all $T, S \in B(X)$, then either:
(1) there is a bounded linear operator $A : X \to X$ such that $\varphi_1(T) = AT(\varphi_2(I)A)^{-1}$ and $\varphi_2(T) = \varphi_2(I)ATA^{-1}$ for all $T \in B(X)$, or
(2) there is a bounded linear operator $A : X^* \to X$ such that $\varphi_1(T) = AT^*(\varphi_2(I)A)^{-1}$ and $\varphi_2(T) = \varphi_2(I)AT^*A^{-1}$ for all $T \in B(X)$. Where I is identity operator on X.

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1. Introduction and Background

Throughout this paper, X denotes infinite-dimensional complex Banach spaces, and $B(X)$ denotes the space of all bounded linear operators on X and its unit

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will be denoted by I. The dual space of X is denoted by X^*. For an operator $T \in B(X)$, the adjoint, the null space and the range of T are denoted by T^*, $ker(T)$ and $R(T)$, respectively. Recall that an operator $T \in B(X)$ is called upper (resp. lower) semi-Fredholm if $R(T)$ is closed and $ker(T)$ (resp. $\frac{X}{R(T)}$) is finite dimensional. The operator T is called semi-Fredholm if it is either upper or lower semi-Fredholm and T is Fredholm if it is both upper and lower semi-Fredholm. If T is a semi-Fredholm operator, then the index, $ind(T)$, is defined by

$$ind(T) = dim(kerT) - codim(R(T)).$$

Obviously, any Fredholm operator has a finite index. An operator $T \in B(X)$ is called a Weyl operator, if it is Fredholm of index zero. The spectrum, the semi-Fredholm domain, the Fredholm domain and the Weyl domain of $T \in B(X)$ are defined, respectively, by

$$\sigma(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \ is \ not \ invertible\},$$

$$\rho_{SF}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \ is \ a \ semi-Fredholm \ operator\},$$

$$\rho_F(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \ is \ a \ Fredholm \ operator\},$$

$$\rho_W(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \ is \ a \ Weyl \ operator\}.$$

Clearly, $\rho_W \subseteq \rho_F(T) \subseteq \rho_{SF}(T)$. The semi-Fredholm domain, the Fredholm domain and the Weyl domain in the spectrum of T is defined by

$$\sigma_1(T) = \{\lambda \in \sigma(T) \ : \ T - \lambda I \ is \ a \ semi-Fredholm \ operator\},$$

$$\sigma_2(T) = \{\lambda \in \sigma(T) \ : \ T - \lambda I \ is \ a \ Fredholm \ operator\},$$

$$\sigma_3(T) = \{\lambda \in \sigma(T) \ : \ T - \lambda I \ is \ a \ Weyl \ operator\},$$

respectively. Obviously, $\sigma_1(T) = \sigma(T) \bigcap \rho_{SF}(T)$ and $\sigma_3(T) \subseteq \sigma_2(T) \subseteq \sigma_1(T)$.

Several authors have studied the linear maps which preserve Fredholm or semi-Fredholm operators (see [1, 3, 4]). Shi and Ji [6] combined the spectrum with semi-Fredholm or Fredholm domain and described additive maps on $B(X)$ preserving the intersection of semi-Fredholm or Fredholm domain with the spectrum. In [2] Hajighasemi and Hejazian showed that if φ is a surjective map on $B(X)$ such that $\sigma_i(\varphi(T)\varphi) = \sigma_i(TS)$ for all $T, S \in B(X)$ then, up to a multiple factor of ± 1, the map φ is either an automorphism or an anti-automorphism.

For a vector $x \in X$ and $f \in X^*$, let $x \otimes f$ stands for the operator of rank at most one defined by $(x \otimes f)y = f(y)x$ for every $y \in X$. Note that, $(x \otimes f)^* = f \otimes \hat{x}$, where $\hat{x}$ denotes the canonical image of x in X^{**}. We denote

$F_1(X)$ the set of all rank-one operators on X and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. It is clear that $x \otimes f \in N_1(X)$ if and only if $f(x) = 0$. We denote by $F(X)$ the set of all finite rank operators in $B(X)$.

The first lemma is an elementary observation that gives the spectral domains of any rank one operator.

Lemma 1. *(See [2, Remark 1.1].) Let $x \in X$ and $f \in X^*$. If $i \in \{1, 2, 3\}$, we have*

$$\sigma_i(x \otimes f) := \begin{cases} \emptyset & \text{if} \quad f(x) = 0, \\ \{f(x)\} & \text{if} \quad f(x) \neq 0. \end{cases}$$

The following result characterizes in term of the spectral domains when two operators are the same.

Lemma 2. *(See [2, Lemma 2.1].) Let $T, S \in B(X)$ and $i \in \{1, 2, 3\}$. The following statements are equivalent*:
(1) $T = S$.
(2) $\sigma_i(TR) = \sigma_i(SR)$ *for all* $R \in B(X)$.
(2) $\sigma_i(TR) = \sigma_i(SR)$ *for all* $R \in F_1(X) \backslash N_1(X)$.

The third lemma gives a spectral characterization of rank one operators in term of the spectral domains $\sigma_i(.)$.

Lemma 3. *([2, Lemma 2.2].) For $i \in \{1, 2, 3\}$ and a nonzero operator $R \in B(X)$, the following statements are equivalent*:
(a) R has rank one.
(b) $\sigma_i(RT)$ contains at most one element for all $T \in B(X)$.
(c) $\sigma_i(RT)$ contains at most one element for every rank two operator $T \in B(X)$.

2. Main Results

The following lemma is a useful observation needed to establish the linearity of surjective maps preserving $\sigma_i(.)$, $i \in \{1, 2, 3\}$.

Lemma 4. *Let $T, S \in B(X)$ and $i \in \{1, 2, 3\}$. For two vectors $x, y \in X$ and a linear functional $f \in X^*$, the following statements hold*:

(1) $\sigma_i((x+y)\otimes f)=\sigma_i(x\otimes f)+\sigma_i(y\otimes f)$
(2) $\sigma_i((T+S)R)=\sigma_i(TR)+\sigma_i(SR)$ *for all* $R\in F_1(X)$.

Proof. First, suppose that $f(x)\neq 0$ and $f(y)\neq 0$. Therefore, by Lemma 1,

$$\sigma_i((x+y)\otimes f)=\{f(x+y)\}=\{f(x)\}+\{f(y)\}=\sigma_i(x\otimes f)+\sigma_i(y\otimes f).$$

Now, suppose that $f(x)\neq 0$ and $f(y)=0$, then

$$\sigma_i((x+y)\otimes f)=\{f(x)\}=\sigma_i(x\otimes f)+\sigma_i(y\otimes f).$$

Similarly, if $f(x)=0$ and $f(y)\neq 0$, then $\sigma_i((x+y)\otimes f)=\sigma_i(x\otimes f)+\sigma_i(y\otimes f)$.
Finally, if $f(x)=0$ and $f(y)=0$, then

$$\sigma_i((x+y)\otimes f)=\emptyset=\sigma_i(x\otimes f)+\sigma_i(y\otimes f).$$

(2) Write $R=x\otimes f$. Then, by part (1),

$$\begin{aligned}\sigma_i((T+S)R)&=\sigma_i((T+S)(x\otimes f))=\sigma_i((Tx+Sx)\otimes f)\\&=\sigma_i(Tx\otimes f)+\sigma_i(Sx\otimes f)\\&=\sigma_i(T(x\otimes f))+\sigma_i(S(x\otimes f))=\sigma_i(TR)+\sigma_i(SR).\end{aligned}$$

□

The following theorem is the main result of this paper.

Theorem 5. *Let* φ_1 *and* φ_2 *be maps from* $B(X)$ *onto* $B(X)$ *which satisfy*

$$\sigma_i(\varphi_1(T)\varphi_2(S))=\sigma_i(TS),\quad (T,S\in B(X))$$

where, $i\in\{1,2,3\}$. *Then either:*
(1) there is a bounded linear operator $A:X\to X$ *such that* $\varphi_1(T)=AT(\varphi_2(I)A)^{-1}$ *and* $\varphi_2(T)=\varphi_2(I)ATA^{-1}$ *for all* $T\in B(X)$, *or*
(2) there is a bounded linear operator $A:X^*\to X$ *such that* $\varphi_1(T)=AT^*(\varphi_2(I)A)^{-1}$ *and* $\varphi_2(T)=\varphi_2(I)AT^*A^{-1}$ *for all* $T\in B(X)$, *where* I *is identity operator on* X.

Proof. The proof of it will be completed after checking several claims.

Claim 1. φ_1 is injective and $\varphi_1(0)=0$.
If $\varphi_1(T)=\varphi_1(S)$ for some $T,S\in B(X)$, we get that

$$\sigma_i(TR)=\sigma_i(\varphi_1(T)\varphi_2(R))=\sigma_i(\varphi_1(S)\varphi_2(R))=\sigma_i(SR)$$

for all $R \in F_1(X)$. By Lemma 2, we see that $T = S$ and φ_1 is injective. For the second part of this claim,

$$\sigma_i(\varphi_1(0)\varphi_2(T)) = \sigma_i(0T) = \emptyset = \sigma_i(0\varphi_2(T))$$

for all $T \in B(X)$. Again by Lemma 2 and the surjectivity of φ_2, we see that $\varphi_1(0) = 0$.

Claim 2. φ_1 and φ_2 preserve rank one operators in both directions.

Let $R = x \otimes f$ be a rank one operator where $x \in X$ and $f \in X^*$. Note that, $\varphi_1(R) \neq 0$, since $\varphi_1(0) = 0$ and φ_1 is injective. Let $T \in B(X)$ be an arbitrary operator. By Lemma 3, $\sigma_i(RT)$ contains at most one element and $\sigma_i(\varphi_1(R)\varphi_2(T)) = \sigma_i(RT) \subseteq \{f(Tx)\}$. Since φ_1 is surjective, we conclude that, $\sigma_i(\varphi_1(R)S)$ has at most one element for every operator $S = \varphi_1(T) \in B(X)$. By Lemma 3, we see that $\varphi_1(R)$ has rank one. Conversely, assume that $\varphi_1(R)$ is rank one for some operator $R \in B(X)$, and note that $R \neq 0$ and that $\sigma_i(\varphi_1(R)\varphi_2(T))$ has at most one element for all $T \in B(X)$. Therefore, $\sigma_i(RT)$ has at most one element for all $T \in B(X)$. Again Lemma 3 tells us that R is rank one and thus φ_1 preserves the rank one operators in both directions. It is clear that, φ_2 preserves rank one operators in both directions in a similar way.

Claim 3. φ_1 is linear.

To establish the linearity of φ_1, let us first show that φ_1 is homogenous. Let R be an arbitrary rank-one operator. For every $\alpha \in \mathbb{C}$ and $T \in B(X)$, we have

$$\begin{aligned}\sigma_i(\alpha\varphi_1(T)\varphi_2(R)) &= \alpha\sigma_i(\varphi_1(T)\varphi_2(R)) \\ &= \alpha\sigma_i(TR) = \sigma_i((\alpha T)R) = \sigma_i(\varphi_1(\alpha T)\varphi_2(R)).\end{aligned}$$

Since φ_2 is surjective and preserve rank one operators in both directions, Lemma 2 shows that $\varphi_1(\alpha T) = \alpha\varphi_1(T)$. Now, let us show that φ_1 is additive. Let R be an arbitrary rank-one operator and $T, S \in B(X)$. By Lemma 4 and the previous claim, we have

$$\begin{aligned}\sigma_i(\varphi_1(T+S)\varphi_2(R)) &= \sigma_i((T+S)R) = \sigma_i(TR) + \sigma_i(SR) \\ &= \sigma_i(\varphi_1(T)\varphi_2(R)) + \sigma_i(\varphi_1(S)\varphi_2(R)) \\ &= \sigma_i((\varphi_1(T) + \varphi_2(S))R).\end{aligned}$$

By Lemma 2, we deduce that $\varphi_1(T+S) = \varphi_1(T) + \varphi_1(S)$ for all $T, S \in B(X)$.

Claim 4. φ_1 and φ_1 have the desired forms.

By the previous claim $\varphi_1 : F(X) \to F(X)$ is a bijective linear map which preserves rank one operators in both directions. Thus by [5, Theorem 3.3], φ_1

has one of the following forms.
(1) There exist bijective linear maps $A : X \to X$ and $B : X^* \to X^*$ such that

$$\varphi_1(x \otimes f) = Ax \otimes Bf, \quad x \in X, \quad f \in X^*.$$

(2) There exist bijective linear maps $A : X^* \to X$ and $B : X \to X^*$ such that

$$\varphi_1(x \otimes f) = Af \otimes Bx, \quad x \in X, \quad f \in X^*.$$

Assume that case (1) occurs. Let $x \in X$ and $f \in X^*$ be arbitrary. Assume first that $f(x) \neq 0$, using Lemma 1,

$$\{0\} \neq \{f(x)\} = \sigma_i(x \otimes f) = \sigma_i(\varphi_1(x \otimes f)\varphi_2(I)) = \sigma_i((Ax \otimes Bf)\varphi_2(I)),$$

which means that $Bf(\varphi_2(I)(Ax) \neq 0$. Then Lemma 1 implies that

$$\{f(x)\} = \sigma_i(x \otimes f) = \sigma_i(\varphi_1(x \otimes f)\varphi_2(I)) = \{(Bf)(\varphi_2(I)Ax)\}.$$

Now, if $f(x) = 0$, we choose a linear functional $g \in X^*$ such that $g(x) \neq 0$. By what has been shown lastly applied to both g and $f + g$, we have

$$g(x) = (Bg)(\varphi_2(I)Ax) \text{ and } (f + g)(x) = (B(f + g))(\varphi_2(I)Ax).$$

Then,

$$\begin{aligned} f(x) + g(x) = (f + g)(x) &= (B(f + g))(\varphi_2(I)Ax) \\ &= (Bf)(\varphi_2(I)Ax) + (Bg)(\varphi_2(I)Ax) \\ &= (Bf)(\varphi_2(I)Ax) + g(x). \end{aligned}$$

Therefore,

$$f(x) = (Bf)(\varphi_2(I)Ax), \quad (x \in X, f \in X^*), \qquad (\star).$$

It is clear that $\varphi_2(I)$ is injective, if not, there is a nonzero vector $y \in X$ such that $\varphi_2(I)y = 0$. Take $x = A^{-1}y$, and let $f \in X^*$ be a linear functional such that $f(x) \neq 0$. By $(\star)$, we have $0 \neq f(x) = (Bf)(\varphi_2(I)Ax) = (Bf)(\varphi_2(I)y) = 0$. This contradiction tells us that $\varphi_2(I)$ is injective. Now, we show that A is continuous. Assume that $(x_n)_n$ is a sequence in X such that $\lim_{n\longrightarrow\infty} x_n = x \in X$ and $\lim_{n\longrightarrow\infty} Ax_n = y \in X$. Then, for every $f \in X^*$, we have $(Bf)(\varphi_2(I)y) = \lim_{n\longrightarrow\infty}(Bf)(\varphi_2(I)Ax_n) = \lim_{n\longrightarrow\infty} f(x_n) = f(x) = (Bf)(\varphi_2(I)Ax)$. Since B is bijective and $f \in X^*$ is an arbitrary linear functional, the closed graph theorem shows that A is continuous. Moreover, we have

$f(x) = (Bf)(\varphi_2(I)Ax) = (\varphi_2(I)A)^*Bf(x)$ for every $x \in X$ and $f \in X^*$, and thus $I_{X^*} = (\varphi_2(I)A)^*B = A^*(\varphi_2(I))^*B$. It follows that $(A^*)^{-1}B^{-1} = (\varphi_2(I))^*$ and therefore $\varphi_2(I)$ is invertible and $B^* = (\varphi_2(I)A)^{-1}$. Hence, we have

$$\varphi_1(x \otimes f) = Ax \otimes Bf = A(x \otimes f)B^* = A(x \otimes f)(\varphi_2(I)^*A)^{-1}$$

for all $x \in X$ and $f \in X^*$. Therefore, for every $T \in B(X)$, we have

$$\begin{aligned}\sigma_i(A(x \otimes f)B^*\varphi_2(T)) &= \sigma_i(\varphi_1(x \otimes f)\varphi_2(T)) = \sigma_i((x \otimes f)T)\\ &= \sigma_i(A(x \otimes f)B^*(B^*)^{-1}TA^{-1}),\end{aligned}$$

for any $x \in X$ and $f \in X^*$. By Lemma 2, we conclude that $\varphi_2(T) = (B^*)^{-1}TA^{-1} = \varphi_2(I)ATA^{-1}$ for all $T \in B(X)$. For the form of φ_1, Observe that

$$\begin{aligned}\sigma_i(AT(\varphi_2(I)A)^{-1}\varphi_2(S)) &= \sigma_i(AT(\varphi_2(I)A)^{-1}(\varphi_2(I)A)SA^{-1})\\ &= \sigma_i(TS) = \sigma_i(\varphi_1(T)\varphi_2(S)),\end{aligned}$$

for all $T \in B(X)$. By Lemma 2, we have $\varphi_1(T) = AT(\varphi_2(I)A)^{-1}$, for every $T \in B(H)$.

If case (2) occurs, similar to the proof of case (1), for all $x \in X$ and $f \in X^*$ we obtain $f(x) = (Bx)(\varphi_2(I)Af)$. Also we have that $A : X^* \to X$, $\varphi_2(I) : X \to X$ and $B : X \to X^*$ are invertible. Moreover, if J denotes the canonical embedding of X in X^{**}, we have $B^*J\varphi_2(I)A = I_{X^*}$ and hence X is reflexive. So for every $x \in X$ and $f \in X^*$

$$\varphi_1(x \otimes f) = Af \otimes Bx = A(f \otimes \hat{x})B = A(f \otimes \hat{x})(\varphi_2(I)A)^{-1}.$$

Therefore, for every $R \in F_1(X)$, we see that $\varphi_1(R) = AR^*(\varphi_2(I)A)^{-1}$. By a proof similar to the above, we get $\varphi_1(T) = AT^*(\varphi_2(I)A)^{-1}$ and $\varphi_2(T) = \varphi_2(I)AT^*A^{-1}$ for all $T \in B(X)$. □

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