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SOLITON SOLUTIONS FOR THE MANHATTAN LATTICE

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Abstract: The article is devoted to the study of the aggregated model of motion in the Manhattan lattice. For such a model, the existence and uniqueness theorem of a soliton solution is established, the ranges of characteristics for which the statements of the theorem are valid are indicated, and the asymptotics of the possible growth of such solutions are obtained. A complete family of bounded soliton solutions is constructed.

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1. Introduction

One of the actively developing areas of research is related to the study of soliton solutions (traveling wave type solutions). Both the questions of the existence

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of soliton solutions and the models in which they arise are important. A large number of works are devoted to this problem [27, 25, 17, 26]. Research methods are also diverse. One of the most frequently used methods of studying such systems is the constructive construction of solutions using the explicit form of the right side, as well as the presence of various symmetries, or its possible periodicity, infinite differentiability, analyticity. Further, by methods of perturbation theory, soliton solutions are also established for the close right-hand sides. A review of works in this direction for infinite-dimensional (homogeneous) systems with Frenkel-Kontorova and Fermi-Pasta-Ulama potentials is given in [26]. At the same time, this approach does not allow us to describe the space of all soliton solutions, as well as the asymptotics of their possible growth.

The presented work is devoted to the study of the motion model in the Manhattan lattice. An aggregated model of such a movement is considered. In such a model, we assume that the number of lattice nodes is sufficiently large. Therefore, the lattice under consideration will be considered complete in all directions and homogeneous. Another assumption concerns the lack of flow details by direction.

The steady flows in such a lattice are described by soliton solutions for an infinite-dimensional ordinary differential equation in the form of a finite-difference analog of a parabolic equation. A formalism is proposed below in which the soliton solutions are in one-to-one correspondence with the solutions of a parametric family of induced functionally differential pointwise equations. The parameter of such a family of equations is the characteristic of the soliton solution.

Within the framework of the proposed formalism, it is possible to describe the entire space of soliton solutions with characteristics in the selected ranges. The possible growth of such solutions is described, where the growth is also related to the magnitude of the characteristic of the soliton solution.

The right part (operator) of an infinite-dimensional differential equation and the right part of a family of induced functional differential equations form a dual pair of "function-operator". Such dualism turns out to be very useful. In some cases, it is possible to use the properties of the finiteness of the phase space of solutions of a family of induced functional differential equations. In other cases, for soliton solutions, it is possible to use the spectral properties of the right-hand side of an infinite-dimensional differential equation. For the dual pair "function-operator", a natural question arises about the existence of a dual pair with the same family of functions, but a simpler operator.

For the model under consideration, a solution to the mentioned question is given. Among soliton solutions, as the most informative, an important class of bounded soliton solutions stands out.

The existence and uniqueness theorem of a soliton solution is established, the ranges of characteristics for which the statements of the theorem are valid are indicated, and the asymptotics of the possible growth of such solutions are obtained. A complete family of bounded soliton solutions is constructed for the model under consideration.

The approach of studying soliton solutions with a given characteristic for an infinite-dimensional dynamical system based on the existence of a *one-to-one correspondence* of such solutions with solutions of a family of induced functional differential equations was used in a number of papers [2, 3, 6, 7, 8, 9, 21, 28, 23, 22].

Beyond the discussion, we leave such an important problem as the question of the solvability of functional differential equations. On this issue, you can refer to the works [19, 20, 16, 9, 1] and a number of other works.

2. Manhattan lattice

The traffic pattern in the Manhattan lattice is such that from each point of the lattice node, movement is possible in three directions along the main highways (forward, left, right) and movement in the opposite direction through the drives. Let's describe an aggregated model of such a movement.

For the group $\Gamma = \mathbb{Z}^2$, $\{\gamma_1, \gamma_2\}$ is the system of its generators, where $\gamma_1 = (1, 0)$, $\gamma_2 = (0, 1)$. Let us consider a one-dimensional finite difference analog of a parabolic equation

$$\dot{y}_\gamma(t) = \sum_{l=1}^{2} \alpha_{\gamma_l}[y_{\gamma_l\gamma} - y_\gamma] + \sum_{l=1}^{2} \beta_{\gamma_l}[y_{\gamma_l^{-1}\gamma} - y_\gamma] + \phi(y_\gamma), \tag{1}$$
$$y_\gamma \in \mathbb{R}, \quad \forall \gamma \in \mathbb{Z}^2, \quad \text{for almost all} \quad t \in \mathbb{R},$$

where the potential ϕ is a continuous function. The solution of such a system is called any vector function $\{y_\gamma(\cdot)\}_{\gamma\in\mathbb{Z}^2}$, the coordinates $y_\gamma(\cdot)$, $\gamma \in \mathbb{Z}^2$ of which are absolutely continuous functions and satisfy the system (1) for almost all $t \in \mathbb{R}$.

For each $\gamma \in \mathbb{Z}^2$ (for each node of a two-dimensional lattice), y_γ denotes the amount of the flow of the transport flow in it. The values $\alpha_{\gamma_l}[y_{\gamma_l\gamma} - y_\gamma]$, $\beta_{\gamma_l}[y_{\gamma_l^{-1}\gamma} - y_\gamma]$, $l = 1, 2$, set the intensity of the flow changes in the node γ depending on the difference of the flow values in the nodes $\gamma, \gamma_l\gamma, \gamma_l^{-1}\gamma$, $l = 1, 2$, respectively. With negative values of such differences, the flows from the node

γ go to the nodes $\gamma_l\gamma, \gamma_l^{-1}\gamma$, $l = 1, 2$, respectively. With positive values of such differences, the flows go from the container to the γ node. The potential ϕ is given by a continuous function and has the form

$$\phi(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \phi(t) > 0, & t \in (0, \triangle), \\ \sigma[t - \triangle], & \sigma < 0, \quad t \in [\triangle, +\infty). \end{cases} \tag{2}$$

The flow level equal to $\triangle$ is critical for the node and is determined by its technical capabilities. When the values of the flow magnitude are less than the critical value of $\triangle$, the flow goes from the container to the node γ. When the flow values are greater than the critical value of $\triangle$, the flow goes from the node γ to the container.

Let us rewrite the system (1) in a different form

$$\dot{y}_\gamma(t) = \sum_{l=1}^{2} \alpha_{\gamma_l} y_{\gamma_l\gamma} - \sum_{l=1}^{2} (\alpha_{\gamma_l} + \beta_{\gamma_l}) y_\gamma + \sum_{l=1}^{2} \beta_{\gamma_l} y_{\gamma_l^{-1}\gamma} + \phi(y_\gamma), \tag{3}$$
$$y_\gamma \in \mathbb{R}, \quad \forall \gamma \in \mathbb{Z}^2, \quad \text{for almost all} \quad t \in \mathbb{R}.$$

If the conditions $\alpha_{\gamma_1} = \beta_{\gamma_1} = \alpha_{\gamma_2} = \beta_{\gamma_2} = 1$ are fulfilled for the technological characteristics of the lattice nodes, then the equation (3) turns out to be a finite difference analog of the heat equation.

For the system under consideration, we will study traveling wave type solutions (soliton solutions) as the most informative class of solutions for the steady-state flow structure.

Definition 1. We will say that the solution $\{y_\gamma(\cdot)\}_{\gamma\in\mathbb{Z}^2}$ of the system (1) defined for all $t \in \mathbb{R}$ has a traveling wave type (soliton solution) if there exists $\tau \geqslant 0$ independent of t and γ, that for all $\gamma \in \mathbb{Z}^2$ and $t \in \mathbb{R}$ the following equality is satisfied

$$y_\gamma(t + \tau) = y_{\gamma_l\gamma}(t), \quad l = 1, 2. \tag{4}$$

The constant τ will be called the *characteristic* of the traveling wave.

The task is to establish in which ranges of the characteristic τ soliton solution exists, what is the asymptotics of such solutions and their dependence on the parameters of the system. Among soliton solutions, an important class of bounded soliton solutions stands out as one of the most informative classes of solutions. Thus, we should study the solutions of the system (3)-(4), which is a boundary value problem with non-local boundary conditions (4).

3. Dual pairs “function-operator”

We will present the boundary value problem under consideration in operator form and construct a pointwise functional differential equation induced by such a boundary value problem. The operator defining the right part of an infinite-dimensional differential equation and the function defining the right part of an induced pointwise functional differential equation form a dual pair “function-operator”. For such a dual pair, statements will be formulated about the correspondence between the solutions of the operator boundary value problem and the induced functional differential equation of pointwise type.

3.1. The operator form of the boundary value problem and the induced functional differential equation

Let us formulate the boundary value problem (3)-(4) in operator form. By $\mathcal{K}^1_{\mathbb{Z}^2}$ we denote the space of sequences with elements $\varkappa = \{x_\gamma\}_{\gamma\in\mathbb{Z}^2}$, $x_\gamma \in \mathbb{R}$, endowed with the Tychonoff topology. We define the linear operator $\mathbb{A}$, the nonlinear operator $\mathbb{F}$ and the group of shift operators $\mathbb{T} = \{T_{\hat\gamma} : \hat\gamma \in \mathbb{Z}^2\}$, acting continuously from the space $\mathcal{K}^1_{\mathbb{Z}^2}$ into itself according to the following rule: for any $\gamma, \hat\gamma \in \mathbb{Z}^2$, $\varkappa \in \mathcal{K}^1_{\mathbb{Z}^2}$, $\varkappa = \{x_\gamma\}_{\gamma\in\mathbb{Z}^2}$,

$$(\mathbb{A}\varkappa)_\gamma = \sum_{l=1}^{2} \alpha_{\gamma_l} x_{\gamma_l\gamma} - \sum_{l=1}^{2} (\alpha_{\gamma_l} + \beta_{\gamma_l}) x_\gamma + \sum_{l=1}^{2} \beta_{\gamma_l} x_{\gamma_l^{-1}\gamma},$$
$$(\mathbb{F}(\varkappa))_\gamma = \phi(x_\gamma), \quad T_{\hat\gamma}\{x_\gamma\}_{\gamma\in\mathbb{Z}^2} = \{x_{\gamma\hat\gamma}\}_{\gamma\in\mathbb{Z}^2}.$$

We define the cyclic group $Q =< \check q >$, $\check q(t) = t + \tau$, as well as the epimorphism $\eta : \mathbb{Z}^2 \to Q$, where $\eta(\gamma_l) = \check q$, $l = 1, 2$ for the generators γ_l, $l = 1, 2$ of the group $\mathbb{Z}^2$. Let’s introduce the notation $\mathbb{G}_{\mathbb{Z}^2} = \mathbb{A} + \mathbb{F}$.

The boundary value problem (3)-(4) has the following equivalent operator representation

$$\dot\varkappa(t) = \mathbb{G}_{\mathbb{Z}^2}(\varkappa), \quad \text{for almost all} \quad t \in \mathbb{R}, \tag{5}$$
$$\varkappa(\eta(\hat\gamma)(t)) = T_{\hat\gamma}\varkappa(t), \quad \forall\hat\gamma \in \{\gamma_1, \gamma_2\}, \quad \forall t \in \mathbb{R}. \tag{6}$$

On the left side of the equation (5) there is the Gateaux derivative. As a *solution of an infinite-dimensional differential equation* (5) is called every vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, each coordinate of which is given by an absolutely continuous function, and the vector function itself satisfies this equation for almost all $t \in \mathbb{R}$.

Since $\{\gamma_1, \gamma_2\}$ are the generators of the group $\mathbb{Z}^2$, the system (5)-(6) is equivalent to the system

$$\dot{\varkappa}(t) = \mathbb{G}_{\mathbb{Z}^2}(\varkappa), \quad \text{for almost all} \quad t \in \mathbb{R}, \tag{7}$$

$$\varkappa(\eta(\hat{\gamma})(t)) = T_{\hat{\gamma}}\varkappa(t), \quad \forall \hat{\gamma} \in \mathbb{Z}^2, \quad \forall t \in \mathbb{R}. \tag{8}$$

The condition (8), which provides a traveling wave condition (soliton solution), means that *time shift is equal to space shift.* It is not difficult to notice that the operator $\mathbb{G}_{\mathbb{Z}^2}$ is permuted with the shift operators $T_{\hat{\gamma}}$, $\hat{\gamma} \in \mathbb{Z}^2$, i.e. for any $\hat{\gamma} \in \mathbb{Z}^2$ there is the equality $T_{\hat{\gamma}}\mathbb{G}_{\mathbb{Z}^2} = \mathbb{G}_{\mathbb{Z}^2}T_{\hat{\gamma}}$. Such a commutation property is a consequence of the spatial homogeneity of the system and the reason for the presence of all canonical properties of soliton solutions.

Let us consider the *induced* functional differential equation of pointwise type

$$\dot{x}(t) = \sum_{l=1}^{2} \alpha_{\gamma_l} x(t+\tau) - \sum_{l=1}^{2} (\alpha_{\gamma_l} + \beta_{\gamma_l}) x(t) + \sum_{l=1}^{2} \beta_{\gamma_l} x(t-\tau)$$
$$+\phi(x(t)), \quad \text{for almost all} \quad t \in \mathbb{R}. \tag{9}$$

Such an equation will be obtained by proving the Theorem 2. It is a representation of the coordinate of the operator equation (7) corresponding to the unit element e of the group $\mathbb{Z}^2$, using the traveling wave condition (8).

Solution of the pointwise type functional differential equation (9) is any absolutely continuous function $x(t)$, $t \in \mathbb{R}$, satisfying this equation almost everywhere.

We transform the functional differential equation of pointwise type (9). To do this, we introduce the notation $\alpha_{\gamma_1} + \alpha_{\gamma_2} = \alpha$, $\beta_{\gamma_1} + \beta_{\gamma_2} = \beta$. Then the equation will be rewritten as

$$\dot{x}(t) = \alpha x(t+\tau) - (\alpha+\beta)x(t) + \beta x(t-\tau) + \phi(x(t)), \tag{10}$$
$$\text{for almost all} \quad t \in \mathbb{R}.$$

The right part of such a pointwise type functional differential equation is given by the mapping $g : \mathbb{R}^3 \to \mathbb{R}$ and has the form

$$g(z_1, z_0, z_{-1}) = \alpha z_1 - (\alpha+\beta)z_0 + \beta z_{-1} + \phi(z_0).$$

The operator $\mathbb{G}_{\mathbb{Z}^2}$, as the right part of an infinite-dimensional ordinary differential equation (7), and the function g, as the right part of an induced functional differential equation of pointwise type (9), *form a dual pair* $(g|\mathbb{G}_{\mathbb{Z}^2})$.

The following important statement is true.

Theorem 2. *Let $(g|\mathbb{G}_{\mathbb{Z}^2})$ be a dual pair. Each solution $x(t)$, $t \in \mathbb{R}$ of a pointwise type functional differential equation* (9) *corresponds to the solution $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$ of the boundary value problem* (7)-(8) *(traveling wave type solution) and vice versa. Such solutions are connected by the relations $x_\gamma(t) = x_e(\eta(\gamma)(t))$, $x_e(t) = x(t)$, $\forall\gamma \in \mathbb{Z}^2$, $\forall t \in \mathbb{R}$.*

Proof. Let $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$ be a solution of the boundary value problem (7)-(8). For such a solution in the infinite-dimensional differential equation (7), the coordinate $x_e(t)$, $t \in \mathbb{R}$, corresponding to the unit element $e = (0, 0)$ of the group $\mathbb{Z}^2$ satisfies the equation

$$\dot{x}_e(t) = \sum_{l=1}^{2} \alpha_{\gamma_l} x_{\gamma_l}(t) - \sum_{l=1}^{2} (\alpha_{\gamma_l} + \beta_{\gamma_l}) x_e(t) + \sum_{l=1}^{2} \beta_{\gamma_l} x_{\gamma_l^{-1}}(t)$$
$$+\phi(x_e(t)), \quad \text{for almost all} \quad t \in \mathbb{R}. \tag{11}$$

From the boundary condition (8) it follows that the system of equalities $x_\gamma(t) = x(\eta(\gamma)(t)$, $\gamma \in \mathbb{Z}^2$ holds for the solution under consideration and, in particular, the equalities $x_{\gamma_l}(t) = x_e(\eta(\gamma_l)(t)) = x_e(t + \tau)$, $x_{\gamma_l^{-1}}(t) = x_e(\eta(\gamma_l^{-1})(t)) = x_e(t - \tau)$, $l = 1, 2$ hold. If these values of $x_{\gamma_l}(t), x_{\gamma_l^{-1}}(t)$, $l = 1, 2$ substitute into the equation (11) and re-assign the variable x_e to the variable x, then such an equation will coincide with the functional differential equation (9).

In the opposite direction: Let $x(\cdot)$ be the solution of a pointwise type functional differential equation (9). Let's define an infinite-dimensional vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, where $x_\gamma(t) = x(\eta(\gamma)(t))$, $\gamma \in \mathbb{Z}^2$. Then, by virtue of the functional differential equation (9), the following system of equalities will be fulfilled

$$\dot{x}_\gamma(t) = \sum_{l=1}^{2} \alpha_{\gamma_l} x_{\gamma_l\gamma}(t) - \sum_{l=1}^{2} (\alpha_{\gamma_l} + \beta_{\gamma_l}) x_\gamma(t) + \sum_{l=1}^{2} \beta_{\gamma_l} x_{\gamma_l^{-1}\gamma}(t)$$
$$+\phi(x_\gamma(t)), \quad \forall\gamma \in \mathbb{Z}^2, \quad \text{for almost all} \quad t \in \mathbb{R}, \tag{12}$$

and the vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$ will be the solution of the infinite-dimensional differential equation (7). On the other hand, the vector function $\varkappa(\cdot)$ defined in a such way satisfies the boundary condition (8). The theorem is proved. □

3.2. Existence and uniqueness theorems of the solution for the dual pair "function-operator"

We transform the functional differential equation of pointwise type (10) from the considered dual pair. We replace the time so that the deviations of the argument become integer, and the characteristic τ is considered as a parameter

$$\dot{\bar{x}}(t) = \tau[\alpha\bar{x}(t+1) - (\alpha+\beta)\bar{x}(t) + \beta\bar{x}(t-1) + \phi(\bar{x}(t))],$$

$$\text{for almost all } t \in \mathbb{R}.$$

Such an equation was investigated in the monograph [9] with minimal restrictions on the potential $\pi(\cdot)$ in the form of the Lipschitz condition (quasi-linear potentials).

Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Consider a transcendental equation with respect to two variables $\tau \in (0, +\infty)$ and $\mu \in (0, 1)$

$$2C_\phi\tau\left(\mu^{-1} + 1\right) = \ln\mu^{-1}, \tag{13}$$

where

$$C_\phi = \max\left\{\alpha + \beta; L_\phi\right\}.$$

The set of solutions of the equation (13) is described by the functions $\mu_1(\tau)$, $\mu_2(\tau)$, given in Fig. 1.

To study the existence and uniqueness of solutions of pointwise type functional differential equations, we propose their localization in the spaces of functions majored by functions of a given exponential growth with the power as a parameter

$$\mathcal{L}_\mu^n C^{(0)}(\mathbb{R}) = \left\{ z(\cdot) : z(\cdot) \in C^{(0)}\left(\mathbb{R}, \mathbb{R}^n\right), \sup_{t\in\mathbb{R}} \|z(t)\mu^{|t|}\|_{R^n} < +\infty \right\},$$
$$\|z(\cdot)\|_\mu^{(0)} = \sup_{t\in\mathbb{R}} \|z(t)\mu^{|t|}\|_{R^n}, \quad \mu \in (0,1). \tag{14}$$

We formulate a theorem of the existence and uniqueness of a solution for the induced functional differential equation (9).

Theorem 3 ([9]). *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial values $a \in \mathbb{R}$, $\tilde{t} \in \mathbb{R}$ and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

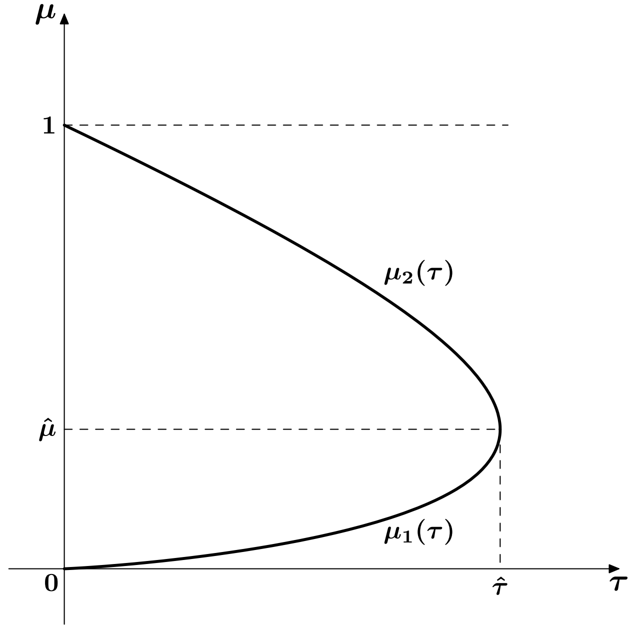

Figure 1: Graphs of functions $\mu_1(\tau), \mu_2(\tau)$.

in the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall \mu,\ \mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, *for the functional differential equation* (9) *there exists a unique solution* $x(t), t \in \mathbb{R}$ *such that it satisfies the initial condition* $x(\tilde{t}) = a$. *Such a solution, as an element of the space* $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, *continuously depends on the initial value* $a \in \mathbb{R}$, *the characteristic* τ *and the potential* $\phi(\cdot)$.

Theorem 3 not only guarantees the existence of a solution but also sets a limit on its possible growth in time t. Moreover, for each $0 < \tau < \hat{\tau}$ spaces $\mathcal{L}^1_{(\sqrt[\tau]{\mu_2(\tau)}-\varepsilon)} C^{(0)}(\mathbb{R})$ for small $\varepsilon > 0$ is much narrower than the spaces $\mathcal{L}^1_{(\sqrt[\tau]{\mu_1(\tau)}+\varepsilon)} C^{(0)}(\mathbb{R})$. The theorem guarantees the existence of a solution in narrower spaces and uniqueness in wider spaces. This property of solutions simultaneously with the Theorem 2 on the one-to-one correspondence of soliton solutions to solutions of a family of induced functional differential equations underlies the proof of the existence and uniqueness theorems of soliton solutions, which will be given below.

Theorem 3 allows reformulation in terms of traveling wave type solutions (soliton solutions) for the original finite difference analogue of the parabolic equation (3). To do this, in the space $\mathcal{K}^1_{\mathbb{Z}^2}$, we define a family of Hilbert subspaces $\mathcal{K}^1_{\mathbb{Z}^2 2\mu}$, $\mu \in (0,1)$

$$\mathcal{K}^1_{\mathbb{Z}^2 2\mu} = \left\{ \varkappa : \varkappa \in \mathcal{K}^1_{\mathbb{Z}^2};\ \sum_{\gamma\in\mathbb{Z}^2} |x_\gamma|^2 \mu^{2|\gamma|} < +\infty \right\},\ |\gamma| = |(i,j)| = |i| + |j|$$

with the norm

$$\|\varkappa\|_{\mathbb{Z}^2 2\mu} = \Big[\sum_{\gamma\in\mathbb{Z}^2} |x_\gamma|^2 \mu^{2|\gamma|} \Big]^{\frac{1}{2}}.$$

Theorem 4. *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial values $\bar\gamma \in \mathbb{Z}^2$, $a \in \mathbb{R}$, $\bar t \in \mathbb{R}$ and characteristic $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat\tau,$$

for the initial infinite-dimensional system of differential equations (3) *there is a unique soliton solution $\varkappa(t) = \{y_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$ (solution of the boundary value problem* (7)-(8)*) with the characteristic τ such that it satisfies the initial condition $y_{\bar\gamma}(\bar t) = a$, for any parameter $\mu, \mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$. Values of the vector function*

$$\varkappa(t) = \{y_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$$

for any $t \in \mathbb{R}$ belong to the space $\mathcal{K}^1_{\mathbb{Z}^2 2\mu}$, and the function

$$\rho(t) = \|\varkappa(t)\|_{2\mu}$$

belongs to the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$. Such a solution continuously depends on the initial value $a \in \mathbb{R}$, the characteristic τ and the potential $\phi(\cdot)$.

Proof. By Theorem 3, for any initial value $a \in \mathbb{R}$, $\tilde t \in \mathbb{R}$ in the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall \mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$ for the functional differential equation (9) there exists a unique solution $x(t)$, $t \in \mathbb{R}$ such that it satisfies the initial condition $x(\tilde t) = a$. Then by Theorem 2, vector-function $\varkappa(t) = \{y_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $y_\gamma(t) = x(\eta(\gamma)(t))$ is a solution of the boundary value problem (7)-(8) (a soliton solution). Let $a, \bar\gamma, \bar t$ be given. If we chose $\tilde t = \eta(\bar\gamma)(\bar t)$, the initial condition $y_\gamma(\bar t) = a$ will be met. It remains to show that $\varkappa(t) \in \mathcal{K}^1_{\mathbb{Z}^2 2\mu}$, $\forall t \in \mathbb{R}$ and $\rho(\cdot) \in \mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall \mu, \mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$. For an arbitrary $t \in \mathbb{R}$ and $\forall \mu, \forall \bar\mu$, $\mu^\tau, \bar\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, $\mu < \bar\mu$ taking into account the evaluation of $|\eta(\gamma)(t)| \leq |\gamma| + |t|$, let's have the following estimations

$$\|\varkappa(t)\|_{\mathbb{Z}^2 2\mu} = \Big[\sum_{\gamma\in\mathbb{Z}^2} |y_\gamma(t)|^2 \mu^{2|\gamma|} \Big]^{\frac{1}{2}} = \Big[\sum_{\gamma\in\mathbb{Z}^2} |x(\eta(\gamma)(t)\,)|^2 \mu^{2|\gamma|} \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_{\gamma\in\mathbb{Z}^2} |x(\eta(\gamma)(t)\,)|^2 \,\bar{\mu}^{2|\gamma|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|} \Big]^{\frac{1}{2}}$$

$$\leq \Big[\sum_{\gamma\in\mathbb{Z}^2} |x(\eta(\gamma)(t)\,)|^2 \;\bar{\mu}^{2(|\gamma|+|t|)}\;\bar{\mu}^{-2|t|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|} \Big]^{\frac{1}{2}}$$

$$\leq \Big[\sum_{\gamma\in\mathbb{Z}^2} |x(\eta(\gamma)(t)\,)|^2 \;\bar{\mu}^{2|\eta(\gamma)(t)\,|}\;\bar{\mu}^{-2|t|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|} \Big]^{\frac{1}{2}}$$

$$\leq \|x(\cdot)\|^{(0)}_{\bar{\mu}}\,\bar{\mu}^{-|t|} \Big[\sum_{\gamma\in\mathbb{Z}^2} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|} \Big]^{\frac{1}{2}}.$$

Since the group $\mathbb{Z}^2$ is commutative (polynomial growth) and $\frac{\mu}{\bar{\mu}} < 1$, the sum is finite and equal to some A. We finally get the estimate

$$\|\varkappa(t)\|_{\mathbb{Z}^2 2\mu} \leq A\,\|x(\cdot)\|^{(0)}_{\bar{\mu}}\,\bar{\mu}^{-|t|}, \quad \forall t\in\mathbb{R}. \tag{15}$$

It follows from the obtained estimate that for $\forall\mu$, $\forall\bar{\mu}$, $\mu^{\tau}, \bar{\mu}^{\tau} \in (\mu_1(\tau), \mu_2(\tau))$, $\mu < \bar{\mu}$, for the vector function $\varkappa(\cdot)$, the conditions $\varkappa(t) \in \mathcal{K}^1_{\mathbb{Z}^2 2\mu}$ hold for any $t\in\mathbb{R}$ and $\rho(\cdot) \in \mathcal{L}^1_{\bar{\mu}}C^{(0)}(\mathbb{R})$. It follows that $\forall\mu$, $\mu^{\tau} \in (\mu_1(\tau), \mu_2(\tau))$ the conditions $\varkappa(t) \in \mathcal{K}^1_{\mathbb{Z}^2 2\mu}$ are valid for any $t\in\mathbb{R}$ and $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$. The existence of a soliton solution is proved. It remains to prove its uniqueness.

Let the vector function $\varkappa(\cdot)$ be a soliton solution (solution of the boundary value problem (7)-(8)) and satisfy the conditions: $y_{\bar{\gamma}}(\bar{t}) = a$, $\varkappa(t) \in \mathcal{K}^1_{\mathbb{Z}^2 2\mu}$, $\forall t\in\mathbb{R}$ and $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$, $\forall\mu$, $\mu^{\tau} \in (\mu_1(\tau), \mu_2(\tau))$. Then the function $x(t) = y_e(t)$, $\forall t\in\mathbb{R}$ will be the solution of the induced functional differential equation (9) and satisfies the initial condition $x(\tilde{t}) = a$, $\tilde{t} = \eta(\bar{\gamma})(\bar{t})$. Moreover, from the condition $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$ it follows that $x(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$. Then by Theorem 3, such a solution $x(\cdot)$ of the equation (9) is the only one, whence the uniqueness of the soliton solution follows. The theorem is proved. □

3.3. Canonical dual pair "function-operator" as a dual pair with the simplest operator

In the previous section, we studied the existence and uniqueness of a soliton solution, as well as for the dual pair $(g|\mathbb{G}_{\mathbb{Z}^2})$ the relationship of soliton solutions of an infinite-dimensional ordinary differential equation with the right-hand side $G_{\mathbb{Z}^2}$ and solutions of a family of induced functional differential equations with the right side of g. The question is to identify a dual pair with the simplest operator $\mathbb{G}_{\mathbb{Z}^2}$ among the dual pairs with the same function g.

For the group $\Gamma = \mathbb{Z}$, by γ_1 we denote the generator, where $\gamma_1 = 1$. Consider a one-dimensional finite difference analog of a parabolic equation

$$\begin{aligned} &\dot{\bar{y}}_\gamma(t) = \alpha[\bar{y}_{\gamma_1\gamma} - \bar{y}_\gamma] + \beta[\bar{y}_{\gamma_1^{-1}\gamma} - \bar{y}_\gamma] + \phi(\bar{y}_\gamma), \\ &\bar{y}_\gamma \in \mathbb{R}, \quad \forall \gamma \in \mathbb{Z}, \quad \text{for almost all} \quad t \in \mathbb{R}. \end{aligned} \tag{16}$$

Let us rewrite the system (16) in a different form

$$\begin{aligned} &\dot{\bar{y}}_\gamma(t) = \alpha\bar{y}_{\gamma_1\gamma} - (\alpha+\beta)\bar{y}_\gamma + \beta\bar{y}_{\gamma_1^{-1}\gamma} + \phi(\bar{y}_\gamma), \\ &\bar{y}_\gamma \in \mathbb{R}, \quad \forall \gamma \in \mathbb{Z}, \quad \text{for almost all} \quad t \in \mathbb{R}, \end{aligned} \tag{17}$$

Definition 5. We will say that the solution $\{\bar{y}_\gamma(\cdot)\}_{\gamma\in\mathbb{Z}}$ of the system (16) defined for all $t \in \mathbb{R}$ has a traveling wave type (soliton solution) if there exists $\tau \geqslant 0$, independent of t and γ, that for all $\gamma \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality is satisfied

$$y_\gamma(t+\tau) = y_{\gamma_l\gamma}(t), \quad l = 1, 2. \tag{18}$$

The constant τ will be called the *characteristic* of the traveling wave.

Let us formulate the boundary value problem (17)-(18) in operator form. By $\mathcal{K}^1_{\mathbb{Z}}$ we denote the space of sequences with elements $\varkappa = \{x_\gamma\}_{\gamma\in\mathbb{Z}}$, $x_\gamma \in \mathbb{R}$, endowed with the Tychonoff topology. We define the linear operator $\mathbb{A}$, the nonlinear operator $\mathbb{F}$ and the group of shift operators $\mathbb{T} = \{T_{\hat{\gamma}} : \hat{\gamma} \in \mathbb{Z}$, acting continuously from the space $\mathcal{K}^1_{\mathbb{Z}}$ into itself according to the following rule: for any $\gamma, \hat{\gamma} \in \mathbb{Z}$, $\varkappa \in \mathcal{K}^1_{\mathbb{Z}}$, $\varkappa = \{x_\gamma\}_{\gamma\in\mathbb{Z}}$

$$\begin{aligned} &(\mathbb{A}\varkappa)_\gamma = \alpha\bar{x}_{\gamma_l\gamma} - (\alpha+\beta)\bar{x}_\gamma + \beta\bar{x}_{\gamma_l^{-1}\gamma}, \quad (\mathbb{F}(\varkappa))_\gamma = \phi(\bar{x}_\gamma), \\ &T_{\hat{\gamma}}\{\bar{x}_\gamma\}_{\gamma\in\mathbb{Z}} = \{\bar{x}_{\gamma\hat{\gamma}}\}_{\gamma\in\mathbb{Z}}. \end{aligned}$$

We define the cyclic group $Q =< \check{q} >$, $\check{q}(t) = t + \tau$, as well as the epimorphism $\bar{\eta} : \mathbb{Z} \to Q$, where $\bar{\eta}(\gamma_1) = \check{q}$ for the generator γ_1 of the group $\mathbb{Z}$. Let us introduce the notation $\mathbb{G}_{\mathbb{Z}} = \bar{\mathbb{A}} + \bar{\mathbb{F}}$.

The boundary value problem (17)-(18) has the following equivalent operator representation

$$\dot{\varkappa}(t) = \mathbb{G}_{\mathbb{Z}}(\varkappa), \quad \text{for almost all} \quad t \in \mathbb{R}. \tag{19}$$

$$\varkappa(\eta(\gamma_1)(t)) = T_{\gamma_1}\varkappa(t), \quad \forall t \in \mathbb{R}. \tag{20}$$

On the left side of the equation (19) there is the Gateaux derivative. As a *solution of an infinite-dimensional differential equation* (19) is called every vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}}$, $t \in \mathbb{R}$, each coordinate of which is given by an absolutely continuous function, and the vector function itself satisfies this equation for almost all $t \in \mathbb{R}$.

Since γ_1 is the generator of the group $\mathbb{Z}$, the system (19)-(20) is equivalent to the system

$$\dot{\varkappa}(t) = \mathbb{G}_{\mathbb{Z}}(\varkappa), \quad \text{for almost all} \quad t \in \mathbb{R}, \tag{21}$$

$$\varkappa(\eta(\hat{\gamma})(t)) = T_{\hat{\gamma}}\varkappa(t), \quad \forall\hat{\gamma} \in \mathbb{Z}, \quad \forall t \in \mathbb{R}. \tag{22}$$

The condition (22), which provides a traveling wave condition (soliton solution), means that *time shift is equal to space shift.* It is not difficult to notice that the operator $\mathbb{G}_{\mathbb{Z}}$ is permuted with the shift operators $T_{\hat{\gamma}}$, $\hat{\gamma} \in \mathbb{Z}$, i.e. for any $\hat{\gamma} \in \mathbb{Z}$ there is the equality $T_{\hat{\gamma}}\mathbb{G}_{\mathbb{Z}} = \mathbb{G}_{\mathbb{Z}}T_{\hat{\gamma}}$.

If for the system (21)-(22) construct an induced pointwise type functional differential equation using the procedure given after the equation (9), then we get the equation (10). The right-hand side of such an equation is also given by the function g. In this case, we will get a dual pair of "function-operator" of the form $(g|\mathbb{G}_{\mathbb{Z}})$. Such a dual pair is called *canonical.* It is a dual pair with the same function g and the simplest operator $\mathbb{G}_{\mathbb{Z}}$.

For such a dual pair, the existence and uniqueness theorem for the induced functional differential equation coincides with Theorem 3. The theorem of existence and uniqueness of a soliton solution for the system (17) is similar to Theorem 4. To formulate it in the space $\mathcal{K}^1_{\mathbb{Z}}$, we define a family of Hilbert subspaces $\mathcal{K}^1_{\mathbb{Z}2\mu}$, $\mu \in (0,1)$

$$\mathcal{K}^1_{\mathbb{Z}2\mu} = \left\{ \varkappa : \varkappa \in \mathcal{K}^1_{\mathbb{Z}};\ \sum_{\gamma\in\mathbb{Z}} |x_\gamma|^2 \mu^{2|\gamma|} < +\infty \right\},\ |\gamma| = |(i,j)| = |i| + |j|$$

with the norm

$$\|\varkappa\|_{\mathbb{Z}2\mu} = \big[\sum_{\gamma\in\mathbb{Z}} |x_\gamma|^2 \mu^{2|\gamma|}\big]^{\frac{1}{2}}.$$

Theorem 6. *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial values $\bar{\gamma} \in \mathbb{Z}$, $a \in \mathbb{R}$, $\bar{t} \in \mathbb{R}$ and characteristic $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial infinite-dimensional system of differential equations (17) there is a unique solution $\varkappa(t) = \{y_\gamma(t)\}_{\gamma\in\mathbb{Z}}$, $t \in \mathbb{R}$ of traveling wave type (soliton *solution) with characteristic τ such that it satisfies the initial condition $y_{\bar{\gamma}}(\bar{t}) = a$, for any parameter μ, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$. Values of the vector function*

$$\varkappa(t) = \{y_\gamma(t)\}_{\gamma \in \mathbb{Z}}$$

for any $t \in \mathbb{R}$ belong to the space $\mathcal{K}^1_{\mathbb{Z}2\mu}$, and the function

$$rho(t) = \|\varkappa(t)\|_{2\mu}$$

belongs to the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$. Such a solution continuously depends on the initial value $a \in \mathbb{R}$, the characteristic τ and the potential $\phi(\cdot)$.

Proof. The proof repeats verbatim the proof of Theorem 4. The only difference is that everywhere the group $\mathbb{Z}^2$ should be replaced by the group $\mathbb{Z}$. □

3.4. Bounded soliton solutions

Bounded soliton solutions occupy a special place among soliton solutions. By virtue of Theorems 2-6, bounded soliton solutions correspond to bounded solutions of the induced functional differential equation. Let us proceed to the study of bounded solutions of a family of induced pointwise type functional differential equation (10)

$$\dot{x}(t) = \alpha x(t+\tau) - (\alpha+\beta)x(t) + \beta x(t-\tau) + \phi(x(t)), \text{ for almost all } t \in \mathbb{R},$$

in which the parameter of the family is the characteristic τ of the soliton solution.

To begin with, we describe *stationary solutions*. There are two such solutions

$$x(t) \equiv 0, \quad x(t) \equiv \triangle, \quad t \in \mathbb{R}.$$

The equation under consideration has a number of symmetries. Due to the autonomy of the equation and the theorem of the existence and uniqueness of the solution (Theorem 3), the solution space *is invariant* with respect to the time shift of solutions. Moreover, due to the one-dimensionality of the phase space, every periodic solution is stationary. Therefore, non-stationary solutions are strictly monotonic.

4. Numerical construction of bounded soliton solutions

4.1. Approximation of the solution of a functional differential equation defined on the line by solutions of a initial-boundary value problem with expanding intervals of the definition

On the basis of Theorem 3 we are going to formulate a proposition on the approximation of solutions of an initial-boundary value problem defined on the whole line by solutions of the initial-boundary value problem defined on the interval $[-k, k]$ as $k \to +\infty$. We consider the initial-boundary value problem on the whole line $B_R = \mathbb{R}$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in \mathbb{R}, \tag{23}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n \tag{24}$$

and for each $k \in \mathbb{Z}_+$ the initial-boundary value problem on the finite interval $B_R = [-k, k]$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in [-k, k] \tag{25}$$

$$\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R} \backslash [-k, k], \quad \varphi(\cdot) \in \mathcal{L}_1^n L_\infty(\mathbb{R}), \tag{26}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n, \tag{27}$$

where $n_j \in \mathbb{R}, j = 1, \ldots, s$.

Theorem 7. *([13]) If, for $\mu \in \cap(0,1)$, the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|n_j|} < \ln \mu^{-1} \tag{28}$$

is satisfied, and (μ_1, μ_2) is the maximum interval of solutions of the inequality (28) then for any $\bar{x} \in \mathbb{R}^n, \varphi(\cdot) \in \mathcal{L}_1^n L_\infty(\mathbb{R})$ the solution $\hat{x}(\cdot)$ of the initial value problem (23)-(24), as an element of the space $\mathcal{L}_\mu^n C^{(0)}(\mathbb{R})$, is approximated by solutions $\hat{x}_k(\cdot)$ of the initial-boundary value problem (25)–(27) as $k \to +\infty$. Moreover, for any arbitrarily small ε, $0 < \varepsilon < \mu_2 - \mu_1$ there exists $C_{f\varphi\varepsilon}$ such that the following estimate takes place

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|_\mu^{(0)} \le C_{f\varphi\varepsilon} \left(\frac{\mu_1}{\mu_2 - \varepsilon} \right)^k .$$

4.2. Numerical experiments

Next, the results of the computational experiments on the study of boundary value problems for systems of functional differential equations of pointwise type (FDEPT) using OPTCON-F software will be presented. The software complex OPTCON-F is designed to obtain a numerical solution of boundary value problems, parametric identification problems and optimal control for dynamical systems described by FDEPT [18]. The proposed technology for solving boundary value problems is based on the Ritz method and spline collocation approaches. To solve the problem we discretized system trajectories on the grid with a constant step and formulate the generalized residual functional, including both weighted residuals of the original differential equation and residuals of boundary conditions.

Let us consider the FDEPT of the following form

$$\dot{x}(t) = \alpha x(t+\tau) - (\alpha+\beta)x(t) + \beta x(t-\tau) + \phi(x(t)), \quad t \in \mathbb{R}, \tag{29}$$

where $\alpha, \beta \in \mathbb{R}, \tau, \triangle \in \mathbb{R}_+$, and

$$\phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1-\exp(-x^2)}{1+\exp(-x^2)}, & \text{if } 0 \le x \le \triangle/2 \\ \frac{\exp(-(x-delta/2)^2)-1}{\exp(-(x-delta/2)^2)+1} + \frac{1-\exp(-(delta/2)^2)}{1+\exp(-(delta/2)^2)}, & \text{otherwise.} \end{cases}$$

Using a time-variable transformation the equation (29) can be rewritten in the form of the following first order equation:

$$\dot{\mathbf{z}}(t) = \tau(\alpha\mathbf{z}(t+1) - (\alpha+\beta)\mathbf{z}(t) + \beta\mathbf{z}(t-1) + \phi(\mathbf{z}(t))).$$

Under this equation, we have the following real parameters: $\alpha, \beta, \tau, \triangle$. In the following example, for a given system, we consider such parameter values that conditions of the existence theorem are satisfied.

Example

We consider dynamical system in the following form:

$$\begin{aligned} &\dot{\mathbf{z}}(t) = 0.1(\mathbf{z}(t+1) - 0.5\mathbf{z}(t) - 0.5\mathbf{z}(t-1) + \phi(\mathbf{z}(t))), \\ &t \in \mathbb{R}, \\ &\text{initial condition:} \\ &\mathbf{z}(0) = c. \end{aligned} \tag{30}$$

In this case, $\alpha = 1, \beta = -0.5, \tau = 0.1, \triangle = 10$, and the equation (13) has on the interval $(0, 1)$ two solutions with approximate values 0.0588 and 0.6771 (the exact values is expressed in terms of the Lambert W-function).

Taking into account the impossibility of considering the numerical solution of the system on an infinite interval, we introduce the parameter k and the corresponding family of expanding initial-boundary value problems

$$\begin{aligned}
&\dot{\mathbf{z}}(t) = 0.1(\mathbf{z}(t+1) - 0.5\mathbf{z}(t) - 0.5\mathbf{z}(t-1) + \phi(\mathbf{z}(t))),\\
&t \in [-k, k],\\
&\text{boundary condition:}\\
&\dot{\mathbf{z}}(t) = 0, \quad t \in (-\infty, -k] \cup [k, +\infty),\\
&\text{initial condition:}\\
&\mathbf{z}(0) = c.
\end{aligned} \tag{31}$$

According to Theorem 7, the solution of the system (31) converges (according to the metric of the space $\mathcal{L}^n_\mu C^{(0)}(\mathbb{R})$ with $\mu \in (\mu_1(\tau), \mu_2(\tau))$) to the solution of the system (30) as $k \to \infty$.

Since the equation (29) is autonomous, the solution space of such equation is invariant with respect to time-variable shifts. Therefore, it suffices to consider a family of solutions of the initial problem (30) with a value of $z(0)$ from zero to the value of $\triangle$. Figure 2 shows the integral curves for different values of the parameter $c = z(0)$ for both the system (31) and the original system (30).

Note that Fig. 2 shows a complete family of bounded solutions (up to the above-mentioned transformation), but at the same time, for some initial conditions there are only unbounded solutions.

5. Conclusion

There were two important assumptions in the Manhattan lattice model under consideration.

First, the number of vertices in the lattice is considered to be large enough, and the procedures for regulating flows in the lattice are uniform throughout the lattice. Therefore, we can consider the lattice complete in all directions. The completeness of the lattice and the unity of flow control procedures determine the spatial uniformity of the system (the right part of the considered infinite-dimensional ordinary differential equation is permutable with a group of shift operators), and with it they guarantee the presence of a rich family of soliton solutions.

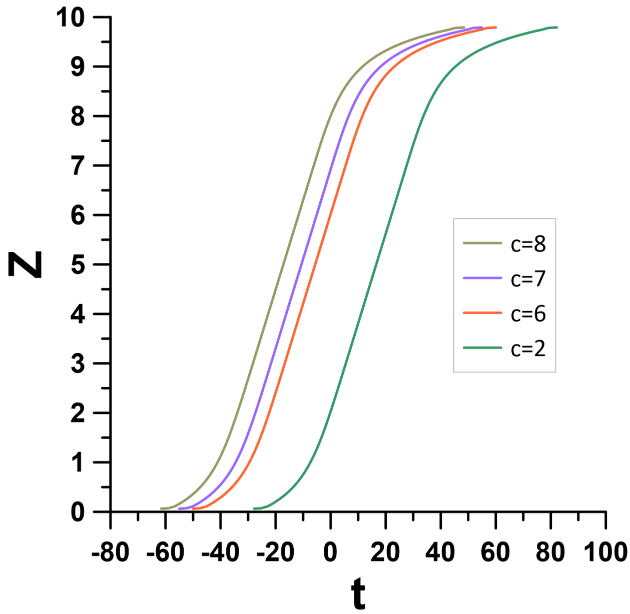

Figure 2: Graphs of bounded solution for various c.

Second, a high level of aggregation of flow characteristics, without detailing flows in directions.

When passing to the consideration of incomplete lattices (lattices with boundaries), the spatial homogeneity of the system is lost (the right part of the considered infinite-dimensional ordinary differential equation is not permutable with a group of shift operators), which leads to a narrowing of the class of soliton solutions, or even their absence. In this case, it is necessary to expand the concept of traveling wave type solutions to quasi-traveling wave type solutions (soliton solutions with a defect). If solutions (absolutely continuous) of the induced family of pointwise type functional differential equations correspond to soliton solutions in the homogeneous model, then in the inhomogeneous model, quasi-traveling waves (soliton solutions with a defect) will correspond to impulse solutions of the induced family of functional differential equations. This approach was implemented for a finite difference analogue of the wave equation in one plastic deformation problem about longitudinal vibrations in an inhomogeneous infinite rod and the heat conduction equation in one problem on transportation [8, 10, 11, 12, 14, 15].

The detailing of flows in directions leads to a complication of the structure of the lattice itself and requires other new approaches to describe soliton solutions.

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References

[1] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, Hindawi Publishing Corporation, New York (2007).

[2] L.A. Beklaryan, A boundary value problem for a differential equation with deviating argument, *Dokl. Akad. Nauk SSSR*, **291**, No 1 (1986), 19-22.

[3] L.A. Beklaryan, A differential equation with deviating argument as an infinite-dimensional dynamical system, In: *Reports in Applied Mathematics*, Vychisl. Tsentr, Moscow (1989), 18.

[4] L.A. Beklaryan, A method for the regularization of boundary value problems for differential equations with deviating argument, *Soviet Math. Dokl.*, **43**, No 2 (1991), 567-571.

[5] L.A. Beklaryan, An optimal control problem for systems with deviating argument and its connection with the finitely generated group of homeomorphisms $\mathbb{R}$ generated by deviation functions, *Soviet Math. Dokl.*, **43**, No 2 (1991), 600-605.

[6] L.A. Beklaryan, Group singularities of differential equations with deviating arguments and metric invariants related to them, *Journal of Mathematical Sciences*, **105**, No 1 (2001), 1799-1811.

[7] L.A. Beklaryan, Equations of advanced-retarded type and solutions of traveling-wave type for infinite-dimensional dynamic systems, *Journal of Mathematical Sciences*, **124**, No 4 (2004), 5098-5109.

[8] L.A. Beklaryan, N.K. Khachatryan, Traveling wave type solutions in dynamic transport models, *Functional Differential Equations*, **13**, No 2 (2006), 125-155.

[9] L.A. Beklaryan, *Introduction to the Theory of Functional Differential Equations. Group Approach*, Factorial Press, Moscow (2007).

[10] L.A. Beklaryan, Quasitravelling waves, *Sbornik: Mathematics*, **201**, No 12 (2010), 1731-1775.

[11] L.A. Beklaryan, N.K. Khachatryan, On one class of dynamic transportation models, *Comput. Math. Math. Phys.*, **53**, No 10 (2013), 1466-1482.

[12] L.A. Beklaryan, Quasi-travelling waves as natural extension of class of traveling waves, *Tambov University Reports. Series Natural and Technical Sciences*, **19**, No 2 (2014), 331-340.

[13] L.A. Beklaryan, A.L. Beklaryan, A.Yu. Gornov, Solutions of traveling wave type for Korteweg-de Vries-type system with polynomial potential, In: *Optimization and Applications 9th International Conference, OPTIMA 2018*, Springer, Switzerland (2019), 291-305.

[14] L.A. Beklaryan, A.L. Beklaryan, Existence of bounded soliton solutions in the problem of longitudinal vibrations of an infinite elastic rod in a field with a strongly nonlinear potential, *Comput. Math. Math. Phys.*, **61**, No 12 (2021), 1980-1994.

[15] L.A. Beklaryan, A.L. Beklaryan, Existence of bounded soliton solutions in the problem of longitudinal oscillations of an elastic infinite rod in a field with a nonlinear potential of general form, *Comput. Math. Math. Phys.*, **62**, No 6 (2022), 904-919.

[16] L.E. El'sgol'ts, S.B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York (1973).

[17] Ya.I. Frenkel, T.A. Contorova, On the theory of plastic deformation and twinning, *Journal of Experimental and Theoretical Physics*, **8**, No 1 (1938), 89-95.

[18] A.Yu. Gornov, T.S. Zarodnyuk, T.I. Madzhara, A.V. Daneeva, I.A. Veyalko, A collection of test multiextremal optimal control problems, In: *Optimization, Simulation, and Control*, Springer, New York (2013), 257-274.

[19] J.K. Hale, *Theory of Functional Differential Equations*, Springer, New York (1977).

[20] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York (1993).

[21] J.P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, *SIAM Journal on Applied Mathematics*, **47**, No 3 (1987), 556-572.

[22] J.W. Cahn, J. Mallet-Paret, E.S. van Vleck, Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice, *SIAM Journal on Applied Mathematics*, **59**, No 2 (1998), 455-493.

[23] J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, *Journal of Dynamics and Differential Equations*, **11**, No 1 (1999), 49-127.

[24] J. Mallet-Paret, The Fredholm alternative for functional-differentional equations mixed type, *Journal of Dynamics and Differential Equations*, **11**, No 1 (1999), 1-47.

[25] T. Miwa, M. Jimbo, E. Date, *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*, Cambridge University Press, UK (2000).

[26] L.D. Pustyl'nikov, Infinite-dimensional non-linear ordinary differential equations and the KAM theory, *Russian Mathematical Surveys*, **52**, No 3 (1997), 551-604.

[27] M. Toda, *Theory of Nonlinear Lattices*, Springer, Berlin, Heidelberg (1989).

[28] B. Zinner, Existence of traveling wavefront solutions for the discrete Nagumo equation, *Journal of Differential Equations*, **96**, No 1 (1992), 1-27.