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# AN INVESTIGATION OF NUMERICAL INTEGRATION USING CHEBYSHEV WAVELETS OF THE SECOND KIND

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**Abstract:** In this research paper, we propose a numerical technique based on Chebyshev wavelets of the second kind for the investigation of numerical integrations. For this purpose, collocation points and the basis functions of Chebyshev wavelets of the second kind have been utilized. The results of numerical experiments are presented, and are compared with exact solutions to confirm the good accuracy of the proposed scheme.

**AMS Subject Classification:** 65N99

**Key Words:** Chebyshev wavelets of the second kind; collocation points; function approximation; numerical integration

## 1. Introduction

Numerical analysis is a branch of mathematics, which is used to find the approximate solutions of various problems arising in real life problems. Numerical integration plays a significant role in numerical analysis. For numerical integration, several techniques have been developed, like Trapezoidal rule, Simpson's rule, Weddle rule, etc. Nowadays wavelets are powerful mathematical tools for solving differential and integral equations numerically. Numerous wavelets families have been established such as Haar wavelets, Hermite wavelets, Chebyshev wavelets, Legendre wavelets, etc. Wavelets based techniques are simple as it

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converts the difficult problems into a system of algebraic equations, which are solved with the aid of any classical techniques like matrix method, Crammer rule, Gauss-Jordan method, Gauss-Seidel method, Gauss-elimination method. From the literature, we have concluded that wavelets based techniques are accurate and efficient. Efficient algorithms based on Haar wavelets have been utilized for solving differential and integral equations in [1, 2, 3, 4, 5, 6, 7]. In [8], an accurate algorithm has been developed using Hermite wavelets to find the numerical solution of fractional Jaulent-Miodek equation associated with energy dependent Schrodinger potential. An efficient technique based on Hermite wavelets has been utilized for solving two- dimensional hyperbolic telegraph equation in [9]. An operational matrix of integration based on Hermite wavelets has been established for solving nonlinear singular initial value problems in [10]. In [11], Chebyshev polynomials have been used to find the solution of two dimensional linear and nonlinear integral equations of the second kind. In [12], Chebyshev wavelets of the second kind has been utilized to find the numerical solutions of fractional nonlinear Fredholm integro-differential equations. In [13], for the solutions of second-order differential equations with singular and Bratu type equations, an efficient technique based on second kind Chebyshev wavelets has been established. Chebyshev wavelets based operational matrix of integration has been developed for solving differential equations in [14]. Fractional order differential equations have been solved with the aid of Chebyshev wavelets of the second kind in [15]. In [16], Chebyshev wavelets of the second kind has been used to find the solutions of convection diffusion equation. Lane-Emden type differential equations have been solved with the help of second kind Chebyshev wavelets in [17]. Chebyshev wavelets based technique has been established for solving partial differential equations with telegraph type boundary conditions in [18].

This research paper is organized as follows: In Section 2, introduction of Chebyshev wavelets of the second kind is presented. Proposed numerical scheme based on Chebyshev wavelets of the second kind for the evaluation of integration has been presented in Section 3. In Section 4, some numerical experiments have been performed to illustrate the accuracy of the proposed numerical scheme. Conclusion is presented in Section 5.

## 2. Chebyshev wavelets of the second kind

In the last few decades, wavelets based numerical techniques have been used extensively for the solution of various problems of science, engineering and technology. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{1/2}\psi(\frac{t-b}{a}), \quad a, b \in R, \quad a \neq 0. \tag{1}$$

The second kind wavelets $\psi_(n, m) = \psi(k, n, m, t)$ have four arguments; $k$ is any positive integer, $n = 1, 2, 3, 4, ..., 2^{(k-1)}$; $m$ is the degree of second kind Chebyshev polynomials and $t$ is normalized time. It is defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2}\tilde{U_m}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where

$$\tilde{U_m}(t) = \sqrt{\frac{2}{\pi}}U_m(t). \tag{3}$$

Here, $m = 0, 1, 2, 3, ..., M-1$ and $M$ is a fixed integer. In relation given by (1) is for orthonormality. Here $U_m(t)$ are the second kind Chebyshev polynomials of degree $m$ which are orthogonal with respect to the weight function $\omega(t) = \sqrt{(1-t^2)}$ on the interval $[-1, 1]$, and satisfy the satisfy the following recursive formula: $U_0(t) = 1, U_1(t) = 2t, U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \ \ m = 1, 2, 3, 4, ....$ Note that in case of second kind Chebyshev wavelets, the weight function has to be dilated and translated as $\omega_n(t) = \omega(2^k t - 2n + 1)$.

## 3. Function approximation

A function $f(x) \in L^2(R)$ defined on the interval $[0, 1)$ may be expanded by second kind Chebyshev wavelets as:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{4}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x)\rangle_{L^2{}_\omega[0,1)} = \int_0^1 f(x)\psi_{n,m}(x)\omega_n(x)dx, \tag{5}$$

in which the symbol $\langle ... \rangle$ denotes the inner product in $L^2{}_{\omega}[0,1)$. If the infinite series is truncated, then it can be written as:

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \cong C^T \psi(x), \tag{6}$$

where $C$ and $\psi$ are matrices of order $2^{k-1}M \times 1$ and are written as:

$$C^T = [c_{1,0}, ..., c_{1,M-1}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}], \tag{7}$$

and

$$\psi(x) = [\psi_{1,0}(x), ..., \psi_{1,M-1}(x), ..., \psi_{2^{k-1},0}(x), ..., \psi_{2^{k-1},M-1}(x)]^T. \tag{8}$$

For $k = 1$ and $M = 6$, the first six basis functions are given as:

$$\begin{cases} \psi_{1,0}(x) = \frac{2}{\sqrt{\pi}}, \\ \psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x - 2), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 3), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 40x - 4), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 336x^2 - 80x + 5), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(1024x^5 - 2560x^4 + 2304x^3 - 896x^2 + 140x - 6). \end{cases} \tag{9}$$

## 4. Proposed technique for numerical integration

Split the interval of integration into $n$ equal parts, each of length $\frac{\beta-\alpha}{n}$ such that $[\alpha = x_0, x_1, x_2, ..., x_n = \beta]$. Approximate the unknown function $f(x)$ into a series of basis functions of Chebyshev wavelets of the second kind as:

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x). \tag{10}$$

Putting the different values of $x$, we obtain

$$f(x_0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x_0), \tag{11}$$

$$f(x_1) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x_1), \tag{12}$$

$$f(x_2) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x_2), \tag{13}$$

$$\vdots$$

$$f(x_n) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x_n). \tag{14}$$

After solving the above system of algebraic equations, we obtain wavelets coefficients. Integrating (10) both sides with respect to $x$, from 0 to 1, we obtain

$$\int_0^1 f(x)dx = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^1 \psi_{n,m}(x)dx. \tag{15}$$

After substituting the values of wavelets coefficients, we obtain the required value of integration.

## 5. Numerical tests

In this section, some numerical examples have been performed to illustrate the accuracy of the proposed method. The obtained numerical results are compared with exact solutions. If the limit of integration is $[c, d]$, then change the limits of integration from $[c, d]$ to $[0, 1]$ as follows.

Consider the integration of the form:

$$I = \int_c^d J(x)dx, \tag{16}$$

where $c$ and $d$ are constants. Applying the transformation

$$x = CX + D, \tag{17}$$

where $C$ and $D$ are unknowns constants that satisfy the conditions $x = c, X = 0$ and $x = d, X = 1$. Therefore from (17), we obtain

$$\begin{cases} c = C(0) + D, \\ d = C(1) + D. \end{cases} \tag{18}$$

Solving these equations, we obtain $C = d - c$ and $D = c$. From (17), we obtain

$$\begin{cases} x = (d-c)X + c, \\ dx = (d-c)dX. \end{cases} \tag{19}$$

Substituting these values in (16), we obtain

$$I = (d-c)\int_0^1 J((d-c)X + c)dX, \tag{20}$$

This implies

$$I = (d-c)\int_0^1 R(X)dX, \tag{21}$$

where $R(X) = J((d-c)X + c)$.

**Example 1:** Consider the integration

$$\int_0^1 J(x)dx, \quad J(x) = \sqrt{x^2+1}. \tag{22}$$

The exact solution is $\frac{1}{\sqrt{2}} + \frac{1}{2}\log|1+\sqrt{2}|$. Divide the interval $[0,1]$ into 6 equal sub-intervals, each of length $h = \frac{1-0}{6}$. The corresponding subintervals are $[0,1/6],[1/6,2/6],...,[5/6,1]$. Expand the given function $J(x) = \sqrt{x^2+1}$ into a series of Hermite wavelets basis functions by taking $k = 1, M = 7$ as follows:

$$\sqrt{x^2+1} = \sum_{m=0}^{6} C_{1,m}\psi_{1,m}(x). \tag{23}$$

Substituting the values of nodes $x = 0, 1/6, 2/6, 3/6, 4/6, 5/6, 1$ in (23), we obtain

$$1 = C_{1,0}\psi_{1,0}(0) + C_{1,1}\psi_{1,1}(0) + C_{1,2}\psi_{1,2}(0) + C_{1,3}\psi_{1,3}(0) + C_{1,4}\psi_{1,4}(0) + C_{1,5}\psi_{1,5}(0) + C_{1,6}\psi_{1,6}(0), \tag{24}$$

$$\sqrt{\frac{37}{36}} = C_{1,0}\psi_{1,0}(1/6) + C_{1,1}\psi_{1,1}(1/6) + C_{1,2}\psi_{1,2}(1/6) + C_{1,3}\psi_{1,3}(1/6) + C_{1,4}\psi_{1,4}(1/6) + C_{1,5}\psi_{1,5}(1/6) + C_{1,6}\psi_{1,6}(1/6), \quad (25)$$

$$\sqrt{\frac{40}{36}} = C_{1,0}\psi_{1,0}(2/6) + C_{1,1}\psi_{1,1}(2/6) + C_{1,2}\psi_{1,2}(2/6) + C_{1,3}\psi_{1,3}(2/6) + C_{1,4}\psi_{1,4}(2/6) + C_{1,5}\psi_{1,5}(2/6) + C_{1,6}\psi_{1,6}(2/6), \quad (26)$$

$$\sqrt{\frac{45}{36}} = C_{1,0}\psi_{1,0}(3/6) + C_{1,1}\psi_{1,1}(3/6) + C_{1,2}\psi_{1,2}(3/6) + C_{1,3}\psi_{1,3}(3/6) + C_{1,4}\psi_{1,4}(3/6) + C_{1,5}\psi_{1,5}(3/6) + C_{1,6}\psi_{1,6}(3/6), \quad (27)$$

$$\sqrt{\frac{52}{36}} = C_{1,0}\psi_{1,0}(4/6) + C_{1,1}\psi_{1,1}(4/6) + C_{1,2}\psi_{1,2}(4/6) + C_{1,3}\psi_{1,3}(4/6) + C_{1,4}\psi_{1,4}(4/6) + C_{1,5}\psi_{1,5}(4/6) + C_{1,6}\psi_{1,6}(4/6), \quad (28)$$

$$\sqrt{\frac{61}{36}} = C_{1,0}\psi_{1,0}(5/6) + C_{1,1}\psi_{1,1}(5/6) + C_{1,2}\psi_{1,2}(5/6) + C_{1,3}\psi_{1,3}(5/6) + C_{1,4}\psi_{1,4}(5/6) + C_{1,5}\psi_{1,5}(5/6) + C_{1,6}\psi_{1,6}(5/6), \quad (29)$$

and

$$\sqrt{2} = C_{1,0}\psi_{1,0}(1) + C_{1,1}\psi_{1,1}(1) + C_{1,2}\psi_{1,2}(1) + C_{1,3}\psi_{1,3}(1) + C_{1,4}\psi_{1,4}(1) + C_{1,5}\psi_{1,5}(1) + C_{1,6}\psi_{1,6}(1). \quad (30)$$

Solving the above system of equations, we obtain the wavelets coefficients. The wavelet coefficients are $1.0304e+000$, $9.3524e-002$, $1.9739e-002$, $-1.7899e-003$, $-3.0005e-005$, $3.7710e-005$, $-4.6792e-006$.

Integrating (23) one time w.r.t. $x$, from 0 to 1, we obtain

$$\int_0^1 \sqrt{x^2+1}dx = \sum_{m=0}^{6} C_{1,m}\int_0^1 \psi_{1,m}(x)dx. \quad (31)$$

Substituting the values of wavelet coefficients into (31), we have

$$\int_0^1 \sqrt{x^2+1}dx = 1.147793929573018, \quad (32)$$

which is nearly same as the exact solution.

**Example 2:** Consider the integration

$$\int_0^1 J(x)dx, \quad J(x) = \frac{1}{2x+1}. \tag{33}$$

Divide the interval $[0,1]$ into 7 equal sub-intervals, each of length $h = \frac{1-0}{7}$. Let $[0,1]$ divided into $[0,1/7],[1/7,2/7],...,[6/7,1]$. Expand the given function $J(x) = \frac{1}{2x+1}$ into a series of Hermite wavelets basis functions by taking $k = 1, M = 8$ as follows:

$$\frac{1}{2x+1} = \sum_{m=0}^{7} C_{1,m}\psi_{1,m}(x). \tag{34}$$

Substituting the values of nodes $x = 0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1$ in (34), we obtain

$$1 = C_{1,0}\psi_{1,0}(0) + C_{1,1}\psi_{1,1}(0) + C_{1,2}\psi_{1,2}(0) + C_{1,3}\psi_{1,3}(0) \\ + C_{1,4}\psi_{1,4}(0) + C_{1,5}\psi_{1,5}(0) + C_{1,6}\psi_{1,6}(0) + C_{1,7}\psi_{1,7}(0), \tag{35}$$

$$\frac{7}{9} = C_{1,0}\psi_{1,0}(1/7) + C_{1,1}\psi_{1,1}(1/7) + C_{1,2}\psi_{1,2}(1/7) + C_{1,3}\psi_{1,3}(1/7) \\ + C_{1,4}\psi_{1,4}(1/7) + C_{1,5}\psi_{1,5}(1/7) + C_{1,6}\psi_{1,6}(1/7) + C_{1,7}\psi_{1,7}(1/7), \tag{36}$$

$$\frac{7}{11} = C_{1,0}\psi_{1,0}(2/7) + C_{1,1}\psi_{1,1}(2/7) + C_{1,2}\psi_{1,2}(2/7) + C_{1,3}\psi_{1,3}(2/7) \\ + C_{1,4}\psi_{1,4}(2/7) + C_{1,5}\psi_{1,5}(2/7) + C_{1,6}\psi_{1,6}(2/7) + C_{1,7}\psi_{1,7}(2/7), \tag{37}$$

$$\frac{7}{13} = C_{1,0}\psi_{1,0}(3/7) + C_{1,1}\psi_{1,1}(3/7) + C_{1,2}\psi_{1,2}(3/7) + C_{1,3}\psi_{1,3}(3/7) \\ + C_{1,4}\psi_{1,4}(3/7) + C_{1,5}\psi_{1,5}(3/7) + C_{1,6}\psi_{1,6}(3/7) + C_{1,7}\psi_{1,7}(3/7), \tag{38}$$

$$\frac{7}{15} = C_{1,0}\psi_{1,0}(4/7) + C_{1,1}\psi_{1,1}(4/7) + C_{1,2}\psi_{1,2}(4/7) + C_{1,3}\psi_{1,3}(4/7) \\ + C_{1,4}\psi_{1,4}(4/7) + C_{1,5}\psi_{1,5}(4/7) + C_{1,6}\psi_{1,6}(4/7) + C_{1,7}\psi_{1,7}(4/7), \tag{39}$$

$$\frac{7}{17} = C_{1,0}\psi_{1,0}(5/7) + C_{1,1}\psi_{1,1}(5/7) + C_{1,2}\psi_{1,2}(5/7) + C_{1,3}\psi_{1,3}(5/7) + C_{1,4}\psi_{1,4}(5/7) + C_{1,5}\psi_{1,5}(5/7) + C_{1,6}\psi_{1,6}(5/7) + C_{1,7}\psi_{1,7}(5/7), \quad (40)$$

$$\frac{7}{19} = C_{1,0}\psi_{1,0}(6/7) + C_{1,1}\psi_{1,1}(6/7) + C_{1,2}\psi_{1,2}(6/7) + C_{1,3}\psi_{1,3}(6/7) + C_{1,4}\psi_{1,4}(6/7) + C_{1,5}\psi_{1,5}(6/7) + C_{1,6}\psi_{1,6}(6/7) + C_{1,7}\psi_{1,7}(6/7), \quad (41)$$

and

$$\frac{1}{3} = C_{1,0}\psi_{1,0}(1) + C_{1,1}\psi_{1,1}(1) + C_{1,2}\psi_{1,2}(1) + C_{1,3}\psi_{1,3}(1) + C_{1,4}\psi_{1,4}(1) + C_{1,5}\psi_{1,5}(1) + C_{1,6}\psi_{1,6}(1) + C_{1,7}\psi_{1,7}(1). \quad (42)$$

Solving the above system of equations, we obtain the wavelets coefficients. The wavelet coefficients are $4.6048e-001$, $-1.7125e-001$, $2.8219e-002$, $-9.8477e-003$, $2.6375e-003$, $-7.0293e-004$, $1.7418e-004$ and $-4.3544e-005$.

Integrating (34) one time w.r.t. $x$, from 0 to 1, we obtain

$$\int_0^1 \frac{1}{2x+1}dx = \sum_{m=0}^{7} C_{1,m} \int_0^1 \psi_{1,m}(x)dx. \quad (43)$$

Substituting the values of wavelet coefficients into (43), we obtain

$$\int_0^1 \frac{1}{2x+1}dx = 0.549322254145040, \quad (44)$$

which is nearly same as the exact solution.

## 6. Conclusion

It is concluded that the Chebyshev wavelets of the second kind are powerful numerical tools for the evaluation of integration. To get the necessary accuracy, the number of collocation points may be raised. The concept of the present method can be extended for the evaluation of two- and three- dimensional integrations arising in various applications of sciences and engineering.

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## References

[1] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems, *IEEE Proceedings: Part D*, **144**, No 1 (1997), 87–94.

[2] U. Lepik, Numerical solution of differential equations using Haar wavelets, *Mathematics and Computers in Simulation*, **68** (2005), 127–143.

[3] U. Lepik, Application of Haar wavelet transform to solving integral and differential equations, *Proceedings of Estonian Academy of Science, Physics, Mathematics*, **56**, No 1 (2007), 28–46.

[4] I. Singh, S. Arora, S. Kumar, Numerical solution of wave equation using Haar wavelet, *International Journal of Pure and Applied Mathematics*, **98**, No 4 (2015), 457–469.

[5] I. Singh, S. Kumar, Haar wavelet collocation method for solving nonlinear Kuramoto–Sivashinsky equation, *Italian Journal of Pure and Applied Mathematics*, **39** (2018), 373–384.

[6] I. Singh, Wavelet based method for solving generalized Burgers type equations, *International Journal of Computational Materials Science and Engineering*, **8**, No 4 (2019), 1–24.

[7] I. Singh, S. Kumar, Modified wavelet method for solving two-dimensional coupled system of evolution equations, *Iranian Journal of Mathematical Sciences and Informatics*, **17**, No 1 (2022), 239–259.

[8] A.K. Gupta, S.S. Ray, An investigation with Hermite wavelets for accurate solution of fractional Jaulent-Miodek equation associated with energy dependent Schrodinger potential, *Applied Mathematics and Computation*, **270** (2015), 458–471.

[9] O. Oruc, A numerical procedure based on Hermite wavelets for two dimensional hyperbolic telegraph equation, *Engineering with Computers*, **34**, No 4 (2018), 741–755.

[10] S.C. Shiralashetti, K. Srinivasa, Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems, *Alexandria Engineering Journal*, **57** (2018), 2591–2600.

[11] Z. Avazzadeh, M. Heydari, Chebyshev polynomials for solving two dimensional linear and nonlinear integral equations of the second kind, *Computational and Applied Mathematics*, **31**, No 1 (2012), 127–142.

[12] L. Zhu, Q. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet, *Communications in Nonlinear Science and Numerical Simulation*, **17**, No 6 (2012), 2333–2341.

[13] W. M. Abd-Elhameed, E.H. Doha, Y.H. Youssri, New spectral second kind Chebyshev wavelets algorithm for solving linear and nonlinear second-order differential equations involving singular and Bratu type equations, *Abstract and Applied Analysis*, **2013** (2013), 1–9.

[14] E. Babolian, F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, *Applied Mathematics and Computation*, **188**, No 1 (2007), 417–426.

[15] Y. Wang, Q. Fan, The second kind Chebyshev wavelet method for solving fractional differential equations, *Applied Mathematics and Computation*, **218**, No 17 (2012), 8592–8601.

[16] F. Zhou, X. Xu, Numerical solution of the convection diffusion equations by the second kind Chebyshev wavelets, *Applied Mathematics and Computation*, **247** (2014), 353–367.

[17] E. H. Doha, W.M. Abd-Elhameed, Y. H. Youssri, Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane-Emden type, *New Astronomy*, **23** (2013), 113–117.

[18] M.H. Heydari, M.R. Hooshmandasl, F. M. Ghaini, A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type, *Applied Mathematical Modelling*, **38**, No 5–6 (2014), 1597–1606.