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LINEAR INDEPENDENT SOLUTIONS AND RECURSION FORMULAS FOR THE QUADRUPLE HYPERGEOMETRIC FUNCTIONS $X_{31}^{(4)}$

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Abstract: Very recently, Younis and Bin-Saad [13] introduced twenty new complete quadruple hypergeometric functions, namely $X_{31}^{(4)}, X_{32}^{(4)}, ..., X_{50}^{(4)}$. In this article, we present the solutions of systems of partial differential equations for hypergeometric function in four variables $X_{31}^{(4)}$. We also obtain some recursion formulas for the function $X_{31}^{(4)}$.

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1. Introduction

Hypergeometric functions have attracted the attention of many researchers due to their importance and applications in diverse areas of mathematical, physical, engineering and statistical sciences, [4, 8, 12]. Multiple hypergeometric functions occur in various fields of pure and applied mathematics such as approximation theory, partition theory, representation theory, group theory, mirror symmetry, difference equations and mathematical physics etc. They possess

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important properties such as recurrence and explicit relations, summation formulae, symmetric and convolution identities, algebraic properties etc. In recent years, several multivariable hypergeometric functions have been considered by many authors (see [1, 3, 5, 7, 9, 10]).

In [13], Younis and Bin-Saad defined twenty complete quadruple hypergeometric functions denote by $X_i^{(4)}(i = 31, 32, ..., 50)$, that have not been included in Exton [6] and Sharma and Parihar [11] set, and then gave their integral representations. Each quadruple hypergeometric function in [6, 11, 13] is of the form

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Theta(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$

where $\Theta(m,n,p,q)$ is a certain sequence of complex parameters and there are twelve parameters in each series $X^{(4)}(.)$ (eight $a's$ and four $c's$). The 1st, 2nd, 3rd and 4th parameters in $X^{(4)}(.)$ are connected with the integers m, n, p and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters in $\Theta(m,n,p,q)$. For example, $X^{(4)}(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_5)$ mean that$(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q$ includes the term. Similarly, $X^{(4)}(a_1, a_1, a_1, a_2, a_1, a_1, a_2, a_3)$ points out the term

$$(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q \text{ and } X^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3)$$

shows the existence of the term $(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}$. Thus, it is possible to form various combinations of indices. Here, $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1) \ (n \in N := \{1, 2, ...\})$$
$$\text{and } (a)_0 = 1.$$

For our purpose, we need to introduce the quadruple hypergeometric function $X_{31}^{(4)}$ defined by

$$\begin{aligned} &X_{31}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, t) \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!} \\ &\left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |u| < 1\right). \end{aligned} \tag{1.1}$$

The integral representation of Laplace-type for the function $X_{31}^{(4)}$ is given by (see [13])

$$\begin{aligned}
&X_{31}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, t\right)\\
&= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+v)} s^{a_1-1}\, v^{a_2-1} {}_0F_1\left(-; c_1; s^2 x\right)\\
&\times {}_0F_1\left(-; c_3; v^2 z\right) \Psi_2\left(a_3; c_2, c_4; sy, vt\right) ds dv \ \ (Re(a_1)>0, Re(a_2)>0)\,, \quad (1.2)
\end{aligned}$$

where ${}_0F_1$ and Ψ_2 are the confluent hypergeometric functions defined, respectively, by (see [12])

$$ {}_0F_1\left(-; c; x\right) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \frac{x^m}{m!}, \quad |x| < \infty$$

and

$$\Psi_2\left(a; b, c; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (c)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < \infty, \quad |y| < \infty.$$

The structure of this paper is as follows. In Section 2, we find the linearly independent solutions of partial differential equations satisfied by the function $X_{31}^{(4)}$. Section 3 provides some recursion formulas for $X_{31}^{(4)}$.

2. Solving the systems of partial differential equations

According to the theory of multiple hypergeometric functions [2], the system of partial differential equations for a quadruple hypergeometric function $X_{31}^{(4)}$ is defined as follows:

$$\begin{cases}
\left(c_1 + x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + 1\right) x^{-1} u \\
\quad - \left(a_1 + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1\right)\left(a_1 + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) u = 0, \\
\left(c_2 + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial y} + 1\right) y^{-1} u \\
\quad - \left(a_1 + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(a_3 + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right) u = 0, \\
\left(c_3 + z\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial z} + 1\right) z^{-1} u \\
\quad - \left(a_2 + 2z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} + 1\right)\left(a_2 + 2z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right) u = 0, \\
\left(c_4 + t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} + 1\right) t^{-1} u \\
\quad - \left(a_2 + 2z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)\left(a_3 + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right) u = 0,
\end{cases} \quad (2.1)$$

where $u = X_{31}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, t)$.

Starting from (2.1) and by making use of some elementary calculations, we define the system of second order partial differential equations

$$\begin{cases} x\left(1-4x\right)u_{xx}-4xyu_{xy}-y^{2}u_{yy}+\left[c_{1}-2\left(2a_{1}+3\right)x\right]u_{x} \\ \quad -2\left(a_{1}+1\right)yu_{y}-a_{1}\left(a_{1}+1\right)u=0, \\ \\ y\left(1-y\right)u_{yy}-2xyu_{xy}-2xtu_{xt}-ytu_{yt}-2a_{3}xu_{x} \\ \quad +\left[c_{2}-\left(a_{1}+a_{3}+1\right)y\right]u_{y}-a_{1}tu_{t}-a_{1}a_{3}u=0, \\ \\ z\left(1-4z\right)u_{zz}-4ztu_{zt}-t^{2}u_{tt}+\left[c_{3}-2\left(2a_{2}+3\right)z\right)]u_{z} \\ \quad -2\left(a_{2}+1\right)tu_{t}-a_{2}\left(a_{2}+1\right)u=0, \\ \\ t\left(1-t\right)u_{tt}-2yzu_{yz}-ytu_{yt}-2ztu_{zt}-a_{2}yu_{t}-2a_{3}zu_{z} \\ \quad +\left[c_{4}-\left(a_{2}+a_{3}+1\right)t\right]u_{t}-a_{2}a_{3}u=0. \end{cases} \tag{2.2}$$

It is noted that four equations of the system (2.2) are simultaneous, because the hypergeometric function $X_{31}^{(4)}$ satisfies the system. Now, in order to find the linearly independent solutions of system (2.2), we will search the solutions in the form

$$u=x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w, \tag{2.3}$$

where w is an unknown function, and α,β,γ and δ are constants, which are to be determined. So, substituting $u=x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w$ into the system (2.2), we get

$$\begin{cases} x\left(1-4x\right)w_{xx}-4xyw_{xy}-y^{2}w_{yy} \\ \quad +\left\{c_{1}+2\alpha-2\left[2\left(a_{1}+2\alpha+\beta\right)+3\right]x\right\}w_{x} \\ \quad -2\left[\left(a_{1}+2\alpha+\beta\right)+1\right]yw_{y}+\alpha\left(c_{1}+\alpha-1\right)x^{-1}w \\ \quad -\left(a_{1}+2\alpha+\beta\right)\left(a_{1}+2\alpha+\beta+1\right)w=0, \\ \\ y\left(1-y\right)w_{yy}-2xyw_{xy}-2xtw_{xt}-ytw_{yt} \\ \quad -2\left(a_{3}+\beta+\delta\right)xw_{x} \\ \quad +\left\{c_{2}+2\beta-\left[\left(a_{1}+2\alpha+\beta\right)+\left(a_{3}+\beta+\delta\right)+1\right]y\right\}w_{y} \\ \quad -\left(a_{1}+2\alpha+\beta\right)tw_{t} \\ \quad +\beta\left(c_{2}+\beta-1\right)y^{-1}w-\left(a_{1}+2\alpha+\beta\right)\left(a_{3}+\beta+\delta\right)]w=0, \\ \\ z\left(1-4z\right)w_{zz}-4ztw_{zt}-t^{2}w_{tt}+\left\{c_{3}+2\gamma\right. \\ \quad \left.-2\left[2\left(a_{2}+2\gamma+\delta\right)+3\right]z\right\}w_{z} \\ \quad -2\left[\left(a_{2}+2\gamma+\delta\right)+1\right]tw_{t}+\gamma\left(c_{3}+\gamma-1\right)z^{-1}w \\ \quad -\left(a_{2}+2\gamma+\delta\right)\left(a_{2}+2\gamma+\delta+1\right)w=0, \\ \\ t\left(1-t\right)w_{tt}-2yzw_{yz}-ytw_{yt}-2ztw_{zt} \\ \quad -\left(a_{2}+2\gamma+\delta\right)yw_{y}-2\left(a_{3}+\beta+\delta\right)zw_{z} \\ \quad +\left\{c_{4}+2\delta-\left[\left(a_{2}+2\gamma+\delta\right)+\left(a_{3}+\beta+\delta\right)+1\right]t\right\}w_{t} \\ \quad +\delta\left(c_{4}+\delta-1\right)t^{-1}w-\left(a_{2}+2\gamma+\delta\right)\left(a_{3}+\beta+\delta\right)w=0. \end{cases} \tag{2.4}$$

We note that system in (2.4) is analogical to system (2.2), therefore we require that the conditions

$$\begin{cases} \alpha\left(c_1+\alpha-1\right)=0 \\ \beta\left(c_2+\beta-1\right)=0 \\ \gamma\left(c_3+\gamma-1\right)=0 \\ \delta\left(c_4+\delta-1\right)=0 \end{cases} \tag{2.5}$$

should be satisfied. It is evident that system (2.5) has the following solutions:

$$\begin{array}{rcccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \alpha: & 0 & 1-c_1 & 0 & 0 & 0 & 1-c_1 & 1-c_1 & 1-c_1 \\ \beta: & 0 & 0 & 1-c_2 & 0 & 0 & 1-c_2 & 0 & 0 \\ \gamma: & 0 & 0 & 0 & 1-c_3 & 0 & 0 & 1-c_3 & 0 \\ \delta: & 0 & 0 & 0 & 0 & 1-c_4 & 0 & 0 & 1-c_4 \end{array}.$$

$$\begin{array}{rcccccccc} & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \alpha: & 0 & 0 & 0 & 1-c_1 & 1-c_1 & 1-c_1 & 0 & 1-c_1 \\ \beta: & 1-c_2 & 1-c_2 & 0 & 1-c_2 & 1-c_2 & 0 & 1-c_2 & 1-c_2 \\ \gamma: & 1-c_3 & 0 & 1-c_3 & 1-c_3 & 0 & 1-c_3 & 1-c_3 & 1-c_3 \\ \delta: & 0 & 1-c_4 & 1-c_4 & 0 & 1-c_4 & 1-c_4 & 1-c_4 & 1-c_4 \end{array}. \tag{2.6}$$

Finally, substituting two solutions of the system (2.6) into (2.4), we find the following linearly independent solutions of the system (2.2) at the origin:

$$u_1(x,y,z,t)=X_{31}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,t), \tag{2.7}$$

$$u_2(x,y,z,t)=x^{1-c_1}X_{31}^{(4)}\left(a_1+2-2c_1,a_1+2-2c_1,a_2,a_2,a_1+2-2c_1,a_3,a_2,\right.$$
$$\left.a_3;2-c_1,c_2,c_3,c_4;x,y,z,t\right), \tag{2.8}$$

$$u_3(x,y,z,t)=y^{1-c_2}X_{31}^{(4)}\left(a_1+1-c_2,a_1+1-c_2,a_2,a_2,a_1+1-c_2,\right.$$
$$\left.a_3+1-c_2,a_2,a_3+1-c_2;c_1,2-c_2,c_3,c_4;x,y,z,t\right), \tag{2.9}$$

$$u_4(x,y,z,t)=z^{1-c_3}X_{31}^{(4)}\left(a_1,a_1,a_2+2-2c_3,a_2+2-2c_3,a_1,a_3,a_2+2-2c_3,\right.$$
$$\left.a_3;c_1,c_2,2-c_3,c_4;x,y,z,t\right), \tag{2.10}$$

$$u_5(x,y,z,t)=t^{1-c_4}X_{31}^{(4)}\left(a_1,a_1,a_2+1-c_4,a_2+1-c_4,a_1,a_3+1-c_4,\right.$$
$$\left.a_2+1-c_4,a_3+1-c_4;c_1,c_2,c_3,2-c_4;x,y,z,t\right), \tag{2.11}$$

$$u_6(x,y,z,t)=x^{1-c_1}y^{1-c_2}X_{31}^{(4)}\left(a_1+3-2c_1-c_2,a_1+3-2c_1-c_2,a_2,a_2,\right.$$
$$\left.a_1+3-2c_1-c_2,a_3+1-c_2,a_2,a_3+1-c_2;2-c_1,2-c_2,c_3,c_4;x,y,z,t\right), \tag{2.12}$$

$$u_7(x,y,z,t)=x^{1-c_1}z^{1-c_3}X_{31}^{(4)}\left(a_1+2-2c_1,a_1+2-2c_1,a_2+2-2c_3,\right.$$
$$\left.a_2+2-2c_3,a_1+2-2c_1,a_3,a_2+2-2c_3,a_3;2-c_1,c_2,2-c_3,c_4;x,y,z,t\right), \tag{2.13}$$

$$u_8(x,y,z,t)=x^{1-c_1}t^{1-c_4}X_{31}^{(4)}\left(a_1+2-2c_1,a_1+2-2c_1,a_2+1-c_4,\right.$$
$$a_2+1-c_4,a_1+2-2c_1,a_3+1-c_4,a_2+1-c_4,a_3+1-c_4;2-c_1,c_2,c_3,$$
$$\left.2-c_4;x,y,z,t\right), \tag{2.14}$$

$$u_9(x,y,z,t) = y^{1-c_2}z^{1-c_3}X_{31}^{(4)}(a_1+1-c_2, a_1+1-c_2, a_2+2-2c_3,$$
$$a_2+2-2c_3, a_1+1-c_2, a_3+1-c_2, a_2+2-2c_3, a_3+1-c_2; c_1, 2-c_2, 2-c_3,$$
$$c_4; x,y,z,t), \tag{2.15}$$
$$u_{10}(x,y,z,t) = y^{1-c_2}t^{1-c_4}X_{31}^{(4)}(a_1+1-c_2, a_1+1-c_2, a_2+1-c_4, a_2+1-c_4,$$
$$a_1+1-c_2, a_3+2-c_2-c_4, a_2+1-c_4, a_3+2-c_2-c_4; c_1, 2-c_2, c_3, 2-c_4;$$
$$x,y,z,t), \tag{2.16}$$
$$u_{11}(x,y,z,t) = z^{1-c_3}t^{1-c_4}X_{31}^{(4)}(a_1, a_1, a_2+3-2c_3-c_4, a_2+3-2c_3-c_4, a_1$$
$$a_3+1-c_4, a_2+3-2c_3-c_4, a_3+1-c_4; c_1, c_2, 2-c_3, 2-c_4; x,y,z,t), \tag{2.17}$$
$$u_{12}(x,y,z,t) = x^{1-c_1}y^{1-c_2}z^{1-c_3}X_{31}^{(4)}(a_1+3-2c_1-c_2, a_1+3-2c_1-c_2,$$
$$a_2+2-2c_3, a_2+2-2c_3, a_1+3-2c_1-c_2, a_3+1-c_2, a_2+2-2c_3,$$
$$a_3+1-c_2; 2-c_1, 2-c_2, 2-c_3, c_4; x,y,z,t), \tag{2.18}$$
$$u_{13}(x,y,z,t) = x^{1-c_1}y^{1-c_2}t^{1-c_4}X_{31}^{(4)}(a_1+3-2c_1-c_2, a_1+3-2c_1-c_2,$$
$$a_2+1-c_4, a_2+1-c_4, a_1+3-2c_1-c_2, a_3+2-c_2-c_4, a_2+1-c_4,$$
$$a_3+2-c_2-c_4; a_2+1-c_4, 2-c_1, 2-c_2, c_3, 2-c_4; x,y,z,t), \tag{2.19}$$
$$u_{14}(x,y,z,t) = x^{1-c_1}z^{1-c_3}t^{1-c_4}X_{31}^{(4)}(a_1+2-2c_1, a_1+2-2c_1,$$
$$a_2+3-2c_3-c_4, a_2+3-2c_3-c_4, a_1+2-2c_1, a_3+1-c_4, a_2+3-2c_3-c_4,$$
$$a_3+1-c_4; 2-c_1, c_2, 2-c_3, 2-c_4; x,y,z,t), \tag{2.20}$$
$$u_{15}(x,y,z,t) = y^{1-c_2}z^{1-c_3}t^{1-c_4}X_{31}^{(4)}(a_1+1-c_2, a_1+1-c_2, a_2+3-2c_3-c_4,$$
$$a_2+3-2c_3-c_4, a_1+1-c_2, a_3+2-c_2-c_4, a_2+3-2c_3-c_4,$$
$$a_3+2-c_2-c_4; c_1, 2-c_2, 2-c_3, 2-c_4; x,y,z,t), \tag{2.21}$$
$$u_{16}(x,y,z,t) = x^{1-c_1}y^{1-c_2}z^{1-c_3}t^{1-c_4}X_{31}^{(4)}(a_1+3-2c_1-c_2, a_1+3-2c_1-c_2,$$
$$a_2+3-2c_3-c_4, a_2+3-2c_3-c_4, a_1+3-2c_1-c_2, a_3+2-c_2-c_4,$$
$$a_2+3-2c_3-c_4, a_3+2-c_2-c_4; 2-c_1, 2-c_2, 2-c_3, 2-c_4; x,y,z,t). \tag{2.22}$$

3. Recursion formulas

Some recursion formulas for the hypergeometric function in four variables $X_{31}^{(4)}$ are established in this section. In what follows, n denotes a non-negative integer in all formulas.

The following abbreviated notations are used in this section. We, for example, write $X_{31}^{(4)}$ for the series

$$X_{31}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, t)$$

and $X_{31}^{(4)}(a_2+n)$ for

$$X_{31}^{(4)}(a_1, a_1, a_2+n, a_2+n, a_1, a_3, a_2+n, a_3; c_1, c_2, c_3, c_4; x, y, z, t).$$

The notation $X_{31}^{(4)}(a_2+n, c_2+n_1)$ stands for

$$X_{31}^{(4)}\left(a_1, a_1, a_2+n, a_2+n, a_1, a_3, a_2+n, a_3; c_1, c_2+n_1, c_3, c_4; x, y, z, t\right),$$

etc.

Theorem 1. *The following recursion formulas hold true for the numerator parameter a_1, a_2, a_3 of the $X_{31}^{(4)}$:*

$$\begin{aligned} X_{31}^{(4)}(a_1+n) = X_{31}^{(4)} &+ \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)X_{31}^{(4)}(a_1+1+n_1, c_1+1) \\ &+ \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{31}^{(4)}(a_1+n_1, a_3+1, c_2+1), \end{aligned} \tag{3.1}$$

$$\begin{aligned} X_{31}^{(4)}(a_1-n) = X_{31}^{(4)} &- \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{31}^{(4)}(a_1+2-n_1, c_1+1) \\ &- \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{31}^{(4)}(a_1+1-n_1, a_3+1, c_2+1), \end{aligned} \tag{3.2}$$

$$\begin{aligned} X_{31}^{(4)}(a_2+n) = X_{31}^{(4)} &+ \frac{2z}{c_3}\sum_{n_1=1}^{n}(a_2+n_1)X_{31}^{(4)}(a_2+1+n_1, c_3+1) \\ &+ \frac{ta_3}{c_4}\sum_{n_1=1}^{n}X_{31}^{(4)}(a_2+n_1, a_3+1, c_4+1), \end{aligned} \tag{3.3}$$

$$\begin{aligned} X_{31}^{(4)}(a_2-n) = X_{31}^{(4)} &- \frac{2z}{c_3}\sum_{n_1=1}^{n}(a_2+1-n_1)X_{31}^{(4)}(a_2+2-n_1, c_3+1) \\ &- \frac{ta_3}{c_4}\sum_{n_1=1}^{n}X_{31}^{(4)}(a_2+1-n_1, a_3+1, c_4+1), \end{aligned} \tag{3.4}$$

$$X_{31}^{(4)}(a_3+n) = X_{31}^{(4)} + \frac{ya_1}{c_2}\sum_{n_1=1}^{n}X_{31}^{(4)}(a_1+1, a_3+n_1, c_2+1)$$

$$+\frac{ta_2}{c_4}\sum_{n_1=1}^{n} X_{31}^{(4)}(a_2+1, a_3+n_1, c_4+1), \tag{3.5}$$

$$X_{31}^{(4)}(a_3-n) = X_{31}^{(4)} - \frac{ya_1}{c_2}\sum_{n_1=1}^{n} X_{31}^{(4)}(a_1+1, a_3+1-n_1, c_2+1)$$
$$-\frac{ta_2}{c_4}\sum_{n_1=1}^{n} X_{31}^{(4)}(a_2+1, a_3+1-n_1, c_4+1). \tag{3.6}$$

Proof. From the definition of hypergeometric function $X_{31}^{(4)}$ and the relation

$$(a_1+1)_{2m+n} = (a_1)_{2m+n}\Big(1+\frac{2m}{a_1}+\frac{n}{a_1}\Big), \tag{3.7}$$

we obtain the following contiguous relation:

$$X_{31}^{(4)}(a_1+1) = X_{31}^{(4)} + \frac{2x}{c_1}(a_1+1)\, X_{31}^{(4)}(a_1+2,\, c_1+1)$$
$$+\frac{ya_3}{c_2} X_{31}^{(4)}(a_1+1, a_3+1, c_2+1). \tag{3.8}$$

To obtain contiguous relation for $X_{31}^{(4)}(a_1+2)$, we replace $a_1 \to a_1+1$ in (3.8) and simplify. This leads to

$$X_{31}^{(4)}(a_1+2) = X_{31}^{(4)} + \frac{2x}{c_1}\sum_{n_1=1}^{2}(a_1+n_1)\, X_{31}^{(4)}(a_1+n_1+1,\, c_1+1)$$
$$+\frac{ya_3}{c_2}\sum_{n_1=1}^{2} X_{31}^{(4)}(a_1+n_1, a_3+1, c_2+1). \tag{3.9}$$

Iterating this process n times, we obtain (3.1). For the proof of (3.2), replace the parameter $a_1 \to a_1-1$ in (3.8). This gives

$$X_{31}^{(4)}(a_1-1) = X_{31}^{(4)} - \frac{2x}{c_1}a_1\, X_{31}^{(4)}(a_1+1\,, c_1+1)$$
$$-\frac{ya_3}{c_2} X_{31}^{(4)}(a_2+1,\, c_2+1). \tag{3.10}$$

Iteratively, we get (3.2).

The recursion formulas from (3.3)-(3.6) can be proved in similar manner. □

Theorem 2. *The following recursion formulas hold true for the numerator parameter a_3 of the $X_{31}^{(4)}$:*

$$X_{31}^{(4)}(a_3+n)=\sum_{N_2\leq n}\binom{n}{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$X_{31}^{(4)}(a_1+n_1,a_2+n_2,a_3+N_2,c_2+n_1,c_4+n_2), \qquad (3.11)$$

$$X_{31}^{(4)}(a_3-n)=\sum_{N_2\leq n}\binom{n}{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}(-y)^{n_1}(-t)^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$X_{31}^{(4)}(a_1+n_1,a_2+n_1,c_2+n_1,c_4+n_2), \qquad (3.12)$$

where $\binom{n}{n_1,n_2}=\dfrac{n!}{n_1!n_2!(n-n_1-n_2)!}$ *and* $N_2=n_1+n_2$.

Proof. The proof of (3.11) is based upon the principle of mathematical induction on $n\in\mathbb{N}$. For $n=1$, the result (3.11) is true obviously by (3.5). Suppose (3.11) is true for $n=m$, that is,

$$X_{31}^{(4)}(a_3+n)=\sum_{N_2\leq m}\binom{n}{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times\ X_{31}^{(4)}(a_1+n_1,a_2+n_2,\ a_3+N_2,\ c_2+n_1,c_4+n_2). \qquad (3.13)$$

Replacing $a_3\mapsto a_3+1$ in (3.13) and using the contiguous relation (3.5) for $n=1$, we get

$$X_{31}^{(4)}(a_3+m+1)=\sum_{N_2\leq m}\binom{n}{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times\Big\{X_{31}^{(4)}(a_1+n_1,a_2+n_2,\ a_3+N_2,c_2+n_1,c_4+n_2)$$
$$+\frac{(a_1+n_1)y}{(c_2+n_1)}X_{31}^{(4)}\,(a_1+n_1+1,a_2+n_2,\ a_3+N_2+1,c_2+n_1+1,c_4+n_2)$$
$$+\frac{(a_2+n_2)t}{(c_4+n_2)}X_{31}^{(4)}(a_1+n_1,a_2+n_2+1,\ a_3+N_2+1,c_2+n_1,c_4+n_2+1)\Big\}. \qquad (3.14)$$

Simplifying, (3.14) takes the form

$$X_{31}^{(4)}\,(a_3+m+1)=\textstyle\sum_{N_2\leq m}\binom{n}{n_1,n_2,n_3}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times X_{31}^{(4)}(a_1+n_1,a_2+n_2,\ a_3+N_2,\ c_2+n_1,c_4+n_2)$$

$$+\sum_{N_2\le m+1}\binom{n}{n_1-1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times X_{31}^{(4)}(a_1+n_1,a_2+n_2,\,a_3+N_2,\,c_2+n_1,c_4+n_2)$$
$$+\sum_{N_2\le m+1}\binom{n}{n_1,n_2-1}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times X_{31}^{(4)}(a_1+n_1,a_2+n_2,\,a_3+N_2,\,c_2+n_1,c_4+n_2). \tag{3.15}$$

Using Pascal's identity in (3.15), we have

$$X_{31}^{(4)}(a_3+m+1)$$
$$=\sum_{N_2\le m+1}\binom{n}{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}y^{n_1}t^{n_2}}{(c_2)_{n_1}(c_4)_{n_2}}$$
$$\times X_{31}^{(4)}(a_1+n_1,a_2+n_2,\,a_3+N_2,\,c_2+n_1,c_4+n_2). \tag{3.16}$$

This establishes (3.11) for $n=m+1$. Hence by induction result given in (3.11) is true for all values of n. The recursion formula (3.12) can be proved in a similar manner. □

Theorem 3. *The following recursion formulas hold true for the denominator parameter c_1, c_2, c_3, c_4 of the $X_{31}^{(4)}$:*

$$X_{31}^{(4)}(c_1-n)=X_{31}^{(4)} \tag{3.17}$$
$$+(a_1)_2\,x\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}X_{31}^{(4)}(a_1+2,\,c_1+2-n_1),$$

$$X_{31}^{(4)}(c_2-n)=X_{31}^{(4)} \tag{3.18}$$
$$+a_1a_3\,y\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}X_{31}^{(4)}(a_1+1,a_3+1,c_2+2-n_1),$$

$$X_{31}^{(4)}(c_3-n)=X_{31}^{(4)} \tag{3.19}$$
$$+(a_2)_2\,z\sum_{n_1=1}^{n}\frac{1}{(c_3-n_1)(c_3+1-n_1)}X_{31}^{(4)}(a_2+2,\,c_3+2-n_1),$$

$$X_{31}^{(4)}(c_4-n)=X_{31}^{(4)} \tag{3.20}$$
$$+a_2a_3\,t\sum_{n_1=1}^{n}\frac{1}{(c_4-n_1)(c_4+1-n_1)}X_{31}^{(4)}(a_2+1,a_3+1,c_4+2-n_1).$$

Proof. Using the definition of hypergeometric function $X_{31}^{(4)}$ and the relation

$$\frac{1}{(c_1-1)_{m+p}} = \frac{1}{(c_1)_{m+p}}\left(1+\frac{m}{c_1-1}+\frac{p}{c_1-1}\right), \qquad (3.21)$$

we have

$$X_{31}^{(4)}(c_1-1) = X_{31}^{(4)} + \frac{(a_1)_2\,x}{c_1(c_1-1)} X_{31}^{(4)}(a_1+2,\, c_1+1) + \frac{(a_2)_2 z}{c_1(c_1-1)} X_{31}^{(4)}(a_2+2,\, c_1+1). \qquad (3.22)$$

Using this contiguous relation to the $X_{31}^{(4)}$ with the parameter c_1-n for n times, we get result (3.17). Recursion formulas (3.18)-(3.20) can be proved in a similar manner. □

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