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PROBLEM OF ANOMALOUS FILTRATION IN NONHOMOGENEOUS POROUS MEDIUM

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Abstract: In this work, the problem of anomalous filtration and solute transport in a two-zone medium with a stripe like source is posed and numerically solved. In one zone, anomalous convective diffusion transfer occurs, and in the other - only diffusion. Here, the influence of anomaly on the transport characteristics is also estimated.

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1. Introduction

Note that mathematical modeling is widely used to study the processes of filtration and solute transport in porous media. However, the existing mathematical models are far from being perfect, a number of known phenomena do not have their own model description [20].

In [21], numerical simulation of problems of two-phase filtration in fractured-porous media is carried out using a dual porosity model with a highly inhomogeneous permeability coefficient. A system of equations is given for the case of two-phase filtration without taking into account capillary and gravitational forces, which is a coupled system of equations for pressure and saturation in a

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porous medium with a system of cracks. Various options for setting the flow functions between a porous medium and fractures are considered. The numerical implementation for approximating the velocity and pressure is based on the finite element method. To discretize the saturation equation by means of the method of introducing artificial diffusion, the classical Galerkin method with upwind approximation is used. The results of numerical calculations for a model problem using various flow functions are presented.

An analysis of diffusion in a complex medium shows that the usual diffusion equation based on Fick's law cannot model the anomalous nature of diffusion mass transfer observed in field and laboratory experiments. New mathematical models of diffusion transport, different from Fick's law, have been proposed and confirmed in the literature. This article gives examples of equations that can be used to describe anomalous mass transfer and discusses some important properties of these equations. Two modes revealed anomalous diffusion. One regime, called subdiffusion, is characterized by a slower propagation of the concentration front, so that the square of the front propagation distance requires more time than in the case of classical Fickian diffusion [10].

Elementary particles under the influence of various force fields of different nature make a complex movement. The trajectories of these particles reproduce geometric objects of complex structure [1].

Nigmatullin [16, 17, 18] was the first to derive the equation of fractional diffusion for media of fractal geometry. Taking into account the comb structure of the medium, he obtained an equation with a fractional derivative with respect to time, simulating the process of "slow" diffusion (subdiffusion). A similar approach was used by Fomin et al. [11] for modeling diffusion in a fractured-porous medium. In a recent study based on the dual porosity model [2], the partial advective diffusion equation in a fractured porous aquifer was derived analytically. An expression for the coefficient before the fractional derivative is obtained and all parameters that can affect its value are determined. It was also shown that the order of the fractional derivative in the advection-diffusion equation depends on the fractal dimension of the pores. The application of these equations for modeling mass transfer in a fractured-porous medium can be found in [9, 12].

In the process of fluid filtration, the equilibrium relationship between the filtration velocity and the pressure gradient can be broken. When filtering inhomogeneous fluids, the substance (or particles) contained in the fluid can linger in the pore space, which leads to a change in the porosity and permeability characteristics of the porous medium, primarily its permeability and porosity [20].

Two-zone media are very common in macroscopically inhomogeneous media. In them, some macroscopic zones may have reservoir properties that differ from the rest of the zone. In such media, the processes of filtration and solute transport proceed with the manifestation of internal mass transfer between different zones. This significantly changes the overall pattern of filtration and mass transfer. The project executors previously solved some problems of filtration and solute transport in two-zone media [4, 5, 7, 8, 13, 14, 19]. Here, the problem of substance transfer is considered in a similar setting, but with allowance for anomalous effects. The transfer equations here, unlike the previous ones, have fractional derivatives. Therefore, the object can be considered as a macroscopically inhomogeneous fractal medium.

In this paper, we consider the filtration and solute transport with a stripe-like source in a two-zone medium with a fractal structure.

2. Statement of the problem

We consider a medium consisting of two zones, i.e.
$R_1\,\{0 \le x < \infty,\ 0 \le y \le l\}$, $R_2\,\{0 \le x < \infty,\ l \le y \le \infty\}$ (Fig.1). Initially, regions R_1 and R_2 are filled with fluid without substance.

In R_1, the system of equations for the solute transport has the form

$$\frac{\partial c}{\partial t} = D_1 \frac{\partial^{\beta_1} c}{\partial x^{\beta_1}} + D_2 \frac{\partial^{\beta_2} c}{\partial y^{\beta_2}} - \frac{\partial (v_x c)}{\partial x} - \frac{\partial (v_y c)}{\partial y}\,. \tag{1}$$

The process of solute transport to R_2 can be described by the diffusion equation in the form

$$\frac{\partial c}{\partial t} = D_3 \frac{\partial^{\beta_3} c}{\partial y^{\beta_3}}, \quad (x, y) \in R_2, \tag{2}$$

where c is the concentration of solid particles in the fluid, v_x, v_y are the components of the filtration velocity, D_1, D_2, D_3 are the longitudinal and transverse diffusion coefficients, β_1, β_2, β_3 are the derivative orders, t is the time.

The anomalous filtration velocity is defined as [3]

$$v_x = -\frac{k_1}{\mu} \frac{\partial^{\gamma_1} p}{\partial x^{\gamma_1}}, \quad v_y = -\frac{k_2}{\mu} \frac{\partial^{\gamma_2} p}{\partial y^{\gamma_2}}, \tag{3}$$

where μ is the viscosity coefficient of the substance, $k = \text{const}$ is the permeability coefficient. The continuity equation of the flow of a compressible fluid through a porous medium can be written as [2]

$$\frac{\partial (\rho m)}{\partial t} + div(\rho \vec{v}) = 0, \tag{4}$$

where m is the porosity coefficient, ρ is the density of the liquid.

We use the equations of state of an elastic fluid and an elastic porous medium [2]

$$\rho = \rho_0(1 + \beta_6(p - p_0)),\ m = m_0 + \beta_c(p - p_0), \tag{5}$$

where β_l is the volume compression coefficient of the liquid, β_m is the elasticity coefficient of the medium, ρ_0 is the initial density of the liquid, p_0 is the initial pressure.

Substituting (3), (5) into (4), we can obtain the piezoconductivity equation with a fractional derivative

$$\frac{\partial p}{\partial t} = \chi_1 \frac{\partial^{\gamma_1+1} p}{\partial x^{\gamma_1+1}} + \chi_2 \frac{\partial^{\gamma_2+1} p}{\partial y^{\gamma_2+1}}\,, \tag{6}$$

where $\chi_1 = \frac{k_1}{\mu\beta^*}$, $\chi_2 = \frac{k_2}{\mu\beta^*}$ is coefficient of piezoconductivity, β^* is coefficient of elasticity of the medium, γ_1, γ_2 are orders of the fractional derivatives.

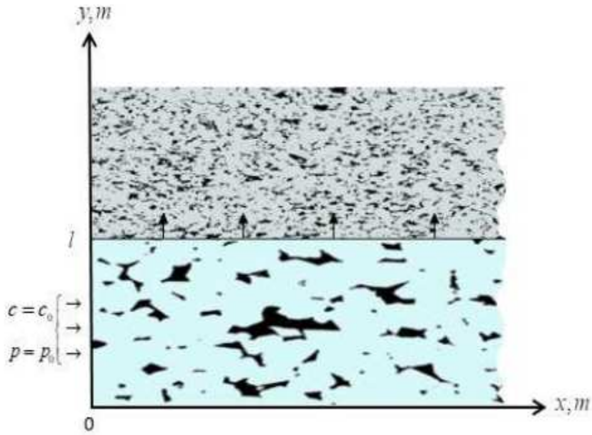

Fig.1. Scheme of filtration and solute transport in a two-zone medium

The initial and boundary conditions of the problem have the form

$$c\,(0, x, y) = 0,\ \ 0 \le x < \infty,\ 0 \le y < \infty, \tag{7}$$

$$c\,(t, 0, y) = 0,\ \ 0 \le y < \delta_1 l,\ \ \delta_2 l < y \le l,\ \ \delta_1 \ne \delta_2,\ \ \delta_1,\ \delta_2 < 1, \tag{8}$$

$$c\,(t, 0, y) = c_0, c_0 = \text{const},\ \ \delta_1 l \le y \le \delta_2 l,\ \ \delta_1, \delta_2 < 1, \tag{9}$$

$$\frac{\partial c}{\partial y}\,(t, x, 0) = 0, \quad 0 \le x < \infty, \tag{10}$$

$$\frac{\partial c}{\partial x}\,(t,\ 0, y) = 0,\ \ 0 \le y < \delta_1 l,\ \ \delta_2 l < y \le l,\ \ \delta_1 \ne \delta_2, \delta_1, \delta_2 < 1, \tag{11}$$

$$\frac{\partial c}{\partial x}(t,\infty,y)=0, \quad 0\le y<\infty, \tag{12}$$

$$\frac{\partial c}{\partial y}(t,x,y)=0\,, \quad y=\infty, \quad 0\le x<\infty \tag{13}$$

$$p\,(0,x,y)=p_0,\ p_0=\text{const}, \tag{14}$$

$$p\,(t,0,y)=p_c,p_c>p_0, \quad p_c=\text{const},\ \delta_1 l\le y\le\delta_2 l,\ \delta_1,\delta_2<1, \tag{15}$$

$$\frac{\partial p}{\partial y}(t,x,0)=0, \quad 0\le x<\infty, \tag{16}$$

$$\frac{\partial p}{\partial y}(t,x,l)=0, \quad 0\le x<\infty, \tag{17}$$

$$\frac{\partial p}{\partial x}(t,\,0,y)=0,0\le y<\delta_1 l,\delta_2 l<y\le l,\delta_1\ne\delta_2,\delta_1,\,\delta_2<1, \tag{18}$$

$$\frac{\partial p}{\partial x}(t,\infty,y)=0, \quad 0\le y\le l. \tag{19}$$

$$c|_{y=l+0}=c|_{y=l-0}\,, \quad 0\le x<\infty, \tag{20}$$

$$D_2\left.\frac{\partial^{\beta_2-1}c}{\partial y^{\beta_2-1}}\right|_{y=l+0}=D_3\left.\frac{\partial^{\beta_3-1}c}{\partial y^{\beta_3-1}}\right|_{y=l-0}, \quad 0\le x<\infty. \tag{21}$$

3. Solution method

Problem (1) - (19) is solved by the finite difference method. To do this, we construct a grid in the area $R_1\bigcup R_2$, $\omega_{h_1h_2\tau}=\omega^+_{h_1h_2\tau}\bigcup\omega^-_{h_1h_2\tau}$, where

$$\omega^1_{h_1h_2\tau}=\big\{(t_k,\,x_i,\,y_j)\,,\ t_k=\tau k,\ \ x_i=ih_1,\ \ y_j=jh_2\,,\ \ k=\overline{0,\,K},$$

$$i=0,1,\ldots,\ \ j=0,1,...,J,\ \ \tau=T/K,\ \ h_2=l/J\}\,,$$

$$\omega^2_{h_1h_2\tau}=\big\{(t_k,\,x_i,\,y_j)\,,\ t_k=\tau k,\ \ x_i=ih_1,\ \ y_j=h_2j,\ \ k=\overline{0,\,K},$$

$$i=0,1,\ldots,\ \ j=J,J+1,...,\ \ \tau=T/K\}\,.$$

In this grid: h_1 - grid step in direction x, h_2 - grid step in direction y in R_1 and R_2, τ - grid step in time, T - maximum time during which the process is studied, K - number of grid intervals of t, J - the number of grid intervals of y in R_1.

Instead of functions $c\,(t,x,y)$, $v\,(t,x,y)$, and $p(t,x,y)$, we will consider network functions whose values at nodes $(t_k,x_i,\,y_j)$, respectively, will be denoted by $c^k_{i\,j}$, $v^k_{i\,j}$, and $p^k_{i\,j}$.

Consider the case $0<\gamma_1,\ \gamma_2\le 1,\ 1<\beta_1,\ \beta_2,\ \beta_3\le 2$.

On grid $\omega_{h_1h_2\tau}$, we approximate the first equation of system (1)-(2) as follows, [6, 15],

$$
\begin{aligned}
&\frac{c_{i\,j}^{k+1/2}-c_{i\,j}^{k}}{0{,}5\,\tau}\\
&\quad=\frac{D_1}{\Gamma(3-\beta_1)h_1^{\beta_1}}\sum_{l=0}^{i-1}\left(c_{i-(l-1),\,j}^{k}-2c_{i-l,\,j}^{k}+c_{i-(l+1),\,j}^{k}\right)\\
&\quad\times\left((l+1)^{2-\beta_1}-l^{2-\beta_1}\right)\\
&\quad+\frac{D_2}{\Gamma(3-\beta_2)h_2^{\beta_2}}\sum_{l=0}^{j-1}\left(c_{i,\,j-(l-1)}^{k}-2c_{i,\,j-l}^{k}+c_{i,\,j-(l+1)}^{k}\right)\\
&\quad\times\left((l+1)^{2-\beta_2}-l^{2-\beta_2}\right)\\
&\quad-\frac{(v_x)_{i\,j}^{k}c_{i\,j}^{k}-(v_x)_{i-1,\,j}^{k}c_{i-1,j}^{k}}{h_1}-\frac{(v_y)_{i\,j}^{k}c_{i\,j}^{k}-(v_y)_{i,\,j-1}^{k}c_{i,j-1}^{k}}{h_2},
\end{aligned}
\tag{22}
$$

$$i=\overline{1,\,I-1},\ \ j=\overline{1,\,J-1},\ \ k=\overline{0,\,K-1},$$

$$
\begin{aligned}
&\frac{c_{i\,j}^{k+1}-c_{i\,j}^{k+1/2}}{0{,}5\,\tau}\\
&\quad=\frac{D_1}{\Gamma(3-\beta_1)h_1^{\beta_1}}\sum_{l=0}^{i-1}\left(c_{i-(l-1),\,j}^{k+1/2}-2c_{i-l,\,j}^{k+1/2}+c_{i-(l+1),\,j}^{k+1/2}\right)\\
&\times\left((l+1)^{2-\beta_1}-l^{2-\beta_1}\right)\\
&\quad+\frac{D_2}{\Gamma(3-\beta_2)h_2^{\beta_2}}\sum_{l=0}^{j-1}\left(c_{i,\,j-(l-1)}^{k+1/2}-2c_{i,\,j-l}^{k+1/2}+c_{i,\,j-(l+1)}^{k+1/2}\right)\\
&\times\left((l+1)^{2-\beta_2}-l^{2-\beta_2}\right)\\
&-\frac{(v_x)_{i\,j}^{k+1/2}c_{i\,j}^{k+1/2}-(v_x)_{i-1,\,j}^{k+1/2}c_{i-1,j}^{k+1/2}}{h_1}-\frac{(v_y)_{i\,j}^{k+1/2}c_{i\,j}^{k+1/2}-(v_y)_{i,\,j-1}^{k+1/2}c_{i,j-1}^{k+1/2}}{h_2},
\end{aligned}
\tag{23}
$$

$$i=\overline{1,\,I-1},\ \ j=\overline{1,\,J-1},\ \ k=\overline{0,\,K-1},$$

$$
\begin{aligned}
&\frac{c_{i\,j}^{k+1/2}-c_{i\,j}^{k}}{0{,}5\,\tau}\\
&\quad=\frac{D_3}{\Gamma(3-\beta_3)h_2^{\beta_3}}\sum_{l=0}^{i-1}\left(c_{i,\,j-(l-1)}^{k}-2c_{i,\,j-l}^{k}+c_{i,\,j-(l+1)}^{k}\right)\\
&\times\left((l+1)^{2-\beta_3}-l^{2-\beta_3}\right),
\end{aligned}
\tag{24}
$$

$$i=\overline{0,\ I},\ \ j=\overline{J,\ J^1-1},\ \ k=\overline{0,K-1}$$

$$
\begin{aligned}
&\frac{c_{i\,j}^{k+1}-c_{i\,j}^{k+1/2}}{0{,}5\,\tau}\\
&\quad=\frac{D_3}{\Gamma(3-\beta_3)h_2^{\beta_3}}\sum_{l=0}^{i-1}\left(c_{i,\,j-(l-1)}^{k+1/2}-2c_{i,\,j-l}^{k+1/2}+c_{i,\,j-(l+1)}^{k+1/2}\right)\\
&\times\left((l+1)^{2-\beta_3}-l^{2-\beta_3}\right),
\end{aligned}
\tag{25}
$$

$$i=\overline{0,\ I},\ \ j=\overline{J,\ J^1-1},\ \ k=\overline{0,K-1}.$$

For the filtration rate component, we use the following schemes

$$(v_x)_{i\,j}^{k+1/2} = -\frac{k_1}{\mu}\frac{p_{i+1\,j}^{k+1/2} - \gamma_1 p_{i,\,j}^{k+1/2}}{(2-\gamma_1)h_1^{\gamma_1}}, \tag{26}$$

$$i = \overline{0,\, I-1},\ \ j = \overline{0,\, J},\ \ k = \overline{0,\, K-1},$$

$$(v_y)_{i\,j}^{k} = -\frac{k_2}{\mu}\frac{p_{i\,j+1}^{k} - \gamma_2 p_{i,\,j}^{k}}{(2-\gamma_2)h_2^{\gamma_2}}, \tag{27}$$

$$i = \overline{0,\, I},\ \ j = \overline{0,\, J-1},\ \ k = \overline{0,\, K-1},$$

$$(v_y)_{i\,j}^{k+1} = -\frac{k_2}{\mu}\frac{p_{i\,j+1}^{k+1} - \gamma_2 p_{i,\,j}^{k+1}}{(2-\gamma_2)h_2^{\gamma_2}}, \tag{28}$$

$$i = \overline{0,\, I},\ \ j = \overline{0,\, J-1},\ \ k = \overline{0,\, K-1}.$$

Equation (4) is approximated as

$$\begin{aligned}
&\tfrac{p_{i\,j}^{k+1/2} - p_{i\,j}^{k}}{0{,}5\,\tau} \\
&\quad= \tfrac{\chi_1}{\Gamma(3-\gamma_1)h_1^{\gamma_1}} \textstyle\sum_{l=0}^{i-1} \left(p_{i-(l-1),\,j}^{k} - 2p_{i-l,\,j}^{k} + p_{i-(l+1),\,j}^{k}\right) \\
&\times \left((l+1)^{2-\gamma_1} - l^{2-\gamma_1}\right) \\
&\quad+ \tfrac{\chi_2}{\Gamma(3-\gamma_2)h_2^{\gamma_2}} \textstyle\sum_{l=0}^{j-1} \left(p_{i,\,j-(l-1)}^{k} - 2p_{i,\,j-l}^{k} + p_{i,\,j-(l+1)}^{k}\right) \\
&\times \left((l+1)^{2-\gamma_2} - l^{2-\gamma_2}\right),
\end{aligned} \tag{29}$$

$$i = \overline{1,\, I-1},\ \ j = \overline{1,\, J-1},\ \ k = \overline{0, K-1},$$

$$\begin{aligned}
&\tfrac{p_{i\,j}^{k} - p_{i\,j}^{k+1/2}}{0{,}5\,\tau} \\
&\quad= \tfrac{\chi_1}{\Gamma(3-\gamma_1)h_1^{\gamma_1}} \textstyle\sum_{l=0}^{i-1} \left(p_{i-(l-1),\,j}^{k+1/2} - 2p_{i-l,\,j}^{k+1/2} + p_{i-(l+1),\,j}^{k+1/2}\right) \\
&\times \left((l+1)^{2-\gamma_1} - l^{2-\gamma_1}\right) \\
&\quad+ \tfrac{\chi_2}{\Gamma(3-\gamma_2)h_2^{\gamma_2}} \textstyle\sum_{l=0}^{j-1} \left(p_{i,\,j-(l-1)}^{k+1/2} - 2p_{i,\,j-l}^{k+1/2} + p_{i,\,j-(l+1)}^{k+1/2}\right) \\
&\times \left((l+1)^{2-\gamma_2} - l^{2-\gamma_2}\right),
\end{aligned} \tag{30}$$

$$i = \overline{1,\, I-1},\ \ j = \overline{1,\, J-1},\ \ k = \overline{0, K-1}.$$

The initial and boundary conditions are approximated as

$$c_{i,j}^{k} = 0,\ \ i = \overline{0, I},\ \ j = \overline{0, J},\ \ k = 0, \tag{31}$$

$$c_{i,j}^{k} = 0,\ \ i = 0, 0 \le y < \delta_1 l,\ \ \delta_2 l < y \le l,\ \ \delta_1 \ne \delta_2,\ \ \delta_1,\, \delta_2 < 1, \tag{32}$$

$$c_{i,j}^{k} = c_0, \quad i = 0, \ , k = \overline{0, K}, \delta_1 l \leq y \leq \delta_2 l, \ \delta_1, \delta_2 < 1, \tag{33}$$

$$\frac{c_{i,j+1}^{k} - c_{i,j}^{k}}{h_2} = 0, \ i = \overline{0,I}, \ \ j = 0, \ \ k = \overline{0, K}, \tag{34}$$

$$\frac{c_{i+1,j}^{k} - c_{i,j}^{k}}{h_1} = 0, \tag{35}$$

$$i = 0, k = \overline{0, K}, 0 \leq y < \delta_1 l, \delta_2 l < y \leq l, \delta_1 \neq \delta_2, \delta_1, \ \delta_2 < 1,$$

$$\frac{c_{i,j}^{k} - c_{i-1,j}^{k}}{h_1} = 0, \ i = I, \ \ j = \overline{0, J^1}, \ \ k = \overline{0, K}, \tag{36}$$

$$\frac{c_{i,j}^{k} - c_{i,j-1}^{k}}{h_2} = 0, \ i = \overline{0,I}, \ \ j = J^1, \ \ k = \overline{0, K}, \tag{37}$$

$$p_{i,j}^{k} = p_0 = \text{const}, \ i = \overline{0, I}, \ \ j = \overline{0, J}, \ \ k = 0, \tag{38}$$

$$p_{i,j}^{k} = p_c, i = 0, \ j = J/2, k = \overline{0, K}, \delta_1 l \leq y \leq \delta_2 l, \delta_1, \delta_2 < 1, \tag{39}$$

$$\frac{p_{i,j+1}^{k} - p_{i,j}^{k}}{h_2} = 0, \ i = \overline{0,I}, \ \ j = 0, \ \ k = \overline{0, K}, \tag{40}$$

$$\frac{p_{i,j}^{k} - p_{i,j-1}^{k}}{h_2} = 0, \ i = \overline{0,I}, \ \ j = J, \ \ k = \overline{0, K}, \tag{41}$$

$$\frac{p_{i+1,j}^{k} - p_{i,j}^{k}}{h_1} = 0, \tag{42}$$

$$i = 0, k = \overline{0, K}, 0 \leq y < \delta_1 l, \delta_2 l < y \leq l, \delta_1 \neq \delta_2, \delta_1, \ \delta_2 < 1,$$

$$\frac{p_{i,j}^{k} - p_{i-1,j}^{k}}{h_1} = 0, \ i = I, \ \ j = \overline{0, J}, \ \ k = \overline{0, K}, \tag{43}$$

$$c_{i,J+0}^{k} = c_{i,J-0}^{k}, \quad i = 0, 1, ..., \quad k = \overline{0, K}, \tag{44}$$

$$D_2 \frac{c_{i,J}^{k} - (\beta_2 - 1) c_{i,J-1}^{k}}{(2 - (\beta_2 - 1)) h_2^{(\beta_2 - 1)}} = D_3 \frac{c_{i,J+1}^{k} - (\beta_3 - 1) c_{i,J}^{k}}{(2 - (\beta_3 - 1)) h_2^{(\beta_3 - 1)}}, \tag{45}$$

where I, J^1 are sufficiently large for which equation $c_{I\,J^1}^{k} = 0$ approximately holds.

The sequence of calculations is as follows: first, p is determined from the difference scheme (29) on the $(k + 1/2)$-layer, then from (26), (28) the components of the filtration rate are calculated, after that c is determined on the $(k + 1/2)$-layer from the difference equations (22). Then, on the $(k + 1)$-layer,

p is determined from the difference scheme (30), the filtration velocity components are calculated from (27), after that, c is determined on the $(k+1)$-layer from the difference equations (25).

For equations (24), (25), we introduce the following notation

$$E_1^k = \frac{(v_x)_{i\,j}^k\, c_{i\,j}^k - (v_x)_{i-1,\,j}^k\, c_{i-1,j}^k}{h_1}, E_2^k = \frac{(v_y)_{i\,j}^k\, c_{i\,j}^k - (v_y)_{i,\,j-1}^k\, c_{i,j-1}^k}{h_2},$$

$$G_1 = \frac{0,5\tau D_1}{\Gamma\left(3-\beta_1\right) h_1^{\beta_1}}, G_2 = \frac{0,5\tau D_2}{\Gamma\left(3-\beta_2\right) h_2^{\beta_2}},$$

$$S_1^k = \sum_{l=0}^{i-1}\left(c_{i-(l-1),\,j}^k - 2c_{i-l,\,j}^k + c_{i-(l+1),\,j}^k\right)\left((l+1)^{2-\beta_1} - l^{2-\beta_1}\right),$$

$$S_2^k = \sum_{l=0}^{j-1}\left(c_{i,\,j-(l-1)}^k - 2c_{i,\,j-l}^k + c_{i,\,j-(l+1)}^k\right)\left((l+1)^{2-\beta_2} - l^{2-\beta_2}\right).$$

Taking into account these notations, equations (22) and (23), respectively, have the form

$$c_{i\,j}^{k+1/2} = c_{i\,j}^k + G_1 S_1 + G_2 S_2 - 0,5\tau\left(E_1^k + E_2^k\right), \tag{46}$$

$$c_{i\,j}^{k+1} = c_{i\,j}^{k+1/2} + G_1 S_1 + G_2 S_2 - 0,5\tau\left(E_1^{k+1/2} + E_2^{k+1/2}\right). \tag{47}$$

The concentration field is determined step by step from (46),(47).

4. Results and discussion

The following values of the initial parameters were used in the calculations: $k_1 = 2\cdot 10^{-13}\, m^{1+\gamma_1}$, $k_2 = 10^{-13}\, m^{1+\gamma_2}$, $\mu = 5\cdot 10^{-3}\, Pa\cdot s$, $\beta^* = 3\cdot 10^{-8}\, Pa^{-1}$, $p_c = 5\cdot 10^5\, Pa$, $p_0 = 10^5\, Pa$ and various values of $D_1 = 5\cdot 10^{-5}\ m^{\beta_1}/s$, $D_2 = 10^{-5}\ m^{\beta_2}/s$, and $D_3 = 10^{-5}\ m^{\beta_3}/s$. Some results are shown in Figs 2-4. Figure 2 shows the distribution of concentration at different points in time. Figure 3 shows the concentration surfaces as the values of β_3 decrease from 2. A comparison of the results shows that a decrease in β_3 from 2 accelerates the diffusion process in the R_2 zone. At the same time, with a decrease in β_3 from 2 in the zone R_1 one can notice a decrease in the concentration values. Figure 4 shows the change in the concentration profile for different values of γ_1 and γ_2. The results obtained show that a decrease in the values of γ_1 and γ_2 from

1 leads to an increase in the effects of anomalous filtration and the diffusion process in both zones.

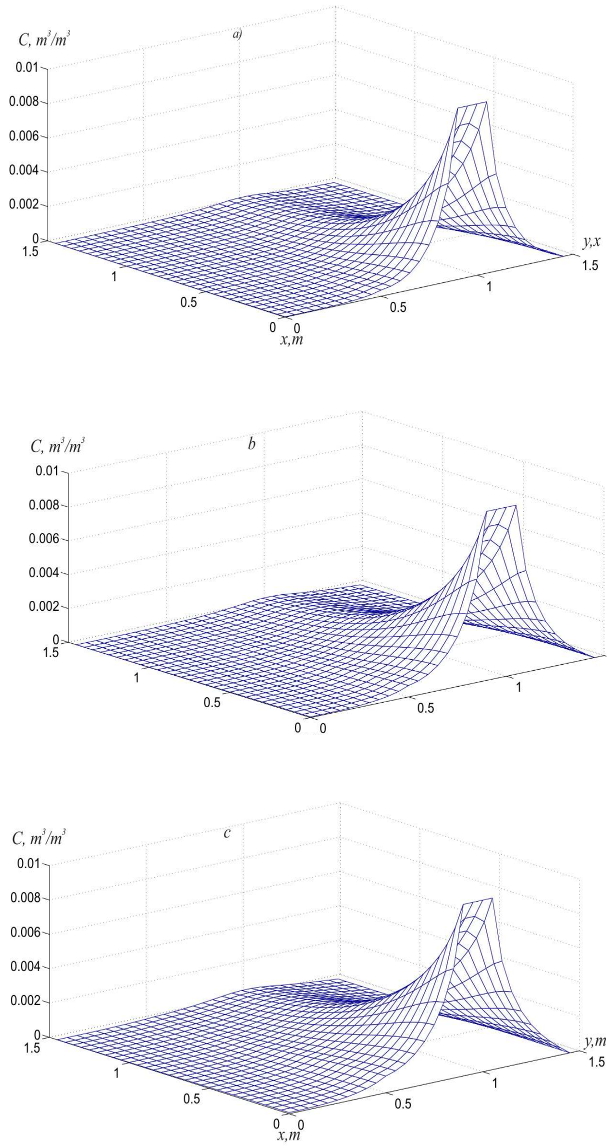

Fig 2. Profiles of c at, $\gamma_1 = 0.8$, $\gamma_2 = 0.8$; $\beta_1 = 1.8$, $\beta_2 = 1.8$; $\beta_3 = 1.6$; $t = 5000\ s()$,$t = 7500\ s(b)$, $t = 10000\ s\ (c)$.

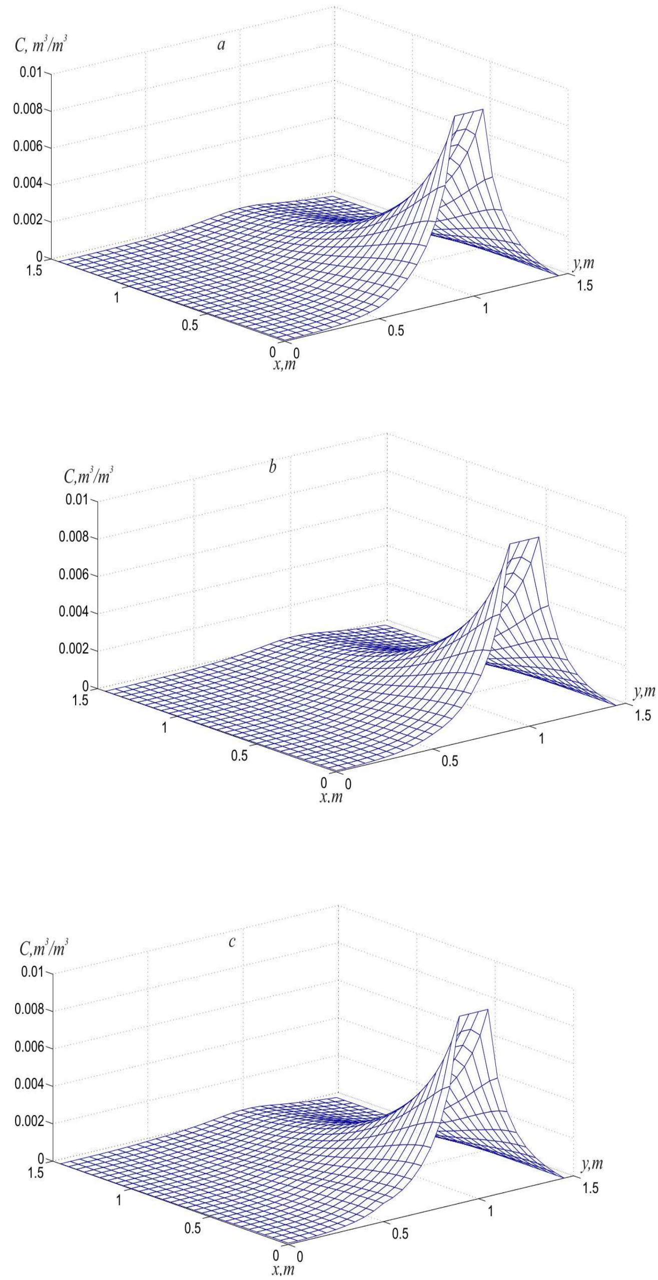

Fig 3. Profiles of c at, $t = 10000$ s, $\gamma_1 = 0.8$,$\gamma_2 = 0.8$; $\beta_1 = 1.8$, $\beta_2 = 1.8$; $\beta_3 = 2()$, $\beta_3 = 1.8$ *(b)*, $\beta_3 = 1.6$ *(c)*

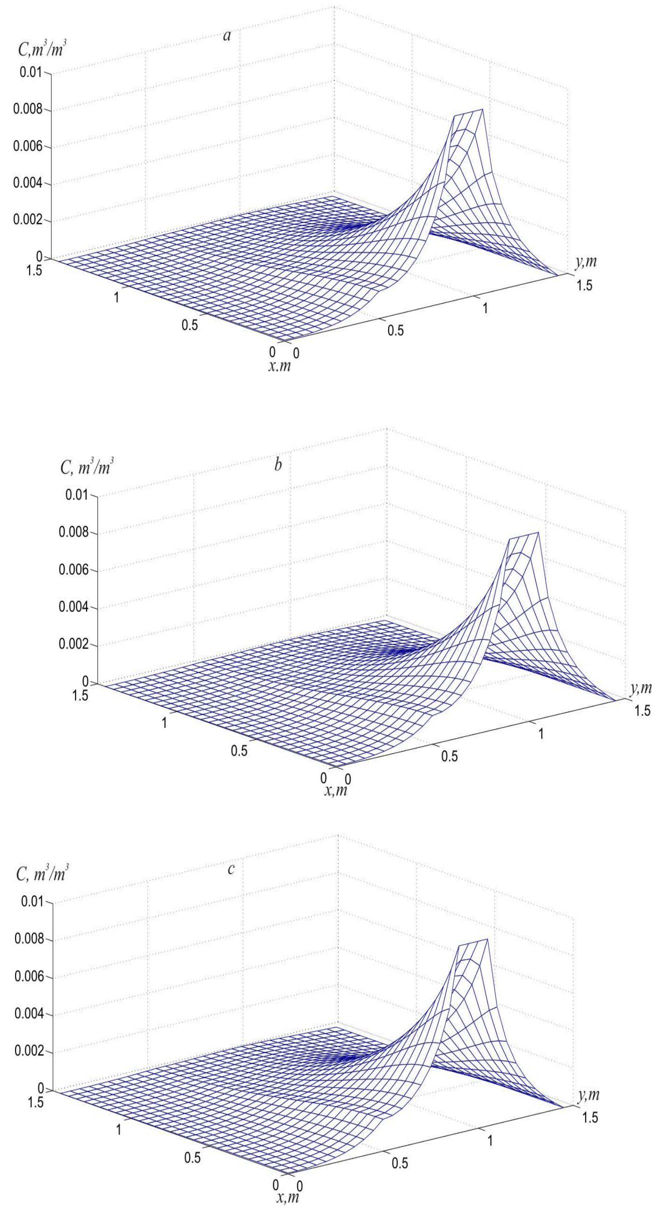

Fig 4. Profiles of c at $t = 10000$ s, $\beta_3 = 1.6$, $\beta_1 = 2$,$\beta_2 = 2$; $\gamma_1 = 1$, $\gamma_2 = 1$ *()*, $\gamma_1 = 0,9$, $\gamma_2 = 0,9$ *(b)*, $\gamma_1 = 0,8$, $\gamma_2 = 0,8$ *(c)*,

5. Conclusion

The calculation results show that a decrease in the derivative order in the filtration velocity equation from 1 leads to an increase in concentration effects. A decrease in the order of the derivative in the diffusion term from 2 leads to an "acceleration" of the diffusion process. The concentration profiles are also shown with decreasing values from 2. Comparison of the results shows that a decrease from 2 accelerates the diffusion process in the zone. At the same time, with a decrease from 2 in the zone, one can notice a decrease in the concentration values.

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