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SIMULTANEOUS AND NON-SIMULTANEOUS BLOW-UP FOR A NON-LOCAL DIFFUSION SYSTEM

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Abstract: In this work, the non-local diffusion system with Neumann boundary conditions

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u(x,t),v(x,t)) \\ v_t(x,t) &= \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + g(u(x,t),v(x,t)) \\ u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \end{aligned} \tag{1}$$

is studied, where $(x,t) \in \Omega \times (0,T)$, Ω is a bounded, connected and smooth domain, f and g are continuous functions and $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, nonnegative and real functions. Existence and uniqueness of solutions is proved. For some particular functions f, g, the simultaneous and non-simultaneous blow-up for solutions is analyzed. Finally, the blow-up rate for the solution is given.

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Key Words: nonlocal diffusion; system equations; Neumann boundary conditions; simultaneous and non-simultaneous blow-up

1. Introduction

The coupled parabolic system

$$u_t = \Delta u + f(u,v), \quad v_t = \Delta v + g(u,v), \tag{2}$$

where $u(x,0)$, $v(x,0)$ are nonnegative bounded functions and f, g continuous

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functions, has been studied by several authors. The existence and uniqueness of solutions and the blow-up phenomena have been analyzed, see for instance, [8], [12], [13], [15], [16] and [17]. For the case $f(u,v) = u^r v^p,\ \ g(u,v) = v^q v^s$, whit $p,q,r,s > 0$, Dickstein and Escobedo in [8], studied (2) considering bounded domains. For the case $f(u,v) = u^r + v^p,\ \ g(u,v) = u^q + v^s$ whit $p,q,r,s > 0$, Souplet and Tayachi in [17], studied the Cauchy problem for (2). Rossi and Souplet in [15], studied (2) considering bounded domains and homogeneous Dirichlet boundary conditions. They show that the non-simultaneous and simultaneous blow-up are possible for the exponent region $r > q+1$ or $s > p+1$, both depending on the initial data.

Equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)dy - u(x,t), \tag{3}$$

and variations of it, have been widely used in the last decade to model diffusion processes, see for instance [1], [2], [6] and [10]. As stated in [10], $J : \mathbb{R}^n \to \mathbb{R}$ is a non-negative, smooth, symmetric radially and strictly decreasing function, with $\int_{\mathbb{R}^n} J(x)dx = 1$, supported in the unitary ball. If $u(x,t)$ is thought as a density at the point x at time t and $J(x-y)$ is thought as the probability distribution of jumping from location y to location x, then $(J*u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^n} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies equation (3). This equation is called nonlocal diffusion equation because the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but also on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation $u_t = \Delta u$ such as the fact that bounded stationary solutions are constant, a maximum principle holds for both of them and even, if J is compactly supported, perturbations propagate with infinite speed.

Bogoya in [3], studied the problem with Neumann boundary condition and source term

$$\begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u(x,t)),\ x\in\Omega,\ t>0, \\ &u(x,0) = u_0(x),\ x\in\Omega, \end{aligned} \tag{4}$$

where $u_0 \in C(\overline{\Omega})$ is nonnegative function and $\Omega \subset \mathbb{R}^N$ a bounded, smooth and connected domain, f a function of u representing reaction term (source). The author proves the existence and uniqueness of solutions. For some conditions on f, it is shown that the solution becomes unbounded at time T (blow-up). For blowing-up solutions, the rate of blow-up (that is the speed at which solutions go to infinity an time T) is also analyze. For $f(u) = e^u$, the author considers the radial case in the ball of radius $r > 0$. It is proved that if radially symmetric initial condition has a unique maximum at the origin, then the solution is radially symmetric and has a unique maximum at the origin which it is the only blow up point.

Perez-Llanos and Rossi in [11], studied the problem (4) for $f(u) = u^p$ with $p > 0$, where $u_0 \in C(\overline{\Omega})$ is nonnegative function. They prove that for, nonnegative and nontrivial solutions, blow up in finite time for the solutions, if and only if, $p > 1$. Moreover, they find that the blow-up rate is the same that the one that holds for the ODE $u'(t) = u^p(t)$, that is, $\lim_{t\to T^-}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_\infty = (1/(p-1))^{1/(p-1)}$.

Bogoya in [4], studied the following nonlocal reaction-diffusion system with Neumann boundary conditions

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + v^p(x,t), \ (x,t)\in\Omega\times(0,T) \\ v_t(x,t) &= \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + u^q(x,t), \ (x,t)\in\Omega\times(0,T) \\ u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{5}$$

with $p,\ q > 0$, $u_0(x),\ v_0(x) \in C(\overline{\Omega})$ nonnegative and nontrivial functions and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ a bounded connected and smooth domain. The author proves the existence and uniqueness of nonnegative solutions (u, v) of (5) and shows that the solution (u, v) is unique if $pq \geq 1$, or if one of the initials conditions is not zero for $pq < 1$. The, the globally existence for the solution of (5) is proved. It is shown that if $pq > 1$ and $u_0,\ v_0$, are nonnegative and nontrivial functions, then the solution (u, v) of (5) blows up in finite time T, if $pq \leq 1$ then the solution (u, v) of (5) exists globally. Finally it is considered the blow-up rate for the solution (u, v) of (5). Bogoya and Gómez in [5] studied the numerical approximation of (5).

General problem: Our first objective in this paper is to study the nonlocal diffusion system

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u(x,t),v(x,t)), \\ v_t(x,t) &= \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + g(u(x,t),v(x,t)), \\ u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{6}$$

where u_0, v_0 are nonnegative bounded functions, $(x,t) \in \Omega\times(0,T)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $f, g : \mathbb{R}+\times\mathbb{R}^+ \to \mathbb{R}$ are continuous functions. Since the integrating is on Ω, is considered that the diffusion takes place only in Ω, sot that, the individuals may not enter nor leave Ω. This is the analogous of what is called in the literature as homogeneous Neumann boundary conditions.

Sum of power terms: A second objective is to the study of (6) with $f(u,v) = u^r + v^p$, $g(u,v) = u^q + v^s$ for $p,q,r,s > 0$, $u_0, v_0 \in C(\Omega)$, $(x,t) \in \Omega \times (0,T)$,

$$\begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + u^r(x,t) + v^p(x,t),\\ &v_t(x,t) = \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + u^q(x,t) + v^s(x,t),\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{aligned} \tag{7}$$

The system (7), can be viewed as a combination of the following two systems: the problem (5) and

$$\begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + u^r(x,t), \ (x,t)\in\Omega\times(0,T)\\ &v_t(x,t) = \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + v^s(x,t), \ (x,t)\in\Omega\times(0,T)\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{8}$$

(8) is uncoupled system and non-simultaneous blowing-up solutions exists, see [11]. In [3], it is studied the coupled system (5), which has only simultaneous blowing-up solutions if $pq > 1$. Therefore it is natural to ask whether the blow-up is simultaneous or not for (7).

It is said that a solution (u,v) blows up in finite time if only if there exists a finite time $T > 0$, such that

$$\limsup_{t\nearrow T} \left(\|u(x,t)\|_{L^\infty(\Omega)} + \|v(x,t)\|_{L^\infty(\Omega)}\right) = \infty.$$

If $T = \infty$, the solution (u,v) is global, i.e. the solution exists for all $t \geq 0$.

It is said that the blow-up is simultaneous if

$$\limsup_{t\nearrow T} \|u(x,t)\|_{L^\infty(\Omega)} = \limsup_{t\nearrow T} \|v(x,t)\|_{L^\infty(\Omega)} = \infty.$$

One can note a priori that there is not a reason for both components of the system to blow up simultaneously. In fact, it could happen that one of the components blows up as $t \to T$, while the other, remains bounded on $[0, T)$. This phenomenon is called non-simultaneous blow up.

The following information is necessary for the study: If (α, β) is the unique solution of

$$\begin{pmatrix} r-1 & p \\ q & s-1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \tag{9}$$

then $\alpha = (1 - s + p)/\Delta,\ \beta = (1 - r + q)/\Delta,\ \Delta = (r-1)(s-1) - pq \neq 0.$

The rest of the paper is organized as follows: In Section 2, we study the general problem (6), the existence and uniqueness of nonnegative solutions (u, v) as well as a comparison principle for the solutions is studied. In Section 3, we study the sum of power terms (7), the global existence and blow-up of solutions is proved, non-simultaneous blows up and the blow-up rate is studied.

2. General problem

In this section, we prove the existence and uniqueness of nonnegative solutions (u, v) of (6) by following the ideas of [8].

Let $t_0 > 0$ be fixed and

$$\begin{aligned} X_{t_0} = C\left([0, t_0] :\ C(\overline{\Omega}) \times C(\overline{\Omega})\right) = \{(u, v) :\\ [0, t_0] \to C(\overline{\Omega}) \times C(\overline{\Omega}) :\quad (u, v) \text{ is continuous}\big\}, \end{aligned}$$

a Banach space with the norm

$$\begin{aligned} |||(u, v)||| &= \max_{0 \le t \le t_0} \|(u(\cdot, t), v(\cdot, t))\|_I, \text{ with } I = L^\infty(\overline{\Omega}) \times L^\infty(\overline{\Omega}), \\ \|(u(\cdot, t), v(\cdot, t))\|_I &= \max_{x \in \overline{\Omega}} |u(x, t)| + \max_{x \in \overline{\Omega}} |v(x, t)|. \end{aligned}$$

Let $\|u\|_\kappa = \max_{0 \le t \le t_0} \|u(\cdot, t)\|_{L^\infty(\overline{\Omega})}$.

Let $P_{t_0} = \{(u, v) \in X_{t_0} : u \ge 0,\ v \ge 0\}$ be a closed subspace of X_{t_0}. We define the operator $\psi : P_{t_0} \to P_{t_0}$ as $\psi_{(u_0, v_0)}(u, v) = (T_{u_0}(u), S_{v_0}(v))$, where

$$\begin{aligned} T_{u_0}(u)(x, t) &= \int_0^t \int_\Omega J(x - y)(u(y, s) - u(x, s))dy\ ds \\ &+ \int_0^t f(u(x, s), v(x, s))ds + u_0(x), \\ S_{v_0}(v)(x, t) &= \int_0^t \int_\Omega J(x - y)(v(y, s) - v(x, s))dy\ ds \\ &+ \int_0^t g(u(x, s), v(x, s))ds + v_0(x). \end{aligned} \tag{10}$$

We begin the study considering the functions f and g locally Lipschitz. The following lemma is very important for the study and its proof is analogous to that given in [4], reason why, we omit here.

Lemma 1. *Let f and g be locally Lipschitz function, (u_0, v_0), $(w_0, z_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and (u, v), $(w, z) \in P_{t_0}$. Then there exists a positive constant $C = C(K_1, K_2, \Omega, J)$ such that*

$$|||\psi_{(u_0,v_0)}(u, v) - \psi_{(w_0,z_0)}(w, z)||| \le Ct_0|||(u, v) - (w, z)||| + \|(u_0, v_0) - (w_0, z_0)\|_I. \tag{11}$$

Theorem 2. *Let f and g be locally Lipschitz function and $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, nonnegative and real functions, then, there exists a unique solution (u, v) of (6) such that $(u, v) \in P_{t_0}$. Moreover, (u, v) can be extended to a maximal interval $[0, T)$ with $T \le \infty$.*

Proof. Following ideas of the proof of Lemma 1, we see that the operator $\psi : P_{t_0} \cap B_r(0, 0) \to P_{t_0} \cap B_r(0, 0)$ is well defined, where

$$r = \max\{\|u_0\|,\ \|v_0\|\} + 1 \quad \text{and} \quad B_r(0, 0) = \{(u, v) : \|(u, v)\|_I < r\ \}. \tag{12}$$

Now, taking $(u_0, v_0) = (w_0, z_0)$ in Lemma 1 and choosing t_0 such that $Ct_0 < 1$, we obtain that $\psi(u, v)$ is a strict contraction of $P_{t_0} \cap B_r(0, 0)$ into itself, therefore, there exists a unique fixed point (u, v) of $\psi(u, v)$ in $P_{t_0} \cap B_r(0, 0)$ by the Banach fixed point theorem, which is the uniqueness of solution to (6), in $\overline{\Omega} \times [0, t_0]$. Uniqueness implies that the solution can be extended to a maximal interval $[0, T)$, with $T \le \infty$.

Remark 3. The solution (u, v) of (6) depend continuously on the initial data. In fact if (u, v) and (w, z) are solutions to (6) with initial data (u_0, v_0) and (w_0, z_0) respectively, then there exists a constant $\widetilde{C} = \widetilde{C}(t_0, K_3, \Omega, J)$ such that

$$|||(u, v) - (w, z)||| \le \widetilde{C}\|(u_0, v_0) - (w_0, z_0)\|_I.$$

Remark 4. $(u,\ v) \in P_{t_0}$ is a solutions of (6),

$$\begin{aligned} u(x,t) &= \int_0^t \int_\Omega J(x-y)(u(y,s)-u(x,s))dy\ ds \\ &\quad + \int_0^t f(u(x,s),v(x,s))ds + u_0(x), \\ v(x,t) &= \int_0^t \int_\Omega J(x-y)(v(y,s)-v(x,s))dy\ ds \\ &\quad + \int_0^t g(u(x,s),v(x,s))ds + v_0(x). \end{aligned} \tag{13}$$

Following the ideas of [8], the existence and uniqueness of the solution of (6) is studied under conditio that f and g, are continuous functions.

Theorem 5. *Let f and g continuous functions and $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ nonnegative and real functions, then, there exists a unique solution (u, v) of (6), such that $(u, v) \in P_{t_0}$.*

We will use the notation $(a, b) \geq (c, d)$ to indicate that $a \geq c$ and $b \geq d$.

Definition 6. Let $\overline{u}, \overline{v} \in C^1([0, T); C(\overline{\Omega}))$. $(\overline{u}, \overline{v})$ is called a supersolution of (6) if

$$\begin{aligned} &\overline{u}_t(x,t) \geq \int_\Omega J(x-y)(\overline{u}(y,t)-\overline{u}(x,t))dy + f(\overline{u}(x,t),\overline{v}(x,t)) \\ &\overline{v}_t(x,t) \geq \int_\Omega J(x-y)(\overline{v}(y,t)-\overline{v}(x,t))dy + g(\overline{u}(x,t),\overline{v}(x,t)) \\ &\overline{u}(x,0) \geq u_0(x), \quad \overline{v}(x,0) \geq v_0(x), \quad x \in \Omega. \end{aligned} \tag{14}$$

Analogously, $(\underline{u}, \underline{v})$, is called a subsolution of (6), if it satisfies the opposite inequalities.

Now, we consider the following hypothesis on f and g:

H_1 : f and g are increasing functions in each variable, i.e. $f(u_2, v_2) \geq f(u_1, v_1)$ and $g(u_2, v_2) \geq g(u_1, v_1)$ for all $0 \leq u_1 \leq u_2$, $0 \leq v_1 \leq v_2$.

The hypothesis H_1 allows us to establish some comparison results, which are given in the following lemma. Its proof is analogous to that given in Lemma 2.2 of [8], a reason why we omit it here.

Lemma 7. *Let f, g, be continuous functions satisfying H_1, and let $(\overline{u}, \overline{v})$, a supersolution of (6) in $\Omega \times (0,T)$. Then, there exists a solution (u,v) of (6) in $\Omega \times (0,T)$ such that $(u,v) \leq (\overline{u}, \overline{v})$. Moreover, if $(\underline{u}, \underline{v})$ is a subsolution of (6) in $\Omega \times (0,T)$, verifying $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, then, there exists a solution (u,v) defined on $\Omega \times (0,T)$ such that $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$.*

As a consequence of these previous results, we have the following corollary, which proof is omitted.

Corollary 8. *Let f and g be locally Lipschitz functions satisfying H_1 and (u,v) the solution of (6), in $\Omega \times (0,T)$.*

1. *If $(\overline{u}, \overline{v})$ is a supersolution of (6), then $(u,v) \leq (\overline{u}, \overline{v})$ in $\Omega \times (0,T)$.*
2. *If $(\underline{u}, \underline{v})$ is a subsolution of (6), then $(\underline{u}, \underline{v}) \leq (u,v)$ in $\Omega \times (0,T)$.*

Definition 9. A solution (u_M, v_M) of (6) in $\Omega \times (0,T)$ is a maximal solution of (6), if given any other solution (u,v) of (6), we have $(u,v) \leq (u_M, v_M)$ in $\Omega \times (0,T)$.

Lemma 10. *Let f, g be continuous functions satisfying H_1. Then, (6) has a maximal solution (u,v) in $\Omega \times (0,T)$.*

Proof. As in Theorem 5, let $(f_n)_n$ and (g_n) decreasing sequences of locally Lipschitz functions, with $f_n \to f$, $g_n \to g$ as $n \to \infty$. Moreover, we can take f_n, g_n satisfying the condition H_1, for all $n \in N$. Let (u_n, v_n) the solution of (1), then $u_M(x,t) = \lim_{n\to\infty} u_n(x,t)$, $v_M(x,t) = \lim_{n\to\infty} v_n(x,t)$, and by Remark (4), we obtain what (u_M, v_M), is a solution of (6), in $\Omega \times (0,T)$. We claim that (u_M, v_M) is the maximal solution of (6). Indeed, if (u,v) is any other solution of (6) with source term f, g then (u,v) is a subsolution of (6) with source term f_n, g_n for all $n \in N$ as $(f_n)_n$ and (g_n) decreasing sequences, therefore $(u,v) \leq (u_n, v_n)$ for all $n \in N$. Letting $n \to \infty$, we get $(u,v) \leq (u_M, v_M)$ in $(x,t) \in \Omega \times (0,T)$.

Remark 11. If $(\overline{u}, \overline{v})$ is a positive supersolution of (6) and $(\underline{u}, \underline{v})$ is a subsolution of (6), then $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$.

The following lemma is very important for the case when f and g are non-Lipchitz functions. The demonstration is very similar to this given in [8], so that we omit here.

Lemma 12. *Let f, g be continuous functions satisfying H_1, with $f(0,0) = g(0,0) = 0$ and $f \neq 0$, $g \neq 0$. Let $(u_0, v_0) = (0,0)$ in Ω and $(\overline{u}, \overline{v})$ a supersolution of (6), such that $(\overline{u}, \overline{v}) > (0,0)$ in $\Omega \times (0,T)$. Then $(\overline{u}, \overline{v}) > (u,v)$ in $\Omega \times (0,T)$ for any solution (u,v) of (6), with $(u_0, v_0) = (0,0)$.*

3. General problem

Now, we analyze the problem (7). As $p, q, r, s > 0$, we note that $f(u,v) = u^r + v^p$, $g(u,v) = u^q + v^s$ satisfy H_1.

3.1 Existence and uniqueness

Theorem 13. *Let u_0, $v_0 \in C(\Omega)$ with $(u_0, v_0) \neq (0,0)$ in Ω. Then there exists a unique solution (u,v) of (7) such that $(u,v) \in P_{t_0}$.*

Proof. It is obtained from Theorem 5.

Theorem 14. *Let u_0, $v_0 \in C(\Omega)$ with $(u_0, v_0) = (0,0)$ for all $x \in \Omega$. If $r \geq 1$, $s \geq 1$ and $pq \geq 1$, then the problem (7), has only the trivial solution. Otherwise, there exists a unique positive solution (u,v) of (7) in $(x,t) \in \Omega \times (0,T)$, which is the maximal.*

Remark 15. The special case $r = s = 0$ leads to study the problem (5), which is studied in [4] and a similar result to Theorem 13 is obtained.

Remark 16. By a previous remark, it follows from the proof of Theorem 14 that if $r \geq 1$, $s \geq 1$, and $pq \geq 1$, then, there is no nontrivial subsolution leaving $(0,0)$.

3.2 Blow-up analysis

In this section, we study conditions under which, the solution of (7) exists globally or blow-up. In what follows, we build on ideas developed in [17].

Theorem 17. *If $\max\{r, s, pq\} > 1$, the solutions of (7), blow-up in finite time.*

Proof. We shall proceed in two cases.

Case 1. Let $pq = \max\{r, s, pq\} > 1$. Let $(\underline{u}, \underline{v})$ the solution of (5), which blow-up in finite time $T > 0$ for $pq > 1$, see [4]. As $(\underline{u}, \underline{v})$ is nonnegative then is a subsolution of (7) in $\Omega \times [0, T)$. If $(0, 0) \neq (\underline{u}_0(x), \underline{v}_0(x)) \leq (u_0(x), v_0(x))$ in Ω, then by Lemma 7 we have that the solution (u, v) of (7) blow-up in finite time.

Case 2. Let $r = \max\{r, s, pq\} > 1$ or $s = \max\{r, s, pq\} > 1$, these two cases can be handled with the same argument, so that, we consider $r > 1$.

Let $\underline{u}$ be the nonnegative solution of (4) with $f(u) = u^r$. As $r > 1$, we have that $\underline{u}$ blow-up in finite time, see [11]. Let $\underline{v}$, the nonnegative solution of (4) with $f = 0$, see [7]. Moreover, we have that $(\underline{u}, \underline{v})$ is a subsolution of (7) in $\Omega \times [0, T)$. If $(0, 0) \neq (\underline{u}_0(x), \underline{v}_0(x)) \leq (u_0(x), v_0(x))$ in Ω, then by Lemma 7 we have that the solution (u, v) of (7) blow-up in finite time.

Theorem 18. *If* $\max\{r, s, pq\} \leq 1$, *then all solutions of (7) are global.*

Proof. Let $\max\{r, s, pq\} \leq 1$ and α, β, Δ, given in (9). Let $\alpha_1 = 2\alpha$, $\beta_1 = 2\beta$. We have $\alpha_1, \beta_1 > 0$, $\alpha_1(1 - r) - \beta_1 p = 2$ and $\beta_1(1 - s) - \alpha_1 q = 2$. Let $\Delta > 0$ and $C > 0$, we considerer the functions $\overline{u} = (t + C)^{\alpha_1}$, $\overline{v} = (t + C)^{\beta_1}$ in $\Omega \times [0, \infty)$. We have

$$\begin{aligned} \overline{u}_t - \int_\Omega J(x - y)(\overline{u}(y, t) - \overline{u}(x, t))dy - \overline{u}^r - \overline{v}^p \\ = \alpha_1(t + C)^{\alpha_1 - 1} - (t + C)^{\alpha_1 - \beta_1 r - 2} - (t + C)^{\alpha_1 - \alpha_1 q - 2} \geq 0, \end{aligned} \tag{15}$$

for C sufficiently large. Similarly, we have

$$\overline{v}_t - \int_\Omega J(x - y)(\overline{v}(y, t) - \overline{v}(x, t))dy - \overline{u}^q - \overline{v}^s \geq 0.$$

Therefore $(\overline{u}, \overline{v})$, is a supersolution of (7). Let $(u_0, v_0) \geq (0, 0)$ with $(u_0, v_0) \leq (\overline{u}(x, 0), \overline{v}(x, 0))$. By Lemma 7 and the uniqueness the result of the Theorems 13 and 14, we have the solution (u, v) of (7) is global.

If $\Delta = 0$, then $(1 - r)(1 - s) = pq$. Let $\overline{u} = Ce^{\mu t}$, $\overline{v} = Ce^{\nu t}$ in $\Omega \times [0, \infty)$ such that $\nu = (1 - r)p^{-1}\mu = q(1 - s)^{-1}\mu$. We have

$$\begin{aligned} \overline{u}_t - \int_\Omega J(x - y)(\overline{u}(y, t) - \overline{u}(x, t))dy - \overline{u}^r - \overline{v}^p = \mu\overline{u} - \overline{u}^r - \overline{v}^p \\ = e^{\mu t}(\mu C - C^r e^{-\nu p t} - C^p e^{-\mu r t}) \geq e^{\mu t}(\mu C - C^r - C^p) \geq 0, \end{aligned} \tag{16}$$

for μ sufficiently large. Similarly, we have

$$\overline{v}_t - \int_\Omega J(x - y)(\overline{v}(y, t) - \overline{v}(x, t))dy - \overline{u}^q - \overline{v}^s \geq e^{\nu t}(\nu C - C^q - C^s) \geq 0. \tag{17}$$

Therefore $(\overline{u}, \overline{v})$ is a supersolution of (7) and the same conclusion for the case $\Delta > 0$ holds.

Next, we will study the conditions under which non-simultaneous blow-up the solution of (7) could occur. Consider the system of ODE

$$
\begin{aligned}
&u'(t) = u^r(t) + v^p(t), \quad v'(t) = u^q(t) + v^s(t), \quad t > 0 \\
&u(0) = a, \ v(0) = b.
\end{aligned}
\tag{18}
$$

Remark 19. 1. If $p = q = r = s = 1$, then $u(t) = C_1 e^{2t}, \ v(t) = C_2 e^{2t}$ is solution of (18).

2. Let $pq > 1$ and (u, v) be nonnegative solutions of (18). We have $u'(t) \geq v^p(t), \ v'(t) \geq u^q(t)$, therefore (u, v) is a supersolution of $w'(t) = z^p(t), \ z'(t) = w^q(t), \ w(0) = c \leq a, \ z(0) = d \leq b$. As the solution (w, z), simultaneous blow-up in finite time $T > 0$ for $qr > 1$, then by Comparison Principle, (u, v) simultaneous blow-up in finite time $T > 0$.

Remark 20. $(u(t), v(t))$ is a flat solution (a solution that does not depend on x) of (7), with initial datum $u(x, 0) = a \geq 0, \quad v(x, 0) = b \geq 0$ if only if $(u(t), v(t))$ is a solution of the system of ODE (18).

The following proposition refers to the flat solution (7) and is analogous to Proposition 2.2(i) of [17], reason for which we omit its demonstration.

Proposition 21. *Let $r > q + 1$. If $c_1 u_0^{1+q-r} + v_0 < c_2 u_0^{(r-1)/(s-1)}$, with $c_1 = \frac{1}{r-1-q}$, $c_2 = \left(\frac{r-1}{s-1}\right)^{1/(s-1)}$, then, for the flat solution $(u(t), v(t))$ of (7), must be u blow-up in finite time $T > 0$, while v remains bounded on $[0, T)$, i.e. satisfies*

$$
\lim_{t \nearrow T} u(t) = \infty \quad \text{and} \quad \sup_{t \in (0,T)} v(t) < \infty. \tag{19}
$$

For $s > p + 1$, the analogue obviously holds by exchanging the roles of u, v.

Theorem 22. *(Non-simultaneous Blow-up) Let (u, v) be a positive blowing-up solution of (7). If $r > q + 1$ or $s > p + 1$, then there exist u_0 and v_0 such that non-simultaneous blow-up occurs.*

Proof. Assume that $r > q+1$, then by Proposition 21, there exists u_0 and v_0 such that $\lim_{t \nearrow T} \|u(t)\|_\infty = \infty$ and $\sup_{\Omega \times (0,T)} v(x, t) < \infty$ for some finite time

$T > 0$. Hence, non-simultaneous blow-up occurs. For $s > p+1$, the analogue of Proposition 21 holds by exchanging the roles of u and v.

3.3 Blow-up rates

In this section, we analyze the blow-up rate of the solutions of (7). We assume that $x = 0 \in \Omega$, and note that for smooth radially symmetric and nondecreasing initial conditions (that is, when $u_0(r),\ v_0(r)$ are C^1 such that $u_0'(r) \le 0,\ v_0'(r) \le 0$) the solutions are also radially symmetric and radially nondecreasing (that is, it holds that $u_r(r,t) \le 0,\ v_r(r,t) \le 0$). Hence, for every $t \in (0,T)$, the maximum of both components is attained at $x = 0$. We state this result as follows. For a proof, we refer to Lemma 4.1 in [11].

Lemma 23. *If $\Omega = B(0;R)$ is a ball and (u_0, v_0) are smooth, radially symmetric, and nondecreasing initial conditions (i.e. $u_0(r),\ v_0(r)$ are C^1 such that $u_0'(r) \le 0,\ v_0'(r) \le 0$) then both components $u,\ v$ of the solution of (7) are radially symmetric and radially nondecreasing (they verify $u_r(r,t) \le 0,\ v_r(r,t) \le 0$ for every $r \in [0,R)$ and every $t > 0$).*

Next, we study the blow-up rate of the solutions of (7).

Theorem 24. *Let (u,v) be a positive blowing-up solution of (7), such that the maximum is reached at $x = 0$ for all $t \in (0,T)$. Let $r < \frac{p(q+1)}{(p+1)}$ and $s < \frac{q(p+1)}{(q+1)}$, then there exists $C_1,\ C_2,\ C_3,\ C_4$ positive constants such that*

$$\begin{aligned} C_1(T-t)^{-\mu} \le u(0,t) \le C_2(T-t)^{-\mu}, \quad 0 < t < T, \\ C_3(T-t)^{-\nu} \le v(0,t) \le C_4(T-t)^{-\nu}, \quad 0 < t < T, \end{aligned} \tag{20}$$

with $\mu = \frac{p+1}{pq-1},\ \nu = \frac{q+1}{pq-1}$.

Proof. Let $u(0,t) = \max_{x\in\overline{\Omega}} u(x,t)$ and $v(0,t) = \max_{x\in\overline{\Omega}} v(x,t)$. By (7), we have

$$\begin{aligned} u_t(0,t) &= \int_\Omega J(0-y)(u(y,t)-u(0,t))dy + u^r(0,t) + v^p(0,t) \\ &\le u^r(0,t) + v^p(0,t), \\ v_t(0,t) &= \int_\Omega J(0-y)(v(y,t)-v(0,t))dy + u^q(0,t) + v^s(0,t) \\ &\le u^q(0,t) + v^s(0,t). \end{aligned} \tag{21}$$

As $1 = \displaystyle\int_{R^N} J(\zeta)d\zeta \geq \int_\Omega J(\zeta)d\zeta$ and (u,v) is a positive solution, we have

$$u_t(0,t) \geq -u(0,t) + v^p(0,t), \quad v_t(0,t) \geq -v(0,t) + u^q(0,t).$$

Therefore, we have that for all $0 < t < T$

$$-u(0,t) + v^p(0,t) \leq u_t(0,t) \leq u^r(0,t) + v^p(0,t) \tag{22}$$

and

$$-v(0,t) + u^q(0,t) \leq v_t(0,t) \leq u^q(0,t) + v^s(0,t). \tag{23}$$

Multiplying the second inequality of (22) by $u^q(0,t)$ and the first inequality of (23) by $v^p(0,t)$, we have

$$u_t(0,t)u^q(0,t) \leq u^{r+q}(0,t) + v_t(0,t)v^p(0,t) + v^{p+1}(0,t),$$

which is equivalent to

$$\left(\frac{u^{q+1}(0,t)}{q+1}\right)_t - u^{r+q}(0,t) \leq \left(\frac{v^{p+1}(0,t)}{p+1}\right)_t + v^{p+1}(0,t).$$

Multiplying the inequality by $(p+1)e^{(p+1)t}$ and integrating on $[0,t]$ with $t < T$, we have

$$u^q(0,t) \leq C(v(0,t))^{(p+1)q/(q+1)}. \tag{24}$$

Replacing the second inequality of (23) by the inequality (24) and as $s < \frac{q(p+1)}{(q+1)}$, we have

$$\begin{aligned} v_t(0,t) &\leq C(v(0,t))^{(p+1)q/(q+1)} + v^s(0,t) \\ &\leq C(v(0,t))^{(p+1)q/(q+1)} + (v(0,t))^{(p+1)q/(q+1)}. \end{aligned}$$

Therefore,

$$v_t(0,t) \leq (C+1)(v(0,t))^{(p+1)q/(q+1)}.$$

Integrating the inequality from above on $[t,T)$, we obtain that $v(0,t) \geq C_3(T-t)^{-\nu}$, where $\nu = \frac{q+1}{pq-1}$. In analogous way, we obtain $u(0,t) \geq C_1(T-t)^{-\mu}$, where $\mu = \dfrac{p+1}{pq-1}$.

Doing a similar analysis to the one developed above, we obtain that there exists a constant $C > 0$, such that, for $0 < t < T$

$$C(v(0,t))^{(p+1)q/(q+1)} \leq u^q(0,t). \tag{25}$$

Replacing the first inequality of (23), by the inequality (25) and as $pq > 1$ we have $(p+1)q/(q+1) > 1$ and

$$C(v(0,t))^{(p+1)q/(q+1)} \leq -v(0,t) + C(v(0,t))^{(p+1)q/(q+1)} \leq v_t(0,t).$$

Integrating the inequality from above on $[t,T)$, we obtain

$$v(0,t) \leq C_4(T-t)^{-\nu}.$$

In analogous way, we obtain

$$u(0,t) \leq C_2(T-t)^{-\mu}.$$

Theorem 25. *Let $r > q+1$ or $s > p+1$ and (u,v) be a positive solution of (7).*
(i) If u blows up, then there exists $C_1,\ C_2$ positive constants, such that

$$C_1(T-t)^{-1/(r-1)} \leq \|u(t)\|_\infty \leq C_2(T-t)^{-1/(r-1)}, \quad 0 < t < T.$$

(ii) If v blows up, then there exists $C_3,\ C_4$ positive constants such that

$$C_3(T-t)^{-1/(s-1)} \leq \|v(t)\|_\infty \leq C_4(T-t)^{-1/(s-1)}, \quad 0 < t < T.$$

Proof. (i) Let $r > q+1$, then by Theorem 22 we have that non-simultaneous blow-up occurs. Let u blow up in finite time $T > 0$ with $\lim_{t\to T^-} \|u(t)\|_\infty = \infty$ and v remains bounded, there exists a constant $C > 0$, such that $v(x,t) \leq C$ for all $(x,t) \in \Omega \times (0,\infty)$. Therefore u a nonnegative solution of the problem

$$\begin{aligned} u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))dy + u^r + b(x,t),& \\ (x,t) \in \Omega \times (0,T),& \\ u(x,0) = u_0, \quad x \in \Omega,& \end{aligned} \tag{26}$$

where $b(x,y) \leq C$, is a positive function. As $r > 1$ and using a similar argument given in [11], we have there exists C_2 a positive constant, such that $\|u(t)\|_\infty \leq C_2(T-t)^{-1/(r-1)}, \quad 0 < t < T$.

Let $U(t) = \sup_{x\in\Omega} u(x,t), \quad V(t) = \sup_{x\in\Omega} v(x,t)$, by Remark 4 we have for $0 < t < t_1 < t$

$$U(t_1) \leq U(t) + \int_t^{t_1} U^r(s)ds \ + \int_0^t V^p(s)ds \leq U(t) + \int_t^{t_1} U^r(s)ds \ + C.$$

As u blows up, there exists a first $t_1 \in [t, T)$, such that $U(t_1) = 2U(t)$. It follows that $U(t_1) = 2U(t) \leq U(t) + (t_1 - t)(2U(t))^r + C$. Therefore $U(t) \leq (T - t)(2U(t))^r + C$ for t close to T, we have $U(t) \geq 2C$. Then $U(t) \geq \frac{1}{2^r}(T - t)^{-1/(r-1)}$. Therefore, $C_1(T - t)^{-1/(r-1)} \leq \|u(t)\|_\infty$. The proof of (ii) proceeds in an analogous way.

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