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NEW OPERATORS VIA SEMI-DELTA-OPEN SETS

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Abstract: Semi-delta-open sets (briefly δ_s-open sets) are a new type of open sets introduced by the authors. The purpose of this paper is to investigate the topological concepts like closure operator, derived set and interior of a set in term of these sets and study their properties. Further, it is shown that the family of semi-delta-open sets forms a topology. In addition, characterizations of semi-delta-open (briefly δ_s-open), semi-delta-closed (briefly δ_s-closed) and semi -delta-continuous functions (briefly δ_s-functions) have been discussed.

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1. Introduction

Many researchers have carried out extensive research on open sets and their new versions in topology for many years. Levine [1] was the first to introduce the concept of semi-open sets. Moreover, the notions of semi-closed set and semi-continuity was also proposed by him. A subset L of the space (G, τ) is termed as semi-open if $L \subseteq Cl(Int(L))$. The complement of a semi-open set is termed as semi-closed set. For a subset L of the space (G, τ), the intersection of all semi-closed supersets of L is termed as semi-closure of L and is denoted

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by $sCl(L)$, also $sCl(L) = L \cup Int(Cl(L))$. Levine's work leads many topologists to use semi-open sets as a substitute to open sets.

Later on, Veličko [2] introduced the notions of δ-continuity and θ-continuity. Along with these terms, he proposed and studied the concepts of δ-closure and θ-closure. δ-closure (respectively θ-closure) of any subset L of a space (G, τ) is defined as the set of all g in G such that $Int(Cl(O)) \cap L \neq \emptyset$ (respectively $Cl(O) \cap L \neq \emptyset$) for each open set O in G containing g. δ-interior (respectively θ-interior) of any subset L of a space (G, τ) is defined as the set of all such $g \in G$ such that $Int(Cl(O)) \subseteq L$ (respectively $Cl(O) \subseteq L$) for some open set O in G containing g. A well-established result is that the collection of all δ-open sets forms a topology called semi-regular topology. In 2008, Renuka et al. [3] worked on γ-spaces using δ_γ- open sets.

Latif [4, 5] explored the various properties of δ-open sets and concepts of $\delta - D$-sets. Recently, Hassan et al. [6] made known the new version of open sets called θ_s-open sets and studied various terms, such as θ_s-continuous, θ_s-open and θ_s-closed functions.

This paper is organized as follows: Definitions and results that have previously been proposed are included in Section 2 and will be applied to support various findings in the next sections. Having introduced this definition of δ_s-open sets by authors [9], Section 3 deals with the properties of δ_s-closure operator, δ_s-derived set and δ_s-interior, and we show that the family of δ_s-open sets forms a topology. In Section 4 characterizations of δ_s-open, δ_s-closed, and δ_s-continuous functions have been discussed.

2. Preliminaries

Throughout this paper, (G, τ) and (H, τ^*) (briefly G and H) represent topological spaces. We denote the closure and the interior of any subset L of space G by $Cl(L)$ and $Int(L)$, respectively.

Definition 1. [1] Let G be a topological space. A subset L of G is termed as semi-open set if $L \subseteq Cl(Int(L))$ and semi-closed set if $Int(Cl(L)) \subseteq L$.

Definition 2. [7] Let L be a subset of topological space G. Then its semi-closure is the intersection of all semi-closed supersets of L, denoted by $sCl(L)$. Here $sCl(L)$ is the smallest semi-closed set containing L. Also, $sCl(L) = L \cup Int(Cl(L))$.

For the following lemma, one may refer to Navalagi and Gurushantanavar [8].

Lemma 3. *For subsets L and M of space G, the following holds for the semi-closure operator:*

(a) $L \subset sCl(L) \subset Cl(L)$.

(b) $sCl(L) \subset sCl(M)$, *if* $L \subset M$.

(c) $sCl(sCl(L)) = sCl(L)$.

(d) $sCl(L \cap M) \subset sCl(L) \cap sCl(M)$.

(e) $sCl(L) \cup sCl(M) \subset sCl(L \cup M)$.

(f) *L is semi-closed if and only if* $sCl(L) = L$.

Definition 4. [9] Let G be a topological space and $L \subseteq G$. Then L is said to be semi-delta-open if for every $g \in L$ there exists an open set O containing g such that $Int[sCl(O)] \subseteq L$.

Definition 5. [9] Let G be a topological space then its subset L is termed as semi-delta -neighbourhood (briefly δ_s-neighbour-hood) of g if there exist a δ_s-open set O in G such that $g \in O \subseteq L$.

Definition 6. [9] Semi-delta-closure (briefly δ_s-closure) of a subset L of a space G is the intersection of all semi-delta-closed supersets of L. We denote semi-delta closure of a set L by $Cl_{\delta_s}(L)$.

Theorem 7. *[9] Let (G, τ) be a topological space. A point $g \in G$ is said to be in δ_s-closure of L if $Int(sCl(O)) \cap L \neq \emptyset$ for any open set O containing g.*

Theorem 8. *[9] Every δ_s-open set is a δ_s-neighbourhood of each of its points.*

Theorem 9. *[9] Let G be a topological space. Then the following statements hold:*

(a) *Empty set and space G are δ_s-closed.*

(b) *Arbitrary intersections of δ_s-closed sets are δ_s-closed.*

(c) Finite union of δ_s-closed sets are δ_s-closed.

Definition 10. [9] Let G be a topological space. A point $g \in G$ is said to be semi-delta-limit point (briefly δ_s-limit point) of a subset L of space G if for every δ_s-open set O containing $g, O \cap (L - \{g\}) \neq \emptyset$. The set of all δ_s-limit points of L is called semi-delta-derived set (briefly δ_s-derived set) of L and is denoted by $D_{\delta_s}(L)$.

Theorem 11. *[9] For any subset L of space G, $Cl_{\delta_s}(L) = L \cup D_{\delta_s}(L)$.*

Definition 12. [9] A point g in G is termed as semi-delta- interior point (briefly δ_s-interior point) of $L \subseteq G$, if there exist a δ_s-open set O containing g such that $O \subseteq L$. The set of all δ_s-interior points of L is called δ_s-interior of L and is denoted by $Int_{\delta_s}(L)$.

Theorem 13. *[9] For subsets L and M of topological space G, the following results hold true.*

(a) $[G - Int_{\delta_s}(L)] = Cl_{\delta_s}(G - L)$.

(b) *L is δ_s-open if and only if $L = Int_{\delta_s}(L)$.*

(c) $Int_{\delta_s}[Int_{\delta_s}(L)] = Int_{\delta_s}(L)$.

(d) $Int_{\delta_s}(L) = [L - D_{\delta_s}(G - L)]$.

(e) $Int_{\delta_s}(L) \cup Int_{\delta_s}(M) \subseteq Int_{\delta_s}(L \cup M)$.

3. Properties of δ_s-Closure, δ_s-Derived set and δ_s-Interior

The properties of semi-delta-closure, semi-delta-derived set, and semi-delta-interiors of a set are the focus of this section. We also show that the family of semi-delta-open sets forms a topology on any non empty set G.

Theorem 14. *Let G be a topological space and $L \subseteq G$. Then*

(a) $Cl(L) \subseteq Cl_{\delta_s}(L)$.

(b) If L is δ_s-closed, then L is closed.

Proof. (a) To prove $Cl(L) \subseteq Cl_{\delta_s}(L)$, let $g \in Cl(L)$ and let O be an open set containing g. Since $g \in Cl(L)$, $O \cap L \neq \emptyset$. Also, $O \subseteq Int(sCl(O))$, we have $Int(sCl(O)) \cap L \neq \emptyset$. Thus, $g \in Cl_{\delta_s}(L)$.
(b) Let L be a δ_s-closed set. Then $Cl_{\delta_s}(L) = L$. Thus by part (a), $Cl(L) = L$. Hence, L is closed. □

Theorem 15. *Let G be a topological space. Then*

(a) If $L \subseteq M \subseteq G$, then $Cl_{\delta_s}(L) \subseteq Cl_{\delta_s}(M)$.

(b) For each subset $L, M \subseteq G$, $Cl_{\delta_s}(L) \cup Cl_{\delta_s}(M) = Cl_{\delta_s}(L \cup M)$.

(c) For each subset $L \subseteq G$, $Cl_{\delta_s}(L)$ is a closed in (G, τ).

(d) For each $L \in \tau_{\delta_s}$, $Cl_{\delta_s}(L) = Cl(L)$.

(e) $g \in Cl_{\delta_s}(L)$ if and only if for each δ_s-open subset O containing g, $O \cap L \neq \emptyset$.

Proof. (a) For each open set O containing g, δ_s-closure of a subset L of G is defined as $Cl_{\delta_s}(L) = \{g \in G : L \cap Int(sCl(O)) \neq \emptyset\} \subseteq \{g \in G : M \cap Int(sCl(O))\} = Cl_{\delta_s}(M)$, as $L \subseteq M$.

(b) By part (a), we have $Cl_{\delta_s}(L) \cup Cl_{\delta_s}(M) \subseteq Cl_{\delta_s}(L \cup M)$. Let $g \notin Cl_{\delta_s}(L) \cup Cl_{\delta_s}(M)$. Then there are two open sets $O, V \in \tau$ such that $g \in O \cap V, Int(sCl(O)) \cap L = \emptyset$ and $Int(sCl(V)) \cap M = \emptyset$. Thus, we have $g \in O \cap V \in \tau$ and

$$\begin{aligned} &Int(sCl(O \cap V)) \cap (L \cup M) \\ &= (Int(sCl(O \cap V)) \cap L) \cup (Int(sCl(O \cap V)) \cap M) \\ &\subseteq (Int(sCl(O)) \cap L) \cup (Int(sCl(V)) \cap M) \\ &= \emptyset \cup \emptyset \\ &= \emptyset. \end{aligned}$$

It follows that $g \notin Cl_{\delta_s}(L \cup M)$.

(c) To show that $G - Cl_{\delta_s}(L) \in \tau$, let $g \in G - Cl_{\delta_s}(L)$, then there is $O \in \tau$ containing g and $Int(sCl(O)) \cap L = \emptyset$. Thus, $O \cap Cl_{\delta_s}(L) = \emptyset$. It follows that $G - Cl_{\delta_s}(L) \in \tau$.

(d) By Theorem 14 (a), for any subset L of G, $Cl(L) \subseteq Cl_{\delta_s}(L)$. Conversely, suppose to the contrary that there is $g \in Cl_{\delta_s}(L) \cap (G - Cl(L))$. Since $G - Cl(L) \in \tau$, we must have $Int(sCl(G - Cl(L))) \cap L \neq \emptyset$. Choose $g \in Int(sCl(G - Cl(L))) \cap L$. Since $L \in \tau_{\delta_s}$, then $(G - Cl(L)) \cap L \neq \emptyset$, a contradiction.

(e) Assuming that for every δ_s-open set O containing g, $O \cap L \neq \emptyset$ and $Int[sCl(O)] \cap L \neq \emptyset$ since $O \subseteq Int(sCl(O))$. This implies that $g \in Cl_{\delta_s}(L)$. Conversely, suppose to the contrary that a point g has δ_s-neighbourhood which does not intersect L. Then the complement of this δ_s-neighbourhood is δ_s-closed superset of L, which does not contain g. Since δ_s-closure of any subset L of space G is the intersection of all δ_s-closed supersets of L, this implies L does not contain g. This completes the proof. □

Theorem 16. *Let G be a topological space. Then family of all semi-delta-open sets (briefly τ_{δ_s}) forms a topology on G.*

Proof. By using Theorem 9(a), we can easily see that $\emptyset$, $G \in \tau_{\delta_s}$. To show that arbitrary union of δ_s-open sets is δ_s-open, let $\{L_\alpha : \alpha \in \Delta\}$ be a family of δ_s-open sets. This implies that $\{G - L_\alpha : \alpha \in \Delta\}$ is a family of δ_s-closed sets. Now, by Theorem 9(b), $G - \cup\{L_\alpha : \alpha \in \Delta\} = \cap\{G - L_\alpha : \alpha \in \Delta\}$ is δ_s-closed set. Hence, $\cup\{L_\alpha : \alpha \in \Delta\} \in \tau_{\delta_s}$.

To show that finite intersection of δ_s-open sets is δ_s-open, let $L, M \in \tau_{\delta_s}$, this implies that $G-L$ and $G-M$ are δ_s-closed. By Theorem 9(c), $G-(L\cap M) = (G - L) \cup (G - M)$ are δ_s-closed sets. Thus, $L \cap M \in \tau_{\delta_s}$. □

Theorem 17. *Let G be a topological space and $L \subseteq G$. Then $L \in \tau_{\delta_s}$ if and only if for each $g \in L$, there is $O \in \tau$ such that $g \in O \subseteq Int(sCl(O)) \subseteq L$.*

Proof. Let $L \in \tau_{\delta_s}$ and $g \in L$. Then $G-L$ is δ_s-closed. Also, $g \notin Cl_{\delta_s}(G-L)$ since $g \notin G-L$ and hence there is $O \in \tau$ and $Int(sCl(O)) \cap (G-L) = \emptyset$, which means that $g \in O \subseteq Int(sCl(O)) \subseteq L$. Conversely, suppose that for each $g \in L$, there is $O \in \tau$ such that $g \in O \subseteq Int(sCl(O)) \subseteq L$ and suppose on the contrary that $L \notin \tau_{\delta_s}$, then $G - L$ is not δ_s- closed and $Cl_{\delta_s}(G - L) \neq G - L$. Choose $g \in Cl_{\delta_s}(G - L) - (G - L)$. Since $g \in L$, there is $O \in \tau$ such that $g \in O \subseteq Int(sCl(O)) \subseteq L$. Thus, $g \in O \in \tau$ and $Int(sCl(O)) \cap (G - L) = \emptyset$. Hence, $g \notin Cl_{\delta_s}(G - L)$, a contradiction. □

Remark 18. Since the collection of all semi-delta-open sets of (G, τ) form a topology with respect to τ, this topological space will be denoted by $(G, \tau_{\delta_s}, \tau)$.

Remark 19. Since every semi-delta open set is open, so $\tau_{\delta_s} \subseteq \tau$. Converse may not be true in general. This can be seen by the following example.

Example 20. Let $(\mathbb{R}, \tau)$ be a topological space with $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$, then $\mathbb{Q}^c \in \tau$ but $\mathbb{Q}^c \notin \tau_{\delta_s}$.

The following Theorem 21 for semi-delta-open sets is an analogue of the lemma given by Mršević and Andrijević [10] for delta- open sets.

Theorem 21. *Let (G, τ) be a topological space and $L \subseteq G$, then the set $Cl_\theta(L)$ is closed in (G, τ_{δ_s}) and thus in (G, τ).*

Proof. Let $g \in Cl_{\delta_s}(Cl_\theta(L))$. Then for every open set O containing g, we have $g \in Int(sCl(O)) \cap Cl_\theta(L)$. Since $g \in sCl(O) \subseteq Cl(O)$, which means that $Cl(O)$ is a closed neighbourhood of g. Hence, $Cl(O) \cap L \neq \emptyset$. Thus, $g \in Cl_\theta(L)$. □

Theorem 22. *For subsets L and M of space G, the following statements hold:*

(a) $D(L) \subseteq D_{\delta_s}(L)$, *where $D(L)$ is derived set of L.*

(b) *If $L \subseteq M$, then $D_{\delta_s}(L) \subseteq D_{\delta_s}(M)$.*

(c) $D_{\delta_s}(L) \cup D_{\delta_s}(M) = D_{\delta_s}(L \cup M)$ *and* $D_{\delta_s}(L \cap M) \subseteq D_{\delta_s}(L) \cap D_{\delta_s}(M)$.

(d) $[D_{\delta_s}(D_{\delta_s}(L)) - L] \subseteq D_{\delta_s}(L)$.

(e) $D_{\delta_s}[D_{\delta_s}(L) \cup L] \subseteq D_{\delta_s}(L) \cup L$.

Proof. (a) The derived set of a subset L of space G is defined as $D(L) = \{\ g \in G : O \cap (L - \{g\}) \neq \emptyset$, for each open set O in G containing $g\}$. Since $O \subseteq sCl(O)$ for any open set O in G, this implies that $O \subseteq Int[sCl(O)]$. Thus, $D(L) \subseteq \{g \in G : Int[sCl(O)] \cap (L - \{g\}) \neq \emptyset$, for each open set O in G containing $g\} = D_{\delta_s}(L)$.

In view of Remark 19, proofs for the parts (b), (c), (d), (e) respectively, are obvious. □

Remark 23. If $D_{\delta_s}(L) = D_{\delta_s}(M)$, then it does not imply that $L = M$. This can be seen by the following example.

Example 24. Let $G = \{a, b, c\}$ with topology $\tau = \{\emptyset, G, \{a\}, \{a, b\}, \{b\}, \{b, c\}\}$. Here $\tau_{\delta_s} = \tau$. Take $L = \{a\}$ and $M = \{c\}$. Then $D_{\delta_s}(L) = D_{\delta_s}(M) = \emptyset$ but $L \neq M$.

Corollary 25. *A subset L of space G is termed as semi-delta- closed (briefly δ_s-closed) if and only if it contains all of its δ_s-limit points.*

Proof. In view of Theorem 11, the proof is obvious. □

Theorem 26. *For subsets L and M of space G, the following statements hold:*

(a) *If $L \subseteq M$, then $Int_{\delta_s}(L) \subseteq Int_{\delta_s}(M)$.*

(b) *$Int_{\delta_s}(L)$ is the largest δ_s-open set contained in L.*

(c) *$Int_{\delta_s}(L \cap M) = Int_{\delta_s}(L) \cap Int_{\delta_s}(M)$.*

(d) *L is δ_s- open if and only if for each $g \in L$, there exist a basic open set M with $g \in M$ such that $Int(sCl(M)) \subseteq L$.*

(e) *$[G - Cl_{\delta_s}(L)] = Int_{\delta_s}(G - L)$.*

(f) *For any $L \subseteq G$, $Int_{\delta_s}(L) \subseteq Int_{\delta}(L)$.*

Proof. In view of Remark 19 the proofs are obvious. □

4. Semi-Delta-Open and Semi-Delta-Continuous Functions

In this section, the characterizations of δ_s-open, δ_s-closed and δ_s-continuous functions have been discussed.

Definition 27. [9] Let (G, τ) and (H, τ^*) be topological spaces. A function $f_1 : (G, \tau) \to (H, \tau^*)$ is termed as δ_s-open if $f_1(L)$ is δ_s-open in (H, τ^*) i.e $f_1(L)$ is open in $(H, \sigma_{\delta_s}, \tau^*)$ for every open set L in (G, τ).

Definition 28. [9] Let (G, τ) and (H, τ^*) be topological spaces. A function $f_1 : (G, \tau) \to (H, \tau^*)$ is termed as δ_s-closed if $f_1(L)$ is δ_s-closed in (H, τ^*) i.e $f_1(L)$ is closed in $(H, \sigma_{\delta_s}, \tau^*)$ for every closed set L in (G, τ).

Definition 29. [9] A function $f_1 : (G, \tau) \to (H, \tau^*)$ is said to be δ_s-continuous if for every open set L in H, $f_1^{-1}(L)$ is δ_s-open in G i.e open in $(G, \tau_{\delta_s}, \tau)$.

Remark 30. Let Cl_{δ_s} be the closure operator in topological space $(G, \tau_{\delta_s}, \tau)$, then the following statements hold true:

(a) A subset L of space G is δ_s-open relative to (G, τ) if and only if it is open in $(G, \tau_{\delta_s}, \tau)$.

(b) A subset L of space G is δ_s-closed relative to (G, τ) if and only if it is closed in $(G, \tau_{\delta_s}, \tau)$.

Theorem 31. *Let (G, τ) and (H, τ^*) be topological spaces and let a function $f_1 : (G, \tau) \to (H, \tau^*)$. Then the following statements are equivalent.*

(a) f_1 is δ_s-open.

(b) $f_1(Int(L)) \subseteq Int_{\delta_s}(f_1(L))$, for every $L \subseteq G$.

(c) $f_1(L)$ is δ_s-open for every basic open set L in G.

(d) For each $g \in G$ and for every open set L in G containing g, there exists an open set O in H containing $f_1(g)$ such that $Int(sCl(O)) \subseteq f_1(L)$.

Proof. $(a) \implies (b)$

Let $L \subseteq G$. Note that $f_1(Int(L)) \subseteq f_1(L)$ and $f_1(Int(L))$ is δ_s-open. In view of Theorem 26 (b), we have $f_1(Int(L)) \subseteq Int_{\delta_s}[f_1(Int(L))] \subseteq Int_{\delta_s}(f_1(L))$.

$(b) \implies (a)$

Let L be an open set in G. Then by hypothesis, $f_1[Int(L)] \subseteq Int_{\delta_s}(f_1(L))$. Since L is open, therefore $Int(L) = L$. Also, $Int_{\delta_s}[f_1(L)] \subseteq f_1(L)$. This implies that $f_1(L) = Int_{\delta_s}(f_1(L))$. Hence, $f_1(L)$ is δ_s-open in H. Thus, f_1 is δ_s-open.

$(b) \implies (c)$

Let L be a basic open set in G. Then using hypothesis, $f_1(L) = f_1(Int(L)) \subseteq Int_{\delta_s}(f_1(L)) \subseteq f_1(L)$. Now, by using Theorem 13 (b), we have $f_1(L)$ is δ_s-open.

$(c) \implies (d)$

Let $g \in G$ and let L be an open set in G containing g. Then there exists a basic open set M containing g such that $M \subseteq L$, which implies that $f_1(g) \in f_1(M) \subseteq f_1(L)$. By assumption, there exists an open set O in H containing $f_1(g)$ such that $Int(sCl(O)) \subseteq f_1(M) \subseteq f_1(L)$.

$(d) \implies (a)$

Let L be an open set in G and let $g \in f_1(L)$. Then there exists $y \in L$ such that $f_1(y) = g$. By assumption, there exists an open set O in H containing

g such that $Int(sCl(O)) \subseteq f_1(L)$. Hence, $f_1(L)$ is δ_s-open in H. Thus, f_1 is δ_s-open. □

Theorem 32. *Prove that a function $f_1 : (G, \tau) \to (H, \tau^*)$ is δ_s-open if and only if for each $g \in G$, and $L \in \tau$ such that $g \in L$, there exists a δ_s-open set $O \subseteq H$ containing $f_1(g)$ such that $O \subseteq f_1(L)$.*

Proof. Follows immediately from Definition 27. □

Theorem 33. *Let $f_1 : (G, \tau) \to (H, \tau^*)$ be δ_s-open function. If $W \subseteq H$ and $F \subseteq G$ is a closed set containing $f_1^{-1}(W)$, then there exists a δ_s-closed set $K \subseteq H$ containing W such that $f_1^{-1}(K) \subseteq F$.*

Proof. Let $K = H - f_1(G - F)$. Since $f_1^{-1}(W) \subseteq F$, we have $f_1(G - F) \subseteq (H - W)$. Since f_1 is δ_s-open, then K is δ_s- closed and $f_1^{-1}(K) = G - f_1^{-1}[f_1(G - F)] \subseteq G - (G - F) = F$. □

Theorem 34. *Let $f_1 : (G, \tau) \to (H, \tau^*)$ be a δ_s-open function and let $O \subseteq H$. Then $f_1^{-1}[Cl_{\delta_s}(Int_{\delta_s}(Cl_{\delta_s}(O)))] \subseteq Cl[f_1^{-1}(O)]$.*

Proof. $Cl[f_1^{-1}(O)]$ is closed in G containing $f_1^{-1}(O)$. Using Theorem 33, there exists a δ_s-closed set $O \subseteq K \subseteq H$ such that $f_1^{-1}(K) \subseteq Cl[f_1^{-1}(O)]$. Thus, $f_1^{-1}[Cl_{\delta_s}(Int_{\delta_s}(Cl_{\delta_s}(O)))] \subseteq f_1^{-1}[Cl_{\delta_s}(Int_{\delta_s}(Cl_{\delta_s}(K)))] \subseteq f_1^{-1}(K) \subseteq Cl[f_1^{-1}(O)]$. □

Theorem 35. *Prove that a function $f_1 : (G, \tau) \to (H, \tau^*)$ is δ_s-open if and only if $Int[f_1^{-1}(M)] \subseteq f_1^{-1}[Int_{\delta_s}(M)]$ for all $M \subseteq H$.*

Proof. Necessity. Let $M \subseteq H$. Since $Int[f_1^{-1}(M)]$ is open in G and f_1 is δ_s-open, then $f_1[Int(f_1^{-1}(M))]$ is δ_s-open in H. Also, we have $f_1[Int(f_1^{-1}(M))] \subseteq f_1[f_1^{-1}(M)] \subseteq M$. Therefore, $f_1[Int(f_1^{-1}(M))] \subseteq Int_{\delta_s}(M)$. Hence, $Int[f_1^{-1}(M)] \subseteq f_1^{-1}[Int_{\delta_s}(M)]$.

Sufficiency. Let $L \subseteq G$. Then $f_1(L) \subseteq H$. By hypothesis, we obtain $Int(L) \subseteq Int[f_1^{-1}(f_1(L))] \subseteq f_1^{-1}[Int_{\delta_s}(f_1(L))]$. Hence, for all $L \subseteq G$, $f_1[Int(L)] \subseteq Int_{\delta_s}[f_1(L)]$. Thus, by Theorem 31 (b), f_1 is δ_s-open. □

Theorem 36. *Let $f_1 : (G, \tau) \to (H, \tau^*)$ be a function. Then a necessary and sufficient condition for f_1 to be δ_s-open is that $f_1^{-1}[Cl_{\delta_s}(M)] \subseteq Cl[f_1^{-1}(M)]$*

for every subset M of H.

Proof. Necessity. Assume that f_1 is δ_s-open. Let $M \subseteq H$ and let $g \in f_1^{-1}[Cl_{\delta_s}(M)]$. Then $f_1(g) \in Cl_{\delta_s}(M)$. Let $O \in \tau$ such that $g \in O$. Since f_1 is δ_s-open, then $f_1(O)$ is δ_s-open set in H. Therefore, $M \cap f_1(O) \neq \emptyset$. Then $O \cap f_1^{-1}(M) \neq \emptyset$. Hence, $g \in Cl[f_1^{-1}(M)]$. Thus, $f_1^{-1}[Cl_{\delta_s}(M)] \subseteq Cl[f_1^{-1}(M)]$.

Sufficiency. Let $M \subseteq H$. Then $(H - M) \subseteq H$. By hypothesis, we have $f_1^{-1}[Cl_{\delta_s}(H-M)] \subseteq Cl[f_1^{-1}(H-M)]$. This implies that $G - Cl[f_1^{-1}(H-M)] \subseteq G - f_1^{-1}[Cl_{\delta_s}(H-M)]$. Hence, $G - Cl[f_1^{-1}(H-M)] \subseteq f_1^{-1}[H - Cl_{\delta_s}(H-M)]$. Now, by using Theorem 10 of [11], we have $Int[f_1^{-1}(M)] \subseteq f_1^{-1}[Int_{\delta_s}(M)]$. By Theorem 35, f_1 is δ_s-open. □

Theorem 37. *Let $(G,\tau), (H,\tau^*)$ and (Z,λ) be topological spaces. If $f_1 : (G,\tau) \to (H,\tau^*)$ is δ_s-continuous function and $g : (H,\tau^*) \to (Z,\lambda)$ is continuous function then $gof_1 : (G,\tau) \to (Z,\lambda)$ is δ_s-continuous map.*

Proof. Let $V \subseteq Z$ be a closed set. Then $Z - V$ is open. Then $(gof_1)^{-1}(Z - V) = f_1^{-1}(g^{-1}(Z-V)) = f_1^{-1}(g^{-1}(Z)-g^{-1}(V))$. Since g is continuous map, this implies that $g^{-1}(V) = M$ is closed in H. Since f_1 is δ_s-continuous, therefore $f_1^{-1}(H - M) = f_1^{-1}(H) - f_1^{-1}(M) = G - f_1^{-1}(M)$ is δ_s-open in G. Hence gof_1 is δ_s-continuous. □

Theorem 38. *Let $f_1 : (G,\tau) \to (H,\tau^*)$ be a bijection. Then the following are equivalent:*

(a) *f_1 is δ_s-closed.*

(b) *f_1 is δ_s-open.*

(c) *f_1^{-1} is δ_s-continuous.*

Proof. $(a) \implies (b)$

Let $O \in \tau$. Then $G - O$ is closed in G. By assumption, $f_1(G - O)$ is δ_s-closed in H. But $f_1(G - O) = f_1(G) - f_1 O) = H - f_1(O)$. Hence, $f_1(O)$ is δ_s-open in H. Thus, f_1 is δ_s-open.

$(b) \implies (c)$

Let $O \subseteq G$ be an open set. Since f_1 is δ_s-open. So $f_1(O) = (f_1^{-1})^{-1}(O)$ is δ_s- open in H. Hence, f_1^{-1} is δ_s-continuous.

$(c) \implies (a)$

Let L be an arbitrary closed set in G. Then $G - L$ is open in G. Since f_1^{-1} is δ_s-continuous, $(f_1^{-1})^{-1}(G - L)$ is δ_s-open in H. But $(f_1^{-1})^{-1}(G - L) = f_1(G-L) = H - f_1(L)$. Hence, $f_1(L)$ is δ_s-closed in H. Thus, f_1 is δ_s-closed. □

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