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ON THE ILL-POSEDNESS OF THE REGULARIZED rBO-ZK EQUATION IN SOBOLEV SPACES $H^s(\mathbf{R}^2)$

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Abstract: In this paper, we study the ill-posedness of the regularized rBO-ZK equation, defined by

$$u_t + a(u^n)_x + (bHu_t + u_{yy})_x - \mu(-\Delta)^\alpha u = 0,$$

where H is the Hilbert transform with respect to x. The values a, b, μ and α are real numbers, with $b > 0$, $\mu > 0$ and $\alpha > \frac{1}{2}$ and $(-\Delta)^\alpha = (-\partial_x^2 - \partial_y^2)^\alpha$.

We show that the associated Cauchy problem is ill posed in Sobolev space $H^s(\mathbf{R}^2)$ for $s < 0$ and $n = 2$. The lack of local well-posedness is in the sense that the dependence of solutions upon initial data fails to be continuous.

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Key Words: regularized Benjamin-Ono-Zakharov-Kuznetsov equation; Cauchy problem; ill-posedness

1. Introduction

In the paper, we are interested in the proof the ill-posedness in Sobolev spaces to the Cauchy problem

$$\begin{cases} u_t + a(u^2)_x + (bHu_t + u_{yy})_x - \mu(-\Delta)^\alpha u = 0, \\ u(0, x, y) = \varphi(x, y), \end{cases} \tag{1}$$

where H is the Hilbert transform with respect to x, defined by

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$$H(f)(x,y) = p.\ v.\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi,y)}{x-\xi}\,d\xi,$$

$f \in \mathcal{S}$, a, b, μ and α are real numbers, with $b > 0$, $\mu > 0$, $\alpha > \frac{1}{2}$ and $(-\Delta)^{\alpha} = (-\partial_x^2 - \partial_y^2)^{\alpha}$.

We say that the initial value problem (IVP) associated to 1 is locally well-posed in a Banach space X if the solution uniquely exists in certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T,T]$ whenever the initial data belongs to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e., we have the continuity of the application $u_0 \to u(t)$ from X to $C([0,T];X)$. If some property in the definition of locally well-posed fails, we say that the IVP is ill-posed. The non-linear evolution equations play a very important role in different areas of science and engineering. It is worth mentioning some of them: fluid mechanics, plasma physics, fiber optics, solid state physics, chemical kinetics, among others. For example, Korteweg-de Vries equation

$$u_t = u_{xxx} + uu_x \qquad (x,t) \in \mathbf{R}^2, \tag{2}$$

that models the behavior of waves in shallow water channels. Other one-dimensional equations are, one, the introduced by Benjamin [2] and by Ono [10],

$$u_t + Hu_{xx} + uu_x = 0, \tag{3}$$

which models the internal waves in stratified fluids deep. The another, is the regularized Benjamin-Ono equation

$$u_t + u_x + uu_x + Hu_{xt} = 0, \tag{4}$$

where $u = u(x,t)$ is a real valued function, with $x,t \in \mathbf{R}$. This equation is a model for the evolution in time of waves with large crests at the interface between two fluids inmiscibles. There are bidimensional versions extending the above equations. In the case of the KdV equation we have the Kadomtsev-Petviashvilli equation, see Kadomtsev [6],

$$(u_t + auu_x + u_{xxx})_x + u_{yy} = 0, \tag{5}$$

which describes waves in thin films of high surface tension. Another is the Zakharov- Kuznetsov equation

$$u_t = (u_{xx} + u_{yy})_x + uu_x, \tag{6}$$

which arises in the study of geophysical fluid dynamics in isotropic sets.

For the case of the equations BO (3) and rBO (4) we have the following bidimensional's versions, the ZK-BO equation

$$u_t = (Hu_x + u_{yy})_x + uu_x, \tag{7}$$

and the rBO-ZK equation

$$u_t + a(u^n)_x + (bHu_t + u_{yy})_x = 0, \quad (x, y) \in \mathbf{R}^2, \quad t > 0. \tag{8}$$

Bona [7], Mammeri [8] and Angulo [1] they studied the well-posedness of the rBO equation (4). Angulo [1] it was also examined the existence and stability of travelling periodic waves of rBO equation. Fonseca [5] was studied the well and ill-posedness of rBO in weighted Sobolev spaces, in particular, there it is obtained a result in unique continuation property of this equation. For equation (8) has been shown the local well-posedness in Sobolev space and the local well-posedness in weighted Sobolev spaces.

For the equation (8) we have the following result.

Theorem 1. *The initial value problem associated to* (8) *is locally well-posed in* $H^s(\mathbf{R}^2)$ *for* $s > 1$.

Proof. We consider the integral equation associated to (8)

$$u = E(t)\varphi + \int_0^t E(t-\tau)B(u^n(\tau))\, d\tau, \tag{9}$$

where

$$E(t) = e^{tA}, \qquad A = -\partial_x(1 + \mathcal{H}\partial_x)^{-1}\partial_y^2, \quad \text{and} \quad B = -\partial_x(1 + \mathcal{H}\partial_x)^{-1}. \tag{10}$$

Since B is a bounded operator in $H^s(\mathbf{R}^2)$, and it is a Banach algebra for $s > 1$, using the Banach fixed point theorem we conclude the result. □

The following theorem is the main result of this paper.

Theorem 2. *Let* $s < 0$. *Then there does not exist* $T > 0$ *such that* (1) *has a unique solution in the interval* $[0, T]$, *and the flow map* $u_0 \to u$ *is of class* C^2 *in zero from* $H^s(\mathbf{R}^2)$ *to* $H^s(\mathbf{R}^2)$.

2. Estimates

In what follows we will show some previous lemmas

Lemma 3. *Let* $p(\xi,\eta)=\frac{\xi\eta^2}{1+|\xi|}$.

$$\int_0^t E(t-\tau)(1+H\partial_x)^{-1}\partial_x[(E(t)\varphi)^2]d\tau = \int_{\mathbf{R}^4} e^{ix\xi+iy\eta+itp(\xi,\eta)}K(\xi,\xi_1,\eta,\eta_1)\,d\xi_1 d\eta_1 d\xi d\eta,$$

where

$$K(\xi,\xi_1,\eta,\eta_1)=\frac{\xi}{1+|\xi|}\widehat{\varphi}(\xi_1,\eta_1)\widehat{\varphi}(\xi-\xi_1,\eta-\eta_1)\left(\frac{e^{it\theta(\xi,\xi_1,\eta,\eta_1)}-1}{i\theta(\xi,\xi_1,\eta,\eta_1)}\right),$$

and

$$\theta(\xi,\xi_1,\eta,\eta_1)=p(\xi_1,\eta_1)+p(\xi-\xi_1,\eta-\eta_1)-p(\xi,\eta). \tag{11}$$

Proof. The proof of this lemma is a common procedure that appears in the literature related to dispersive equations (see for example Lemma 4 in Molinet [9] and Bona [3]).

From the definition of convolution between functions and Fubini's theorem, we have that

$$\begin{aligned}
&\int_0^t E(t-\tau)(1+H\partial_x)^{-1}\partial_x[(E(t)\varphi)^2]d\tau\\
&=\int_0^t\left[\int_{\mathbf{R}^2} e^{ix\xi+iy\eta}((1+H\partial_x)^{-1}\partial_x E(t-\tau)[(E(t)\varphi)^2])^\wedge(\xi,\eta)\,d\xi d\eta\right]d\tau\\
&=\int_0^t\left[\int_{\mathbf{R}^2} e^{ix\xi+iy\eta}e^{i(t-\tau)p(\xi,\eta)}\frac{\xi}{1+|\xi|}[e^{i\tau p(\cdot,\cdot)}\widehat{\varphi}*e^{i\tau p(\cdot,\cdot)}\widehat{\varphi}](\xi,\eta)\,d\xi d\eta\right]d\tau\\
&=\int_{\mathbf{R}^4} e^{ix\xi+iy\eta+itp(\xi,\eta)}\frac{\xi}{1+|\xi|}\Psi(\xi,\xi_1,\eta,\eta_1)\left(\int_0^t e^{i\tau\theta(\xi,\xi_1,\eta,\eta_1)}d\tau\right)d\xi_1 d\eta_1 d\xi d\eta\\
&=\int_{\mathbf{R}^4} e^{ix\xi+iy\eta+itp(\xi,\eta)}K(\xi,\xi_1,\eta,\eta_1)\,d\xi_1 d\eta_1 d\xi d\eta,
\end{aligned}$$

where

$$\Psi(\xi,\xi_1,\eta,\eta_1)=\widehat{\varphi}(\xi_1,\eta_1)\widehat{\varphi}(\xi-\xi_1,\eta-\eta_1).$$

□

The function $\theta(\xi, \xi_1, \eta, \eta_1)$ in (11) is called *the resonant.*

The next two lemmas may be proved by a simple direct calculus.

Lemma 4. *Let $s < 0$, $\alpha > 0$, $N > 0$ and*

$$\widehat{\varphi}(\xi, \eta) = \alpha^{-1} N^{-s} (\chi_{I_1}(\xi, \eta) + \chi_{I_2}(\xi, \eta)), \tag{12}$$

where

$$I_1 = [-N - \alpha, -N] \times [\alpha, 2\alpha], \qquad I_2 = [N, N + \alpha] \times [\alpha, 2\alpha],$$

and χ_I is a characteristic function on the set I. Then, $\|\varphi\|_{H^s(\mathbf{R}^2)} \leq 2$.

Lemma 5. *If $(\xi_1, \eta_1) \in I_1$ y $(\xi - \xi_1, \eta - \eta_1) \in I_2$ we have that*

$$|\theta(\xi, \xi_1, \eta, \eta_1)| \leq C\alpha^2.$$

Next, we prove an important theorem.

Theorem 6. *Consider the Cauchy problem* (1) *and $s < 0$ y $T > 0$. Then there does not exist a subspace X_T continuously included in $C([0, T]; H^s(\mathbf{R}^2))$ such that, for some constant $C > 0$, we have that*

$$\|E(t)\varphi\|_{X_T} \leq C\|\varphi\|_{H^s(\mathbf{R}^2)}, \tag{13}$$

for all $\varphi \in H^s(\mathbf{R}^2)$, and

$$\left\| \int_0^t E(t - \tau)[(1 + H\partial_x)^{-1}\partial_x(u^2(\tau))]\, d\tau \right\|_{X_T} \leq C\|u\|_{X_T}^2, \tag{14}$$

for $u \in X_T$.

Proof. By contradiction, we suppose that there exist a subspace X_T continuously included in $C([0, T]; H^s(\mathbf{R}^2))$ such that (13) and (14) are satisfied. If, we replace $u(t) = E(t)\varphi$ in (14), then X_T is in $C([0, T]; H^s(\mathbf{R}^2))$ and by (13), follows that

$$\left\| \int_0^t E(t - \tau)[(1 + H\partial_x)^{-1}\partial_x(E(\tau)\varphi)^2]\, d\tau \right\|_{H^s} \leq C\|\varphi\|_{H^s}^2. \tag{15}$$

We will proof that φ defined in Lemma 4 does not satisfies (15) for N large enough. By Lemma 3, we have that

$$\int_0^t E(t - \tau)(1 + H\partial_x)^{-1}\partial_x[(E(\tau)\varphi)^2] d\tau$$

$$= g_1(t,x,y) + g_2(t,x,y) + g_3(t,x,y), \tag{16}$$

where g_i, $i = 1,2,3$, and

$$\widehat{g_1}(t,\xi,\eta) = \frac{1}{\alpha^2 N^{2s}} e^{itp(\xi,\eta)} \frac{\xi}{1+|\xi|} \int_{I_{11}(\xi,\eta)} \frac{e^{it\theta(\xi,\xi_1,\eta,\eta_1)} - 1}{i\theta(\xi,\xi_1,\eta,\eta_1)} \, d\xi_1 d\eta_1,$$

$$\widehat{g_2}(t,\xi,\eta) = \frac{1}{\alpha^2 N^{2s}} e^{itp(\xi,\eta)} \frac{\xi}{1+|\xi|} \int_{I_{22}(\xi,\eta)} \frac{e^{it\theta(\xi,\xi_1,\eta,\eta_1)} - 1}{i\theta(\xi,\xi_1,\eta,\eta_1)} \, d\xi_1 d\eta_1,$$

further,

$$\widehat{g_3}(t,\xi,\eta) = \frac{1}{\alpha^2 N^{2s}} e^{itp(\xi,\eta)} \frac{\xi}{1+|\xi|} \int_{I_{12}(\xi,\eta)\cup I_{21}(\xi,\eta)} \frac{e^{it\theta(\xi,\xi_1,\eta,\eta_1)} - 1}{i\theta(\xi,\xi_1,\eta,\eta_1)} \, d\xi_1 d\eta_1,$$

with

$$I_{ij}(\xi,\eta) = \{(\xi_1,\eta_1)\in \mathbf{R}^2 \mid (\xi_1,\eta_1) \in I_i, (\xi-\xi_1, \eta-\eta_1) \in I_j, \} \;\; i,j = 1,2. \tag{17}$$

On the other hand, we have that the following supports are disjoint,

$$supp(\widehat{g_1}) \subset [2N, 2N+2\alpha] \times [2\alpha, 4\alpha],$$

$$supp(\widehat{g_2}) \subset [-2N-2\alpha, -2N] \times [2\alpha, 4\alpha],$$

and

$$supp(\widehat{g_3}) \subset [-\alpha, \alpha] \times [2\alpha, 4\alpha],$$

for N large enough and α small. Therefore,

$$\left\| \int_0^t E(t-\tau)(1+H\partial_x)^{-1}\partial_x[(E(t)\varphi)^2] d\tau \right\|_{H^s} \geq \|g_i(t,\cdot,\cdot)\|_{H^s}, \tag{18}$$

for $i = 1,2,3$. If we take $\alpha = N^{-\epsilon}$, with $\epsilon > 0$ small enough, we have that $(\xi,\eta) \in supp(\widehat{g_3})$ y $(\xi_1,\eta_1) \in I_{12} \cup I_{21}$

$$\left| \frac{e^{it\theta(\xi,\xi_1,\eta,\eta_1)} - 1}{i\theta(\xi,\xi_1,\eta,\eta_1)} \right| = |t| + O(N^{-\epsilon}), \tag{19}$$

by Lemma 5,

$$|\theta(\xi,\xi_1,\eta,\eta_1)| \leq CN^{-2\epsilon} \leq CN^{-\epsilon}.$$

Thus,

$$\|g_3(t,\cdot,\cdot)\|_{H^s}^2 = \int_{supp(\widehat{g_3})} (1+\xi^2+\eta^2)^s |\widehat{g_3}(t,\xi,\eta)|^2 \, d\xi d\eta$$

$$
\begin{aligned}
&= \frac{1}{\alpha^4 N^{4s}} \int_{\mathbf{R}^2} (1+\xi^2+\eta^2)^s \frac{|\xi|^2}{(1+|\xi|)^2} \\
&\quad \times \left| \int_{I_{12}\cup I_{21}} \frac{e^{it\theta}-1}{i\theta}\, d\xi_1 d\eta_1 \right|^2 d\xi d\eta \\
&\geq C \frac{|t|^2\alpha^4}{\alpha^4 N^{4s}} \int_{-\alpha/2}^{3\alpha/2} \int_0^{3\alpha/4} (1+\xi^2+\eta^2)^s \frac{\xi^2}{(1+|\xi|)^2}\, d\xi d\eta \\
&\geq C \frac{|t|^2}{N^{4s}} \int_{-\alpha/2}^{3\alpha/2} \int_0^{3\alpha/4} \xi^2 \, d\xi d\eta \\
&\geq C \frac{|t|^2}{N^{4s}} \int_{-\alpha/2}^{3\alpha/2} \int_0^{3\alpha/4} \xi^2 \, d\xi d\eta \\
&\geq C|t|^2 N^{-4s}\alpha^4 \\
&\geq C|t|^2 N^{-4(s+\epsilon)}.
\end{aligned}
$$

Finally, by (15), (18), and by Lemma 4, we deduce that

$$
4 \geq \|\varphi\|^2_{H^s(\mathbf{R}^2)} \geq \|g_3(t)\|_{H^s((R)^2)} \geq CN^{-4(s+\epsilon)},
$$

This is a contradiction for $s<0$ and N large. □

3. Main result

In this section we will prove the main theorem.

Theorem 7. *Let $s<0$. Then, there does not exist $T>0$ such that (1) has a unique solution in the interval $[0,T]$, and such that the flow map $u_0 \to u$ is of class C^2 in zero from $H^s(\mathbf{R}^2)$ to $H^s(\mathbf{R}^2)$.*

Proof. Consider the Cauchy problem

$$
\begin{cases} u_t + (u^2)_x + (Hu_t + u_{yy})_x = 0, & (x,y)\in \mathbf{R}^2, t>0 \\ u_\epsilon(0,x,y) = \epsilon\varphi(x,y), \end{cases} \tag{20}
$$

and ϵ small enough. We suppose that $u_\epsilon(t,x,y)$ is a unique local solution of (20) and that the flow map associated is of class C^2 in zero from $H^s(\mathbf{R}^2)$ to $H^s(\mathbf{R}^2)$. Then,

$$
u_\epsilon(t,x,y) = \epsilon E(t)\varphi(x,y) + \int_0^t E(t-\tau)(1+H\partial_x)^{-1}\partial_x(u_\epsilon^2(\tau,x,y))d\tau. \tag{21}
$$

If we take the derivate with respect to ϵ, then we have that

$$\frac{\partial^2 u_\epsilon(t,x,y)}{\partial \epsilon^2}\Big|_{\epsilon=0} = 2\int_0^t E(t-\tau)(1+H\partial_x)^{-1}\partial_x(E(\tau)\varphi)^2 d\tau. \tag{22}$$

Since the flow map is of class C^2, then

$$\left\| \int_0^t E(t-\tau)[(1+H\partial_x)^{-1}\partial_x(E(\tau)\varphi)^2]\, d\tau \right\|_{H^s(\mathbf{R}^2)} \leq C\|\varphi\|^2_{H^s}(\mathbf{R}^2).$$

This is a contradiction by Theorem 6.

This completes the proof. □

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