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THE SYSTEM OF d-GENERATORS OF d-SKEW FIELD OF $Sp(n)-$INVARIANT d-RATIONAL FUNCTIONS

Kobyljon Muminov[1 §], Saidaxbor Juraboyev[2]

[1]National University of Uzbekistan named after Mirzo Ulugbek
Tashkent - 100174, UZBEKISTAN

[2]Fergana State University
Fergana - 150100, UZBEKISTAN

Abstract: Let H^n be an n dimensional (the left) vector space over the skew-field of quaternion numbers, and $Sp(n)$ be a group of symplectic transformations of H^n. Also, the skew-field of all $Sp(n)-$invariant non-commutative differential rational functions denoted by $\Re[\![x,\bar{x}]\!]^{Sp(n)}$. In the paper an explicit description of a finite generating system in the differential skew-field $\Re[\![x,\bar{x}]\!]^{Sp(n)}$.

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1. Introduction

Let V be a finite-dimensional vector space over a field k (of real or complex numbers) with basis $x_1, ..., x_n$ and let $k[V] = k[x_1, ..., x_n]$ denote the commutative polynomial ring of rank n over k. Let $GL(V)$ be a general linear group of linear transformations V. If G is a finite subgroup of $GL(V)$, then there is induced homogeneous action of G on $k[V]$, the commutative polynomial ring.

Let G be a subgroup of $GL(V)$. We shall study the algebra of invariant with respect to the action of the group G, e.g.,

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§Correspondence author

$$k[V]^G = \{f \in k\,[V] : \; g \cdot f = f \;\; \text{for all } g \in G\}\,.$$

In the course of the invariant theory, *the problems of describing the generators of the algebra* $k[V]^G$ *and finding the defining relations between them* are considered (for example, see, [21], p.144). In particular, the problem related to the finite generation of the algebra $k[V]^G$ is known as *Hilbert's 14th problem.* This problem was solved positively by Hilbert-Nagata-Mumford theorem for many algebraic linear groups, including reductive groups (see, [15], [19]). However, in general, e.g., for any algebraic linear group $G \in GL\,(V)$, the isn't solved positively. In the study of Hilbert's 14th problem, H. Weyl's works are commendable (see, [24]). In his works, he shoved fundamental theorems of the Invariant Theory and their methods proving under action of some classical groups.

The differential analogue of the above problems were studied by R.G. Aripov [1], I.V. Chilin [3], Dj. Khadjiyev [9], K.K. Muminov [16], and obtained the positively solutions of this problem with respect to the action *orthogonal, pseudo-orthogonal and symplectic groups.* At present, the results obtained are applied to *differential geometry, non-Euclidean geometry and other important fields of science* (see, [10], [17], [18], [20]).

Also, the algebra of non-commutative invariants is widely studied by scientists. In particular, for free associative algebras of non-commutative invariants, positive solutions of analogues of many problems in the commutative case, obtained. Usually $G-$invariant free associative algebras of finite rank are denoted by $k\langle V\rangle^G$. Problems such as the description of the generators of the algebra $k\langle V\rangle^G$, the determination of a finite or infinite number of them, and the determination of relation between them represent a non-commutative analogue of Hilbert's 14th problem. In about it, many important facts, and analogues of the main theorems are given in the works of such scientists as G. Almkvists [2], M. Domokos, V. Drensky [5], E. Formanek [7], V.K. Kharchenko [11], A.N. Koryukin [14].

In this paper, we study the differential analogue of Hilbert's 14th problem for the case $k = \Re$, $V = H^n$ and $G = Sp\,(n)$, where $\Re$ is a center of the skew-field quaternion numbers, H^n is a n dimensional vector space over H and $Sp\,(n)$ is a group of symplectic (compact symplectic) transformations of the space H^n.

This article is organized as follows: In Section 2, the quaternion number, the group of symplectic transformations in quaternion space are Gram matrix are introduced briefly. Also, the some properties of these notions are given by remarks and propositions. In Section 3, the preliminary notions of the theory of non-commutative invariants are described, and the system of generators of the ring of $Sp(n)-$invariant non-commutative polynomials is shown. Also, in this section it is studied too, which the skew-field of the non-commutative rational functions. Using the results of Sections 3, the system of generators of a differential skew-field of $Sp(n)-$invariant differential rational functions is restored and expounded in detail in Section 4. Section 5 is the final part.

2. Preliminaries

2.1. Symplectic group

Let H denote the set of quaternion numbers. We write

$$H = \{q = t + xi + yj + zk \,|\, t, x, y, z \in R\},$$

where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Conjugation and modulus is given respectively by

$$\bar{q} = \overline{(t + xi + yj + zk)} = t - xi - yj - zk,$$

$$|q| = \sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2}.$$

Then $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ for $q_1, q_2 \in H$.

$$\mathrm{Re}\,(q) = \frac{1}{2}\,(q + \bar{q})\,, \mathrm{Re}\,(q_1 q_2) = \mathrm{Re}\,(q_2 q_1) = \mathrm{Re}\,(\bar{q}_1 \bar{q}_2) = \mathrm{Re}\,(\bar{q}_2 \bar{q}_1)\,.$$

A pure quaternion is of the form

$$Pu\,(q) = xi - yj - zk = \frac{1}{2}\,(q - \bar{q})\,,$$

and an inverse of the quaternion q is of the form $q^{-1} = \frac{\bar{q}}{|q|}$. Also, the set H is *a skew field* under the operations addition and multiplications (see, [4]).

Let H^n be an n dimensional linear space over the skew field H (multiplication of numbers is defined on the left), where H is a skew field of quaternion numbers. The elements of H^n will be represented as n dimensional row-vector $x = (\zeta_1, \zeta_2, ..., \zeta_n)$, where $\zeta_l \in H$, $l = \overline{1, n}$. By $GL\,(H^n)$, denote the group of all invertible linear transformations of the space. We consider the metric function

$\langle x, y\rangle : H^n \times H^n \to H$, which satisfy the following conditions for $\forall x, y, z \in H^n$ and $\lambda, \mu \in H$:

$$\begin{cases} \langle x, x\rangle \geq 0,\ \langle x, x\rangle = 0 \Leftrightarrow x = \theta; \\ \langle x, y\rangle = \overline{\langle y, x\rangle}; \\ \langle x, \lambda y + \mu z\rangle = \langle x, y\rangle\, \bar{\lambda} + \langle x, z\rangle\, \bar{\mu}, \end{cases}$$

where $\bar{q}$ means the conjugate of a quaternion $q = a + bi + cj + dk$.

Principally, we obtain the metric function as a bilinear form as follows:

$$\langle x, y\rangle = \zeta_1\bar{\eta}_1 + \zeta_2\bar{\eta}_2 + ... + \zeta_n\bar{\eta}_n. \tag{1}$$

It is known that the symplectic group $Sp\,(n)$ with respect to the function $\langle x, y\rangle$ is defined as a subgroup of as follows (see, [4], p. 35):

$$Sp\,(n) = \{\sigma \in GL\,(H^n) :\ \langle \sigma x, \sigma y\rangle = \langle x, y\rangle\,,\ x, y \in H^n\,\}\,, \tag{2}$$

Let $GL\,(n, H)$ be a group of the invertible square quaternion matrices of order n, i.e.,

$$GL\,(n, H) = \{g \in M\,(n, H) :\ d\det g \neq 0\}\,,$$

where $d\det g = c\mathrm{det}_i\left(g\bar{g}^T\right) = r\mathrm{det}_j\left(g\bar{g}^T\right)$, $\bar{g}^T-$ Hermitian conjugate of the transpose of a matrix g (see, [12]). It is plain that the relation $\sigma x \leftrightarrow xg$ is true for all $x \in H^n$ and $\sigma \in GL\,(H^n)$, where $g \in GL\,(n, H)$. In the case , the symplectic group $Sp\,(n)$ is defined as follows

$$Sp\,(n) = \left\{g \in GL\,(n, H) : g\bar{g}^T = E\right\}\,,$$

where E is identity element of the group $GL\,(n, H)$.

2.2. Gram matrices and it's elementary properties.

It is known that the function $\langle x, y\rangle$ expresses of scalar product in the space V (see, [13], p.11). Let $\{x_1, x_2, ..., x_m\}$ be a set of arbitrary vectors in H^n. The matrix

$$\begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & ... & \langle x_1, x_m\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & ... & \langle x_2, x_m\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle x_m, x_1\rangle & \langle x_m, x_2\rangle & ... & \langle x_m, x_m\rangle \end{pmatrix}$$

for some finite natural number m will be called Gram matrix of the vectors $x_1, x_2, ..., x_m$ and denote by $\Gamma\,(x_1, x_2, ..., x_m)\,(m)$ (see, [23], p.49). Obviously that Gram matrix expresses Hermitian matrix.

Remark 1. If the condition $a_{ij} = \bar{a}_{ji}$ $(i \neq j)$ is hold for elements of matrix $A = (a_{ij})_{i,j=1}^{n} \in M(n, H)$, then the matrix A will be called Hermitian matrix (see,[12], p.854).

It means that the properties of Hermitian matrix and its determinants are valid for the Gram matrix and its determinants. We state some properties of the Gram matrix in the following.

Proposition 1. *If the set $B = \{x_1, x_2, ..., x_m\}$ is orthogonal system of vectors $x_1, ..., x_m \in H^n$, then $\Gamma(x_1, ..., x_m)(m)$ is a diagonal quaternion matrix of order m.*

Proposition 2. *If the set $B = \{x_1, x_2, ..., x_n\}$ is orthonormal basis of the space H^n, then $\Gamma(x_1, x_2, ..., x_n)(n)$ is an identity quaternion matrix of order n.*

Proposition 3. *If the set $B = \{x_1, x_2, ..., x_n\}$ is a set of basis vectors for H^n, then*

$$\langle x, y\rangle = ([x]_B)\,\Gamma(x_1, x_2, ..., x_n)(n)\left(\overline{[y]}_B\right)^T,$$

where $\forall x, y \in H^n$.

Corollary 1. *If the set B is orthonormal basis, then $\langle x, y\rangle = [x]_B[\bar{y}]^T{}_B$.*

Proposition 4. $\det\Gamma(x_1, ..., x_m)(m) = \det\bar{\Gamma}^T(x_1, ..., x_m)(m)$.

Proposition 5. *For arbitrary a set of vectors $a_1, a_2, ..., a_n \in H^n$ and the scalar number $\lambda \in H$, the equality*

$$\det\Gamma(a_1, ..., a_k, ..., a_l, ..., a_n)(n) = \det\Gamma(a_1, ..., a_k + \lambda a_l, ..., a_l, ..., a_n)(n)$$

holds.

This property follows from the properties of Hermitian matrix (see, [12], [8]).

Corollary 2. *If a set of vectors $a_1, a_2, ..., a_s \in H^n$ generated from a set of vectors $x_1, x_2, ..., x_s \in H^n$ by orthogonalization, then the equality*

$$\det\Gamma(x_1, ..., x_s)(s) = \det\Gamma(a_1, ..., a_s)(s) = |a_1|^2|a_2|^2...|a_s|^2$$

is true.

Corollary 3. *If a set of vectors* $x_1, x_2, ..., x_s \in H^n$ *is linear independent, then the relation* $\det \Gamma(x_1, x_2, ..., x_s)(s) > 0$, *otherwise the relation* $\det \Gamma(x_1, x_2, ..., x_s)(s) = 0$ *is true.*

3. Theory of non-commutative invariants.

Let K be any field of characteristic zero, and let V be a finite dimensional vector space over the field K, with basis $x_1, x_2, ..., x_n$. Also, let

$$K\langle V\rangle = K\langle x_1, ..., x_n\rangle = K \oplus V \oplus (V \otimes V) \oplus V^{\otimes 3} \oplus ...$$

denote the free associative algebra (or tensor algebra) of rank n. Naturally that the elements of the algebra $K\langle V\rangle$ is represented with in form a polynomial with a non-commutative variables $x_1, ..., x_n$ (see, [7], p-88).

Let G be a subgroup of $GL(V)$, where $GL(V)$ is a group of all invertible linear transformations in V. As an action of the group G to the space V is defined in form $(v, g) = v.g$, where $g \in G$, $v \in V$, also an action of G to the algebra $K\langle V\rangle$ is defined in form $(g, f) = f(v.g)$, where $g \in G$, $f \in K\langle V\rangle$, $v \in V$.

Definition 4. The polynomial $f \in K\langle V\rangle$ is called $G-$invariant, if the equality $f(v.g) = f(v)$ holds for all $g \in G$ (see, [6]).

It is known that a set of all $G-$invariant polynomials is a sub-algebra to $K\langle V\rangle$ and denote by $K\langle V\rangle^G$ i.e.,

$$K\langle V\rangle^G = \{f \in K\langle V\rangle : \ f(v.g) = f(v), \ \forall g \in G, \ \forall v \in V\}.$$

Let S be a set, which consisted of elements $K\langle V\rangle^G$.

Definition 5. The set S is called the system of generators of the algebra $K\langle V\rangle^G$, if the smallest sub-algebra in $K\langle V\rangle^G$ containing the set S corresponds to $K\langle V\rangle^G$ (see, [1], p.7).

The problem *of describing the generating system of the algebra of invariants* expresses the main problem of the Invariant Theory. We will consider this problem for the cases $K = \Re$, $V = H^n$, $G = Sp(n)$, in the following.

Let $\Re$ be such a commutative, unit sub-ring of the skew field H that the equality $ax = xa$ is valid for $\forall a \in \Re$, $\forall x \in H$. We also denote by $\Re\langle x_1, ..., x_n\rangle^{Sp(n)}$, the ring of $Sp\,(n)\,-$invariant polynomials of arbitrary positive (integer) degree with quaternion vector-variable over the ring $\Re$, where $x_l \in H^n$, $l = \overline{1, n}$. When studying the system of generators of the ring, we use from the operation $* : \Re\,\langle x_1, ..., x_n\rangle \to \Re\,\langle x_1, ..., x_n\rangle$, which will satisfy the following conditions, together with operations of the ring $\Re\langle x_1, ..., x_n\rangle^{Sp(n)}$:

i) $(f^*)^* = f,\ \forall f \in \Re\,\langle x_1, ..., x_n\rangle\,;$

ii) $(a \cdot f)^* = a^* \cdot f^*,\ \forall a \in \Re,\ \forall f \in \Re\,\langle x_1, ..., x_n\rangle\,;$

iii) $(f + g)^* = f^* + g^*,\ \forall f, g \in \Re\,\langle x_1, ..., x_n\rangle\,;$

iii) $(f \cdot g)^* = g^* \cdot f^*,\ \forall f,\, g \in \Re\,\langle x_1, ..., x_n\rangle\,.$

In particular, we obtain as the Hermitian conjugate of the operation $*$, in the ring of polynomials with quaternion variables. Also, we replace of the notation f^* with $\bar{f}$, and notation $\Re\,\langle x_1, ..., x_n\rangle$ with $\Re\,\langle x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n\rangle$. Obviously, this ring represents the free algebra of rank n^2 with quaternion variables $x_{lm} \in H$, where $l, m = \overline{1, n}$.

Theorem 6. *All the elements of the ring*

$$\Re\langle x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n\rangle^{Sp(n)}$$

are generated by applying the operations of ring and Hermitian conjugate to the bilinear forms $\langle x_l\,,\, x_m\rangle$.

Proof. Let us say the vectors $x_1, x_2, ...$ will be given in the space V, and the vectors $\xi_1, \xi_2, ...$ will be given in the space V^*, where V^* is a adjoin space to V. Obviously, we can express any polynomial $f\,[x_1, x_2, ...\,|\xi_1, \xi_2, ...]$ by the form $P\,\{(x_m\,|\xi_n\,)\}$, where

$$(x_m\,|\xi_n\,) = \sum_{k=1}^{n} x_{lk}\xi_{km}, \quad l, m = \overline{1, n}.$$

Hence, to prove Theorem 6, it is enough to show that the product $(x_l\,|\xi_m\,)$ can be expressed in the linear form $\langle x_l, x_m\rangle$.

Let a pair of linear independent sets of vectors $x_1, x_2, ..., x_n \in V$ and $\xi_1, \xi_2, ..., \xi_n \in V^*$, also let $Sp\,(n)\,-$invariant polynomial $f\,[x_1, x_2, ..., x_n\,|\xi_1, \xi_2, ...\,, \xi_n]$ are given, i.e.,

$$f\left[x_1,x_2,...,x_n\left|\xi_1,\xi_2,...\right.,\xi_n\right] = f\left(\begin{array}{c|c} x_{11},x_{12},...,x_{1n} & \xi_{11},\xi_{12},...,\xi_{1n} \\ x_{21},x_{22},...,x_{2n} & \xi_{21},\xi_{22},...,\xi_{2n} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ x_{n1},x_{n2},...,x_{nn} & \xi_{21},\xi_{22},...,\xi_{2n} \end{array}\right). \quad (3)$$

Naturally, in this case, it is possible to establish a one-value correspondence between the sets of vectors $\{x_1,...,x_n\}$ and $\{\xi_1,...,\xi_n\}$ in the form $\{\xi_1,\xi_2,...,\xi_n\} \leftrightarrow \left\{x_{\pi(1)},x_{\pi(2)},...,x_{\pi(n)}\right\}$, where

$$\pi:\begin{pmatrix} 1 & 2 & ... & n \\ i_1 & i_2 & ... & i_n \end{pmatrix} \to \begin{pmatrix} 1 & 2 & ... & n \\ j_1 & j_2 & ... & j_n \end{pmatrix},\ i_k,\, j_k=\overline{1,n},\ k=\overline{1,n}.$$

Using from the transformation $\sigma \in Sp\,(V)$, we can pass the vector arguments $x_1,x_2,...,x_n$ to the vectors $e_1,e_2,...,e_n$, which the standard basis of vectors in V. Then, we have the equalities

$$\sigma x_1 = x_1 g = e_1,\ \sigma x_2 = x_2 g = e_2,\ ...,\sigma x_n = x_n g = e_n,$$

where $g \in Sp\,(n)$. In general, these equalities can be written in form the matrix equation $Xg=E$, where $X=(x_{ij})_{i,j=1}^n$, and E is an identity matrix. From this, the relation $g=X^{-1}$ will follow. It is known that the matrix g is an element of the group $Sp\,(n)$. Hence, the equality $g=X^{-1}=\bar{X}^T$ is true. In turn, when transforming the set $\{x_1,x_2,...,x_n\}$ into the set $\{e_1,e_2,...,e_n\}$, respectively, the vectors $\xi_1,\xi_2,...,\xi_n$ will change to the set of vectors $\xi_1',\xi_2',...,\xi_n'$ and will be defined as follows:

Based on the correspondence

$$\{\xi_1,\xi_2,...,\xi_n\} \leftrightarrow \left\{x^*_{\pi(1)},x^*_{\pi(2)},...,x^*_{\pi(n)}\right\}$$

we have the correspondence

$$\left\{\xi_1',\xi_2',...,\xi_n'\right\} \leftrightarrow \left\{\left(\sigma x_{\pi(1)}\right)^*,\left(\sigma x_{\pi(2)}\right)^*,...,\left(\sigma x_{\pi(n)}\right)^*\right\} = \left\{g^*x^*_{\pi(1)},g^*x^*_{\pi(2)},...,g^*x^*_{\pi(n)}\right\}.$$

From this, we have the equality

$$\Xi = X\cdot X_\pi^* = \left(\sum_{i=1}^n x_{mi}\bar{x}_{\pi(l)i}\right)_{m,l=1}^n = \left(\left\langle x_m,x_{\pi(l)}\right\rangle\right)_{m,l=1}^n, \quad (4)$$

where $\Xi = \left\{\xi'_{lm}\right\}_{l,m=1}^{n}$. It follows from equality (4) that arbitrary element ξ'_{lm} is defined in form $\langle x_l, x_m\rangle$. Then, we have the equality

$$f\left(x_1, ..., x_n \left|\xi_1, ..., \ \xi_n\right.\right) = f\left(e_1, ..., e_n \left|\xi'_1, ..., \ \xi'_n\right.\right) = P\left\{\langle x_i, x_j\rangle\right\}.$$

This implies that any $G-$invariant polynomial $f\left(x_1, ..., x_n, \bar{x}_1, ..., \bar{x}_n\right)$ is expressed by bilinear forms $\langle x_l, x_m\rangle$. Theorem 8 is proved. □

Corollary 7. *The generating system of the ring* $\Re\langle x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n\rangle^{Sp(n)}$ *is expressed by bilinear forms* $\langle x_l, x_m\rangle$.

3.1. The skew field of $Sp(n)-$invariant noncommutative rational functions

Let K be any field of characteristic 0, and let $x = (x_1, ..., x_n)$ be a n-tuple of noncommutative indeterminate $x_1, ..., x_n$. It is known that a n.c (noncommutative) polynomial is a formal linear combination of words in x with coefficients in K. For example, $3x_1^4 - 2x_1x_2 + 2x_2x_1 - 4$, $4x_1^2 - x_1x_2x_3 + x_1x_3x_2 - 5$. We denote the free associative algebra of n.c polynomials an n generators $K\left\langle x_1, ..., x_n\right\rangle$. *A n.c rational expression* is a syntactically valid combination of n.c polynomials, arithmetic expression, $+$, $\cdot$, $^{-1}$, and parentheses, i.e., $\left(1 - x_3 + 2x_2x_3^{-1}\right)^{-1}$, $x_1^{-1} + x_2^{-1} - 3x_3(x_1 - x_2)^{-1}$. This expressions can be naturally evaluated an n-tuples of matrices. An expressions is called *non-degenerate* if it is valid to evaluate it on at least any such tuple of matrices. Two non-degenerate expressions with same evaluations whenever they are both defined are *equivalent*. *A n.c rational function* is an equivalent class of a non-degenerate rational expressions. They from the free skew field $K\left[\!\left[x_1, ..., x_n\right]\!\right]$, which is the universal skew field of fraction of the free algebra $K\left\langle x_1, ..., x_n\right\rangle$ (see, [22]). In what follows, we will studythe skew field $K\left[\!\left[x_1, ..., x_n\right]\!\right]$ in the case $K = \Re$ and $x_l \in H^n$, $l = \overline{1, n}$. It is known that the invertible element exist for an arbitrary non-zero element of the commutative ring with unity element, and the equality $q^{-1} = \frac{\bar{q}}{|q|^2}$ holds for every quaternion number $q \in H$, $(q \neq 0)$. Then, the non-zero element of the skew field $\Re\left[\!\left[x_1, ..., x_n\right]\!\right]$ can be expresses in form $a^{-1} \cdot b$ or $b \cdot a^{-1}$, where $a, b \in \Re\left\langle x_1, ..., x_n\right\rangle$. For example,

$$x_{11}^{-1} - x_{12}^{-1} = \frac{\bar{x}_{11}}{|x_{11}|^2} - \frac{\bar{x}_{12}}{|x_{12}|^2} = \frac{|x_{12}|^2\bar{x}_{11} - |x_{11}|^2\bar{x}_{12}}{|x_{11}|^2|x_{12}|^2} =$$
$$= (x_{12}\bar{x}_{12}\bar{x}_{11} - x_{11}\bar{x}_{11}\bar{x}_{12})\left(|x_{11}|^2|x_{12}|^2\right)^{-1}.$$

Let G be an arbitrary subgroup of $GL(n,H)$. The skew field of rational invariants, denoted by $\Re[\![x_1,...,x_n]\!]^G$, is the skew field of elements of $\Re[\![x_1,...,x_n]\!]$ that are invariant under the action of G, that is

$$\Re[\![x_1,...,x_n]\!]^G = \{r \in \Re[\![x_1,...,x_n]\!] : r(xg) = r(x) \text{ for all } g \in G\}.$$

It is known that the relation

$$\Re[\![x_1,...,x_n]\!]^G \subset \Re[\![x_1,...,x_n]\!]$$

is true, for $G \in GL(n,H)$. Let G be the group $Sp(n)$. In this case, we write of $\Re[\![x_1,...,x_n]\!]^G$, by $\Re[\![x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\!]^{Sp(n)}$.

Theorem 8. *Any element of the skew field*

$$\Re[\![x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\!]^{Sp(n)}$$

is rationally expressed by $Sp(n)-$invariant n.c polynomials.

Proof. Let $f[\![x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\!]$ be an element of the skew field $\Re[[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]]^{Sp(n)}$. According to the above statement, we have the expression

$$f[\![x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\!] = q^{-1}[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\,p[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n],$$

where $p[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]$ and $q[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]$ are elements of $\Re[[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]]^{Sp(n)}$.

Also, according to the definition of a $G-$invariant rational function, the equality

$$f[\![x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}]\!] = f[\![x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\!]$$

is true, for $\forall g \in Sp(n)$. From this we obtain the following:

$$\begin{aligned} q^{-1}[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}]\,p[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}] \\ = q^{-1}[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\,p[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]. \end{aligned} \tag{5}$$

From (5), we obtain the following expression:

$$\begin{aligned} p[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}] &= \\ &= \left\{q[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}]\,q^{-1}[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]\right\} \\ &\quad \times p[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n]. \end{aligned} \tag{6}$$

It is clear that the relations

$$\left.\begin{array}{l}\deg\left(p\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]\right)\leq\deg\left(p\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\right)\\ \deg\left(q\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]\right)\leq\deg\left(q\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\right)\end{array}\right\} \tag{7}$$

are always true, where $\deg f$ is degree of the polynomial f. From expressions (5)-(7) we will have the equalities

$$\deg\left(p\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]\right)=\deg\left(p\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\right),$$

$$\deg\left(q\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]\right)=\deg\left(q\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\right).$$

According to these equalities we can say that the product $q\left[x_0g,...,x_ng,\overline{x_0g},...,\overline{x_ng}\right]q^{-1}\left[x_0,...,x_n,\bar{x}_0,...,\bar{x}_n\right]$ does not depend on any of the variables $x_1,x_2,...,x_n$. Thus, we can denote it as

$$q\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]q^{-1}\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]=\lambda\left(g\right).$$

From this we obtain the equalities

$$\left.\begin{array}{l}p\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]=\lambda\left(g\right)p\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\\ q\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]=\lambda\left(g\right)q\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]\end{array}\right\}. \tag{8}$$

It is known from invariant theory that a function that satisfies equality (8) is called *a relative invariant with the multiplier* $\lambda(g)$ (see, [24], p.25). Also, the function $\lambda\left(g\right)$ is called *a characteristic multiplier*, and satisfies the following conditions for an arbitrary g in G:

1. $\lambda\left(g_1g_2\right)=\lambda\left(g_2\right)\lambda\left(g_1\right)$;
2. $\lambda\left(e\right)=1$, where e is a unity element of the group G;
3. $\lambda\left(g_1\right)\neq\lambda\left(g_2\right)$ for all $g_1,g_2\in G$ that $g_1\neq g_2$.

It is known from theory of invariants, that if g is an element of the group $GL\left(n,K\right)$, then the equality $\lambda\left(g\right)=\left(\det g\right)^m$ is true, where K is any field (see, [24], p. 26). But, if $K=H$ then the equality $\lambda\left(g\right)=\left(\det g\right)^m$ is not true. Because $\det g$ don't simultaneously satisfy conditions 1)- 3). Accordingly, for $g\in GL\left(n,H\right)$ we get a function $\lambda\left(g\right)=\left(\mathrm{ddet} g\right)^m$ satisfying conditions 1)-3) as a function $\lambda\left(g\right)$, where $\mathrm{ddet} g=\det\left(g\bar{g}^T\right)$. It is plain that if the matrix g be a element of the group $Sp\left(n\right)$ then $\mathrm{ddet} g=1$. This implies $\lambda\left(g\right)=1$ and the equalities

$$p\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]=p\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right];$$

$$q\left[x_1g,...,x_ng,\overline{x_1g},...,\overline{x_ng}\right]=q\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]$$

are hold, i.e., the polynomials $p\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]$ and $q\left[x_1,...,x_n,\bar{x}_1,...,\bar{x}_n\right]$ are $Sp\left(n\right)$-invariants. Thus, any $Sp\left(n\right)$-invariant rational functions are expressed rationally with $Sp\left(n\right)$-invariant polynomials. The theorem is proved. □

Due to Theorem 6 and Theorem 8 the corollary follows:

Corollary 9. *All the $Sp\left(n\right)-$invariant n.c rational function are rationally expressed by bilinear forms $\langle x_l, x_m\rangle$.*

4. The system generating of the differential skew field $\Re\left[\!\left[x,\bar{x}\right]\!\right]^{Sp(n)}$.

Let K be an arbitrary commutative ring and let $d : K \to K$ be its differential, i.e., the conditions 1) $d\left(a+b\right)=d\left(a\right)+d\left(b\right)$; 2) $d\left(a\cdot b\right)=d\left(a\right)b+ad\left(b\right)$; 3) $d\left(1_K\right)=0_K$ for $\forall a,\ b,\ 1_K\in K$.

Consider a quaternion-valued function of real variable $f : R \to H$ (x is a real variable) such that

$$f\left(t\right)=f_1\left(t\right)+f_2\left(t\right)i+f_3\left(t\right)j+f_4\left(t\right)k.$$

The first derivative of a quaternion function $f\left(t\right)$ with respect to the real variable t we denote by

$$f'\left(t\right):=\frac{df\left(t\right)}{dt}=\frac{df_1\left(t\right)}{dt}+\frac{df_2\left(t\right)}{dt}i+\frac{df_3\left(t\right)}{dt}j+\frac{df_4\left(t\right)}{dt}k.$$

It easy to prove the following proposition on properties of the derivative of a quaternion functions.

Proposition 6. (see, [13], Prop.2.1) *If $q : R \to H$ and $r : R \to H$ are differentiable, then $\left(q\pm r\right)\left(t\right)$, $qr\left(t\right)$ and for any integer $n\geq 1$, $q^n\left(t\right)$ are differentiable and*

i_1) $\left(q\pm r\right)'\left(t\right)=q'\left(t\right)\pm r'\left(t\right)$;

i_2) $\left(q\cdot r\right)'\left(t\right)=q'\left(t\right)r\left(t\right)+q\left(t\right)r'\left(t\right)$;

i_3) $\left[aq\left(t\right)\right]'=d\left(a\right)q\left(t\right)+aq'\left(t\right)$;

$i_4)$ $[q^n(t)]' = \sum_{\lambda=0}^{n-1} q^{\lambda}(t)\, q'(t)\, q^{n-1-\lambda}(t);$

$i_5)$ $\left[q^{-1}(t)\right]' = -q^{-1}(t)\, q'(t)\, q^{-1}(t).$

Let $x = (x_1, ..., x_n)$ be such a vector function that its components is quaternion function with real variables. The differential of the vector function x is denoted by dx, and it is defined in the form

$$dx = (dx_1, dx_2, ..., dx_n) \texttt{ or } x' = \left(x_1', x_2', ..., x_n'\right).$$

In the skew field $\Re [\![x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n]\!]$ we consider the operation

$$\delta : \Re [\![x_1, ..., x_n]\!] \to \Re [\![x_1, ..., x_n]\!]$$

which satisfies the following conditions:

$j_1)$ $\forall x_l \in H^n$ for all $\delta(x_l) = x_{l+1};$

$j_2)$ $\delta(ax_l) = d(a)\, x_l + ax_{l+1}$ for all $\forall a \in \Re$ and $\forall x_l \in H^n$.

It is clear that if the vector function $x_l \in H^n$ is a vector function all of whose components are functions of a real variable, then the operation δ can be considered as an operation differentiable. In this case, the skew field $\Re[[x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n]]$ is called a differential skew field ($d-$skew field), if we consider the operation δ together. Now we insert the notations $x = x^{(0)}$, $x_l = x^{(l)}$, $\delta\left(x^{(l)}\right) = x^{(l+1)}$. Using the notations, we can write the skew field $\Re [\![x_1, ..., x_n; \bar{x}_1, ..., \bar{x}_n]\!]$ in the form $\Re [\![x, \bar{x}]\!]$. In the d-skew field $\Re [\![x, \bar{x}]\!]$, the notion $G-$invariance and the notion of a system of $G-$invariant generators are defined similarly to the notions in the previous subsections. In the following, we consider of the problem describing system of $d-$generators of the $d-$skew field $\Re[\![x, \bar{x}]\!]^G$.

Theorem 10. *Let $G = Sp(n)$. Then any element of the skew field is generated with invariant n.c polynomials in the form $\left\langle x^{(l)}, x^{(m)}\right\rangle$, $l, m \in Z_0^+$ by applying operations of a skew field and differentiation.*

Theorem 10 represents a differential analogue of Corollary 9.

Theorem 11. *Let be $G = Sp(n)$. Then the system of $G-$invariant n.c polynomials in the form*

$$\left\langle x^{(r-1)}, x^{(r-1)}\right\rangle,\ \left\langle x^{(r-1)}, x^{(r)}\right\rangle,\ r = \overline{1, n} \tag{9}$$

is a finite system of generators in the d-skew field $\Re[\![x, \bar{x}]\!]^G$.

Proof. According to Theorem 10 that any n.c the $d-$rational function $f\,[\![x,\bar{x}]\!] \in \Re[\![x,\bar{x}]\!]^{Sp(n)}$ is expressed $d-$rationally in terms of $Sp\,(n)-$invariant polynomials of the form $\left\langle x^{(l)}, x^{(m)}\right\rangle$,
$l,m \in Z_0^+$. Therefore to prove Theorem 11 it suffices to show that the polynomials $\left\langle x^{(l)}, x^{(m)}\right\rangle$ is expressed by elements of system (9). To do this, we use the following properties and lemmas:

Proposition 7. *The equality* $\overline{\left\langle x^{(l)}, x^{(m)}\right\rangle} = \left\langle x^{(m)}, x^{(l)}\right\rangle$ *is true for* $\forall x^{(l)}, x^{(m)} \in H^n$.

Proof.

$$\overline{\left\langle x^{(l)}, x^{(m)}\right\rangle} = \overline{\left(x_1^{(l)}\bar{x}_1^{(m)} + ... + x_n^{(l)}\bar{x}_n^{(m)}\right)}$$
$$= x_1^{(m)}\bar{x}_1^{(l)} + ... + x_1^{(m)}\bar{x}_n^{(l)} = \left\langle x^{(m)}, x^{(l)}\right\rangle.$$

□

Proposition 8. *For arbitrary vectors* $x^{(l)}, x^{(m)} \in H^n$, *the equality*

$$\left\langle x^{(l)}, x^{(m)}\right\rangle' = \left\langle x^{(l+1)}, x^{(m)}\right\rangle + \left\langle x^{(l)}, x^{(m+1)}\right\rangle$$

holds.

Proof.

$$\left\langle x^{(l)}, x^{(m)}\right\rangle' = \left(x_1^{(l)}\bar{x}_1^{(m)} + ... + x_n^{(l)}\bar{x}_n^{(m)}\right)'$$
$$= \left(x_1^{(l+1)}\bar{x}_1^{(m)} + x_1^{(l)}\bar{x}_1^{(m+1)} + ... + x_n^{(l+1)}\bar{x}_n^{(m)} + x_n^{(l)}\bar{x}_n^{(m+1)}\right)$$
$$= \left(x_1^{(l+1)}\bar{x}_1^{(m)} + ... + x_n^{(l+1)}\bar{x}_n^{(m)}\right) + \left(x_1^{(l)}\bar{x}_1^{(m+1)} + ... + x_n^{(l)}\bar{x}_n^{(m+1)}\right)$$
$$= \left\langle x^{(l+1)}, x^{(m)}\right\rangle + \left\langle x^{(l)}, x^{(m+1)}\right\rangle.$$

□

Proposition 9 *Let A be a linearly independent set of the vectors $x, x^{(1)}, ..., x^{(n-1)}$ in H^n. Then, the following relations are always hold:*

$a)$ $\det\Gamma\left(x, x^{(1)}, ..., x^{(n-1)}\right)(n) \neq 0$ *for the elements of the set;*

$b)$ $\det\Gamma\left(x, x^{(1)}, ..., x^{(n-1)}, y\right)(n+1) = 0$ *for the elements of the set and any non-zero vector y in H^n;*

c) $\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, y, z\right)(n+1) = 0$, *for the elements of the set and arbitrary non-zero vectors* y, z *in* H^n, *i.e.*,

$$\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, y, z\right)(n+1) =$$
$$= \begin{vmatrix} \left\langle x, x^{(1)}\right\rangle & \left\langle x, x^{(2)}\right\rangle & ... & \langle x, z\rangle \\ \left\langle x^{(1)}, x^{(1)}\right\rangle & \left\langle x^{(1)}, x^{(2)}\right\rangle & ... & \left\langle x^{(1)}, z\right\rangle \\ \vdots & \vdots & ... & \vdots \\ \left\langle y, x^{(1)}\right\rangle & \left\langle y, x^{(2)}\right\rangle & ... & \langle y, z\rangle \end{vmatrix} = 0;$$

d) $\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, y\right)(n+1) \neq 0$, *for the elements of the set and arbitrary non-zero vectors* y *in* H^n.

Proposition 9 follows from properties of the determinant of Gram matrix. Using the above properties we will prove the following lemmas.

Lemma 1. *Any* $Sp(n)-$*invariant n.c* $d-$*polynomials in the form* $\left\langle x^{(l)}, x^{(m)}\right\rangle$ *are expressed rationally in terms of elements of the system*

$$\left\langle x^{(r-1)}, x^{(r-1)}\right\rangle, \ \left\langle x^{(r-1)}, x^{(r)}\right\rangle, \ 1 \leq r \leq \left[\frac{l+m}{2}\right], \ r \in N. \qquad (10)$$

Proof. To prove Lemma 1, we consider separately the following cases:

Case 1. Let be $l \leq m$. In this case, we apply the principle of mathematical induction with respect to the difference $m - l = h$:

1. For $h = 0$, $h = 1$ the assertion in Lemma 1 is true;

2. Let $h = 2$. Then, the assertion in Lemma 1 follows from the equality

$$\left\langle x^{(l)}, x^{(l+2)}\right\rangle = \left\langle x^{(l)}, x^{(l+1)}\right\rangle' - \left\langle x^{(l+1)}, x^{(l+1)}\right\rangle;$$

3. Suppose the assertion in Lemma 1 is be true for all $h \leq s$ i.e., for k satisfying the condition $l \leq r \leq \left[\frac{2l+s}{2}\right]$, $\left\langle x^{(l)}, x^{(l+s)}\right\rangle$ is expressed in terms of non-commutative d-polynomials of the form $\left\langle x^{(r-1)}, x^{(r-1)}\right\rangle$ and $\left\langle x^{(r-1)}, x^{(r)}\right\rangle$;

4. Now let us check that the assertion in Lemma 1 is also true for $h = s+1$: according to Proposition 8 the equality

$$\left\langle x^{(l)}, x^{(l+s+1)}\right\rangle = \left\langle x^{(l)}, x^{(l+s)}\right\rangle' - \left\langle x^{(l+1)}, x^{(l+s)}\right\rangle$$

holds; here the polynomial $\left\langle x^{(l)}, x^{(l+s)}\right\rangle$ satisfy of assertion in Lemma 1 to according supposition; furthermore, the d-polynomial $\left\langle x^{(l+1)}, x^{(l+s)}\right\rangle$ also satisfy of the assertion in Lemma 1 to according the condition $l+s-l-1 = s-1 < s$;

thus, the polynomial $\langle x^{(l+1)}, x^{(l+s)} \rangle$ is expressed in terms to the d-polynomials $\langle x^{(r_1-1)}, x^{(r_1-1)} \rangle$ and $\left\langle x^{(k'-1)}, x^{(k')} \right\rangle$ for all r_1, where
$l+1 \leq r_1 \leq \left[\frac{2l+s+1}{2}\right]$, $r_1 \in N$; in this case, it is not difficult to determine that the following conditions hold for k':

$$r_1 = \begin{cases} \leq r \text{ for all } s = 2\kappa, \ \kappa \in N, \\ \leq r+1 \text{ for all } s = 2\kappa+1, \ \kappa \in N. \end{cases}$$

From these it follows that the assertion in Lemma 1 is true for the d-polynomial $\langle x^{(l)}, x^{(l+s+1)} \rangle$. Hence, according to the principle of mathematical induction the assertion in Lemma 1 is true for all h.

Case 2. For this case, the assertion in Lemma 1 follows from Case 1 using by the equality $\langle x^{(l)}, x^{(m)} \rangle = \overline{\langle x^{(m)}, x^{(l)} \rangle}$. Lemma 1 is proved. □

Lemma 2. *All $Sp(n)-$invariant $d-$polynomials in the form*

$$\left\langle x^{(r-1)}, x^{(r-1)} \right\rangle, \ \left\langle x^{(r-1)}, x^{(r)} \right\rangle, \ \ r \in N$$

are $d-$rationally expressed in terms of elements of system (9).

Proof. According to the assertion in Lemma 1, the assertion in Lemma 2 is true for $1 \leq r \leq n$. In the following, we will prove the lemma only $r = n+1$. All other cases (i.e., $r = n+s, \ s \in \{2, 3, ...\}$) can be shown by the principle of mathematical induction.

Let be $r = n+1$. In this case, according to the part b) of Property 9 the equality $\det \Gamma \left(x, x^{(1)}, ..., x^{(n)}\right)(n+1) = 0$ is hold, i.e.,

$$\begin{vmatrix} \langle x, x \rangle & \langle x, x^{(1)} \rangle & ... & \langle x, x^{(n)} \rangle \\ \langle x^{(1)}, x \rangle & \langle x^{(1)}, x^{(1)} \rangle & ... & \langle x^{(1)}, x^{(n)} \rangle \\ \vdots & \vdots & ... & \vdots \\ \langle x^{(n)}, x \rangle & \langle x^{(n)}, x^{(1)} \rangle & ... & \langle x^{(n)}, x^{(n)} \rangle \end{vmatrix} = 0. \tag{11}$$

All elements of the determinant $\det \Gamma \left(x, x^{(1)}, ..., x^{(n)}\right)(n+1)$, except for $\langle x^{(n)}$, $x^(n) \rangle$, are expressed by the elements of system (9). Because for the order of derivatives of these elements satisfies the condition $\max \left\{ \left[\frac{l+m}{2}\right] \right\} = \left[\frac{2n-1}{2}\right] = n-1$, where l, m is the order of the derivative. Furthermore, since the set of vectors $x, x^{(1)}, ..., x^{(n-1)}$ in H^n, the relation $\det \Gamma \left(x, x^{(1)}, ..., x^{(n-1)}\right)(n) \neq 0$ is valid. It follows that $R_{n+1n+1} \neq 0$ also holds, where R_{n+1n+1} is the first minor of $\det \Gamma \left(x, x^{(1)}, ..., x^{(n)}\right)(n+1)$. Hence, the matrix corresponding to R_{n+1n+1} is invertible. This allows the element $\langle x^{(n)}, x^{(n)} \rangle$ of the determinant

$\det\Gamma\left(x, x^{(1)}, ..., x^{(n)}\right)(n+1) = 0$ to be expressed using the equality (11) in term of the remaining elements of the matrix. Therefore, the $d-$polynomial $\left\langle x^{(n)}, x^{(n)}\right\rangle$ is $d-$rationally expressed the elements of system (9); also according to part c) of Proposition 9 the equality

$$\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, x^{(n)}, x^{(n+1)}\right)(n+1) = 0$$

is true for a set linearly independent of the vectors $x, x^{(1)}, ..., x^{(n-1)}$ and arbitrary vectors $x^{(n)}, x^{(n+1)}$, i.e.,

$$\begin{vmatrix} \left\langle x, x^{(1)}\right\rangle & \left\langle x, x^{(2)}\right\rangle & ... & \left\langle x, x^{(n+2)}\right\rangle \\ \left\langle x^{(1)}, x^{(1)}\right\rangle & \left\langle x^{(1)}, x^{(2)}\right\rangle & ... & \left\langle x^{(1)}, x^{(n+2)}\right\rangle \\ \vdots & \vdots & ... & \vdots \\ \left\langle x^{(n+1)}, x^{(1)}\right\rangle & \left\langle x^{(n+1)}, x^{(2)}\right\rangle & ... & \left\langle x^{(n+1)}, x^{(n+2)}\right\rangle \end{vmatrix} = 0. \qquad (12)$$

According to part d) of Proposition 9, the double determinant of the first minor R'_{n+1n+1} of

$$\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, x^{(n)}, x^{(n+1)}\right)(n+1)$$

is non-zero. Hence, the matrix corresponding to R'_{n+1n+1} is invertible. Therefore, according to the assertion in Lemma 1 and the case in above all elements of $\det\Gamma'\left(x, x^{(1)}, ..., x^{(n-1)}, x^{(n)}, x^{(n+1)}\right)(n+1)$, except from $\left\langle x^{(n)}, x^{(n+1)}\right\rangle$ are expressed by the elements of system (9). Thus, the polynomial $\left\langle x^{(n)}, x^{(n+1)}\right\rangle$ is also expressed in terms of the elements of system (9). The lemma is proved. □

An assertion in Theorem 11 follows from assertions of Lemma 2. Theorem 11 is proved. □

5. Conclusion

In conclusion, we can state the following corollary from Theorem 11.

Corollary 12. *The d-skew-field of $Sp(n)-$invariant d-rational functions over $\Re$ has a finite number of d-generators, and their number is equal to $2n$.*

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