# NEW BOUNDS FOR THE MAXIMAL EIGENVALUES
# OF POSITIVE DEFINITE MATRICES

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**Abstract:**   In this paper, we improve earlier bounds on the extremal eigenvalues of positive definite matrices by introducing an increasing function and by considering a vector function of the eigenvalues. For various choices of the monotonic function we are able to obtain bounds for the extremal eigenvalues in terms of the traces of the matrix and its powers. Our bounds are a function of two parameters achieved by using Jensen's inequality. These bounds are relatively simple to compute.

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**Key Words:**   positive definite matrix; eigenvalues; bounds

## 1. Introduction

The maximal eigenvalues of an Hermitian matrix $\mathbf{A}$ play an important part in communication analysis, solution of linear systems, spectral analysis and approximation. Of particular importance are the condition number, spectral radius $\rho(\mathbf{A})$ and spectral bounds. In such cases approximation to these suffice as for a large dense matrix it is inefficient to determine the eigenvalues exactly or

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numerically. Some methods of locating the spectrum are based on Gerschgorin disks, ovals of Cassini [2], Brauer bounds [3], interlacing property for Hermitian matrices and Rayleigh's quotient [7]. For positive definite Toeplitz matrices Dembo [5] provided useful bounds for the extremal eigenvalues. Wolkowicz and Styan [10] provided classical bounds using traces. Recently an interval containing the eigenvalues of real symmetric matrices was provided by Huang and Xu [8] using the trace($\mathbf{A}$) and trace($\mathbf{A}^2$). Here we improve the bounds previously found by Sharma et al. [9]

## 2. Theory

**Definition 1.** Let $\boldsymbol{\lambda} = (\lambda_i) \in \mathbf{R}^n$ be the vector of eigenvalues of a positive definite matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$ and $f : (0, \infty) \longrightarrow (0, \infty)$ be an increasing function. Order the eigenvalues such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Define $\mathbf{f}(\boldsymbol{\lambda}) = [f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)]^t$ and

$$m = \frac{\langle \mathbf{f}(\boldsymbol{\lambda}), \mathbf{e} \rangle}{n}$$
$$= \frac{\text{trace}(f(\mathbf{A}))}{n},$$

the average value of $\mathbf{f}(\boldsymbol{\lambda})$, where $\mathbf{e} = [1\ 1, \cdots, 1]^t$.

**Lemma 2.** *The following inequality is satisfied by* $\mathbf{f}(\boldsymbol{\lambda})$

$$\sum_{i=1}^{n} \langle \mathbf{f}(\boldsymbol{\lambda}) - m\mathbf{e}, \mathbf{e_i} \rangle^{2r} \geq \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1}} \langle f(\boldsymbol{\lambda}) - m\mathbf{e}, \mathbf{e_j} \rangle^{2r}, \qquad (1)$$

*where* $\mathbf{e_i}$ *is the unit vector in* $\mathbf{R}^n$, *with unity in the* $i_{th}$ *position and zeroes elsewhere,* $r \in \mathbb{N}$ *the natural numbers and* $\langle ., . \rangle$ *is the usual innerproduct.*

*Proof.*

$$\sum_{i=1}^{n} (f(\lambda_i) - m)^{2r} = (f(\lambda_j) - m)^{2r} + \sum_{\substack{i=1 \\ i \neq j}}^{n} (f(\lambda_i) - m)^{2r}. \qquad (2)$$

To obtain a lower bound on the second term in (2) we use Jensen's inequality [6]:

$$\left[\sum_{\substack{i=1 \\ i \neq j}}^{n}(f(\lambda_i) - m)^2\right]^r = (n-1)^r \left[\sum_{\substack{i=1 \\ i \neq j}}^{n}\frac{1}{n-1}(f(\lambda_i) - m)^2\right]^r$$

$$\leq (n-1)^{r-1}\sum_{\substack{i=1 \\ i \neq j}}^{n}(f(\lambda_i) - m)^{2r}. \tag{3}$$

Hence

$$\sum_{\substack{i=1 \\ i \neq j}}^{n}(f(\lambda_i) - m)^{2r} \geq \frac{1}{(n-1)^{r-1}}\left[\sum_{\substack{i=1 \\ i \neq j}}^{n}(f(\lambda_i) - m)^2\right]^r. \tag{4}$$

From the definition of $m$ we have

$$0 = \langle \mathbf{f}(\boldsymbol{\lambda}), \mathbf{e}\rangle - mn$$

$$= \sum_{i=1}^{n}(f(\lambda_i) - m)$$

$$= (f(\lambda_j) - m) + \sum_{\substack{i=1 \\ i \neq j}}^{n}(f(\lambda_i) - m). \tag{5}$$

Hence

$$|f(\lambda_j) - m| \leq \sum_{\substack{i=1 \\ i \neq j}}^{n}|f(\lambda_i) - m|. \tag{6}$$

Squaring (6) and applying Jensen's inequality [6] once again we obtain

$$(f(\lambda_j) - m)^2 \leq \left(\sum_{\substack{i=1 \\ i \neq j}}^{n}|f(\lambda_i) - m|\right)^2$$

$$= (n-1)^2 \left(\sum_{\substack{i=1 \\ i \neq j}}^{n}\frac{1}{n-1}|f(\lambda_i) - m|\right)^2$$

$$\leq (n-1) \sum_{\substack{i=1 \\ i \neq j}}^{n} (f(\lambda_i) - m)^2.$$

Hence

$$\left( \sum_{\substack{i=1 \\ i \neq j}}^{n} (f(\lambda_i) - m)^2 \right)^r \geq \frac{1}{(n-1)^r} (f(\lambda_j) - m)^{2r}. \tag{7}$$

Substitute (4) and (7) into (2) to finally get

$$\sum_{i=1}^{n} (f(\lambda_i) - m)^{2r} \geq \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1}} (f(\lambda_j) - m)^{2r} \tag{8}$$

which is equivalent to (1).                                                    □

**Theorem 3.**   *Define* $\mathbf{B} = f(\mathbf{A}) - m\mathbf{I}$, *then the extremal eigenvalue satisfies the bounds*

$$f(\lambda_1) \leq m + \left[ \frac{(n-1)^{2r-1}}{1 + (n-1)^{2r-1}} \ trace\left(\mathbf{B}^{2r}\right) \right]^{\frac{1}{2r}}, \tag{9}$$

$$f(\lambda_n) \geq m - \left[ \frac{(n-1)^{2r-1}}{1 + (n-1)^{2r-1}} \ trace\left(\mathbf{B}^{2r}\right) \right]^{\frac{1}{2r}}. \tag{10}$$

*Proof.* Substitute $j = 1$ and $j = m$ in (8) from Lemma 2 and use the fact that

$$\sum_{i=1}^{n} (f(\lambda_i) - m)^{2r} = \text{trace}(\mathbf{B}^{2r}).$$

□

**Theorem 4.**   *The lower and upper bounds for* $f(\lambda_1)$ *and* $f(\lambda_n)$ *are given by*

$$f(\lambda_1) \geq m + \frac{trace\left(\mathbf{B}^2\right)}{n} \left[ \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1} \ trace\left(\mathbf{B}^{2r}\right)} \right]^{\frac{1}{2r}}, \tag{11}$$

$$f(\lambda_n) \leq m - \frac{trace\left(\mathbf{B}^2\right)}{n} \left[ \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1} \ trace\left(\mathbf{B}^{2r}\right)} \right]^{\frac{1}{2r}}. \tag{12}$$

*Proof.* We use the fact that for real numbers $f(\lambda_i)$, $i = 1, 2, ..., n$ the variance satisfies the inequality ([1])

$$\frac{1}{n} \sum_{i=1}^{n} (f(\lambda_i) - m)^2 \leq (f(\lambda_1) - m)(m - f(\lambda_n)),$$

or

$$\frac{1}{n} \operatorname{trace}(\mathbf{B}^2) \leq (f(\lambda_1) - m)(m - f(\lambda_n)). \tag{13}$$

$\square$

We prove only (12) as (11) is proved similarly. From (9) and (13) we have

$$m - f(\lambda_n) \geq \frac{\operatorname{trace}(\mathbf{B}^2)}{n(f(\lambda_1) - m)}$$

$$\geq \frac{\operatorname{trace}(\mathbf{B}^2)}{n} \left[ \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1} \operatorname{trace}(\mathbf{B}^{2r})} \right]^{\frac{1}{2r}}$$

from which the result follows.

## 3. Results

**Example 5.** Consider the test matrix [10]

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}$$

with spectrum $\sigma(\mathbf{A}) = \{1.4257, 4.7754, 6.423, 9.3759\}$ and choose $f(x) = x^k$, $k \in \mathbb{N}$. We summarize results for $k, r = 1, 2, 3$ in Table 1. For $k = 1$, we obtain the bounds of [9]. If equation (10) gives a negative value for $f(\lambda_n)$ we do not take the $k_{th}$ root but rather bound $\lambda_n$ below by zero.

**Example 6.** Consider the positive definite difference matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

with spectrum $\sigma(\mathbf{A}) = \{0.3820, 1.3820, 2.6180, 3.6180\}$ We choose $f(x)$ as in example 5. Results are summarized for $k = 1, 2, 3$ and $r = 1, 2, 3, 4$ in Table 2.

Table 1: Bounds Example 5

| $k$ | $r$ | $\lambda_1$ | $\lambda_n$ |
|---|---|---|---|
| 1 | 1 | $[7.1583, 10.4749]$ | $[0.5251, 3.8417]$ |
|   | 2 | $[7.2586, 10.1912]$ | $[0.8088, 3.7414]$ |
|   | 3 | $[7.3473, 9.9659]$ | $[1.0341, 3.6527]$ |
| 2 | 1 | $[7.5375, 9.6666]$ | $(0, 4.4928]$ |
|   | 2 | $[7.5975, 9.5315]$ | $(0, 4.3907]$ |
|   | 3 | $[7.6403, 9.4409]$ | $(0, 4.3157]$ |
| 3 | 1 | $[7.8461, 9.4672]$ | $(0, 4.8978]$ |
|   | 2 | $[7.8718, 9.4151]$ | $(0, 4.8305]$ |
|   | 3 | $[7.8865, 9.3862]$ | $(0, 4.7913]$ |

For both examples it is clear that the bounds for $\lambda_1$ get better as $k$ and $r$ increases. Also for $\lambda_n$ the bounds are much better for $k = 1$ and larger values of $r$. However it may not be prudent to use large $k$ and large $r$ as the computation of powers of $\mathbf{A}$ may be too expensive. However for sparse matrices this is not an impediment. Also the usage of non polynomial functions is prohibited due to the complexity of evaluating $f(\mathbf{A})$.

## 4. Conclusion

We have provided useful bounds for the extremal eigenvalues of positive definite matrices based on two parameters. These bounds are a useful addition to the arsenal of tools already available to locate the spectrum.

## References

[1] R. Bhatia, C. Davis, A better bound on the variance, *Amer. Math. Monthly*, **107** (2000), 353–357.

[2] A. Brauer, Limits for the characteristic roots of a matrix VII, *Duke Math. J.*, **25** (1958), 583–590.

[3] A. Brauer, A. C. Mewbom, The greatest distance between two characteristic roots of a matrix, *Duke Math. J.*, **26** No 4 (1959), 653–661.

Table 2: Bounds Example 6

| $k$ | $r$ | $\lambda_1$ | $\lambda_n$ |
|---|---|---|---|
| 1 | 1 | $[2.7071, 4.1213]$ | $(0, 1.2929]$ |
| | 2 | $[2.7825, 3.9168]$ | $[0.0832, 1.2175]$ |
| | 3 | $[2.8260, 3.8159]$ | $[0.1841, 1.1740]$ |
| | 4 | $[2.8501, 3.7645]$ | $[0.2355, 1.1499]$ |
| 2 | 1 | $[2.8985, 3.7687]$ | $(0, 1.6121]$ |
| | 2 | $[2.9402, 3.6782]$ | $(0, 1.5347]$ |
| | 3 | $[2.9597, 3.6395]$ | $(0, 1.4968]$ |
| | 4 | $[2.9667, 3.6261]$ | $(0, 1.4828]$ |
| 3 | 1 | $[3.0316, 3.6739]$ | $(0, 1.8309]$ |
| | 2 | $[3.0499, 3.6377]$ | $(0, 1.7790]$ |
| | 3 | $[3.0577, 3.6227]$ | $(0, 1.7556]$ |
| | 4 | $[3.0597, 3.6191]$ | $(0, 1.7497]$ |

[4] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **20** (1967), 197–228.

[5] A. Dembo, Bounds on the extreme eigenvalues of positive definite Toeplitz matrices, *IEEE Trans. Inform. Theory*, **34** (1988), 352–355.

[6] X. Gao, M. Sitharam, A. Roitberg, Bounds on the Jensen gap, and implications for mean-concentrated distributions, *The Australian J. of Math. Anal. and Appl.*, **16**, No 2 (2019).

[7] R. Horn, C.A. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge (2012).

[8] T.Z. Huang, C.X. Xu, Bounds for the extreme eigenvalues of symmetric matrices, *ZAMM*, **83**, No 3 (2003), 214–216.

[9] R. Sharma, R. Kumar, R. Saini, Note on Bbounds for eigenvalues using traces, *arXiv:1409.0096v1* (Functional Analysis) (2014).

[10] H. Wolkowicz, G.P.H. Styan, Bounds for eigenvalues using traces, *Linear Algebra Appl.*, **29** (1980), 471–506.