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TRAVELLING WAVE SOLUTIONS FOR A DIFFUSIVE PREY-PREDATOR MODEL WITH ONE PREDATOR AND TWO PREYS

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Abstract: The paper discusses about a three-species diffusive prey predator system, in which the predator is a generalist that can thrive on two distinct prey species. The occurrence of invasion waves in the model is investigated in this paper. Schauder's fixed point theorem proves the existence of travelling semi fronts $\omega > \omega^*$, where ω is the minimal wave speed. Harnack's inequality for positive super solutions on $\mathbb{R}$ is established to solve the reducibility problem that arises in the proofs. The boundedness is demonstrated and LaSalle's invariance principle proves that such waves connect coexistence equilibrium. The rescaling method and limit arguments are used to determine the existence of a travelling front with $\omega = \omega^*$. The Laplace transform demonstrates the nonexistence of moving fronts with speed $0 < \omega < \omega*$.

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1. Introduction

In ecology, the predator-prey model has become a popular topic of study [1],

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[2], [3]. Lotka and Volterra developed the first order differential equations of the predator-prey type in 1925 and 1926, respectively. An example of a simple Lotka-Volterra model ordinary differential equation is shown below,

$$\begin{aligned}\frac{dm_1}{dt} &= c_1 m_1 - \gamma m_1 n,\\ \frac{dn}{dt} &= \delta m_1 n - \rho n,\end{aligned}$$

where $\delta = w_1\gamma$. With a logistic (or Pearl-Verhulst) growth limitation on the prey, Lotka-Volterra equations becomes,

$$\begin{aligned}\frac{dm_1}{dt} &= c_1 m_1 \left(1 - \frac{m_1}{\mathbb{K}_1}\right) - \gamma m_1 n,\\ \frac{dn}{dt} &= \delta m_1 n - \rho n.\end{aligned}$$

These are the most basic mathematical models of predator-prey interactions. Because of their importance in science and engineering, systems of nonlinear PDE's have received a lot of interest in recent years. The reaction-diffusion systems are among the most commonly confronted types of parabolic partial differential equations.

Mathematicians have given importance to spatiotemporal patterns and their interactions [4]. A simple diffusion-reaction model can be obtained by introducing a diffusive term and carrying capacity,

$$\begin{aligned}\frac{\partial m_1}{\partial t} &= D_1 \frac{\partial^2 m_1}{\partial x^2} + c_1 m_1 \left(1 - \frac{m_1}{\mathbb{K}_1}\right) - \alpha m_1 m_2 - \gamma m_1 n,\\ \frac{\partial m_2}{\partial t} &= D_2 \frac{\partial^2 m_2}{\partial x^2} + c_2 m_2 \left(1 - \frac{m_2}{\mathbb{K}_2}\right) - \beta m_1 m_2 - \delta m_2 n,\\ \frac{\partial n}{\partial t} &= w_1 \gamma m_1 n + w_2 \delta m_2 n - \rho n.\end{aligned} \tag{1}$$

The description of parameters is given in Table 1. The focus of this research is to prove that a travelling wave solution exists for a large class of non-monotone reaction diffusion systems. Many researchers have investigated the existence of travelling wave solutions in certain case studies of the prey-predator model. Gardner [7] used a variation of the Conley index to prove the existence of travelling wave solutions connecting stable spatially homogenous solutions. Dunbar [5], [6] studied on whether a diffusive model may include travelling wave solutions.

Table 1: Description of Parameters

Parameter	Description
$m_1(t)$	Prey 1 population
$m_2(t)$	Prey 2 population
$n(t)$	Predator population
c_1, c_2	Intrinsic growth rate
$\mathbb{K}_1, \mathbb{K}_2$	Carrying capacity of preys
D_1, D_2	Diffusion coefficients of two preys
D_3	Diffusion coefficient of predator
α,β	Competition rates between two preys
γ, δ	The rate of predation on preys by predator
w_1, w_2	Conversion efficiencies of preys
ρ	Predator's natural death rate

Now we will look at how to review for the existence of travelling wave solutions. Liang and Zhao [8], Wu and Zou [9] provides the existence of travelling wave solutions for monotonic systems. Dunbar [10] proposed the shooting method, which is widely used to illustrate the presence of non-monotonic systems. Furthermore, Huang [11] is the developer of this method, which substantially simplifies its implementation. The Schauder's fixed-point theorem is used to demonstrate the existence of travelling wave solutions [12], [13], [14], [15]. However, generating a pair of adequate upper–lower solutions for the application of Schauder's fixed-point theorem to non-monotonic models is difficult.

Model (1) has four equlibrium points,

$$
\begin{aligned}
&E_0 = (0, 0, 0),\\
&E_1 = (\mathbb{K}_1, 0, 0),\\
&E_2 = (0, \mathbb{K}_2, 0)\,,\\
&E_* = \left(\frac{c_1\mathbb{K}_1c_2 + \alpha\mathbb{K}_1\mathbb{K}_2c_2}{c_1c_2 - \alpha\beta\mathbb{K}_1\mathbb{K}_2}, \frac{c_1\mathbb{K}_1c_2 - \beta\mathbb{K}_1\mathbb{K}_2c_1}{c_1c_2 - \alpha\beta\mathbb{K}_1\mathbb{K}_2}, 0\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
(A) =& c_1 - \frac{\gamma c_2}{\delta}\left(1 - \frac{\rho}{w_2\delta\mathbb{K}_2}\right) > 0,\\
& c_2 - \frac{\delta c_1}{\gamma}\left(1 - \frac{\rho}{w_1\gamma\mathbb{K}_1}\right) > 0, \gamma w_1\mathbb{K}_1 + \delta w_2\mathbb{K}_2 > \rho.
\end{aligned}
$$

Then it implies that if and only if (A) holds, system (1) yields a unique positive equilibrium $E^* = (m_1^*, m_2^*, n^*)$, where

$$
\begin{aligned}
m_1^* &= \frac{\mathbb{K}_1 \left(\gamma c_2 \rho + \mathbb{K}_2 \delta \left(c_1 w_2 \delta - \gamma c_2 w_2 - \alpha \rho\right)\right)}{\alpha \gamma w_1 \mathbb{K}_1 \mathbb{K}_2 \delta + \beta \gamma w_2 \mathbb{K}_1 \mathbb{K}_2 \delta - \gamma^2 c_2 w_1 \mathbb{K}_1 - \delta^2 c_1 w_2 \mathbb{K}_2}, \\
m_2^* &= \frac{\mathbb{K}_2 \left(\delta c_1 \rho + \mathbb{K}_1 \gamma \left(c_2 w_1 \gamma - \delta c_1 w_1 - \beta \rho\right)\right)}{\alpha \gamma w_1 \mathbb{K}_1 \mathbb{K}_2 \delta + \beta \gamma w_2 \mathbb{K}_1 \mathbb{K}_2 \delta - \gamma^2 c_2 w_1 \mathbb{K}_1 - \delta^2 c_1 w_2 \mathbb{K}_2}, \\
n^* &= \frac{c_1 c_2 \left(\gamma w_1 \mathbb{K}_1 + \delta w_2 \mathbb{K}_2 - \rho\right) + \mathbb{K}_1 \mathbb{K}_2 \left(\alpha \beta \rho - \alpha \gamma c_2 w_1 - \beta \delta c_1 w_2\right)}{\alpha \gamma w_1 \mathbb{K}_1 \mathbb{K}_2 \delta + \beta \gamma w_2 \mathbb{K}_1 \mathbb{K}_2 \delta - \gamma^2 c_2 w_1 \mathbb{K}_1 - \delta^2 c_1 w_2 \mathbb{K}_2}.
\end{aligned}
\tag{2}
$$

The paper is organized as follows: We verify the existence of travelling semi fronts for $\omega > \omega^*$ in Section 3. The existence of a travelling front with wave speed $\omega = \omega^*$ is determined in Section 4. The non-existence of moving fronts is demonstrated in Section 5 using the Laplace Transform.

2. Main results

The solutions of (1) are of the form

$$
\begin{aligned}
m_1 (, t) &= M_1 (+\omega t), m_2 (, t) = M_2 (+\omega t), \\
n (, t) &= N (+\omega t).
\end{aligned}
$$

The wave speed parameter ω is positive and the functions m_1, m_2, n are functions of a single variable $\tau = +\omega t$. If $M_1(\tau), M_2(\tau), N(\tau)$ are defined for all $\tau \in \mathbb{R}$ and are referred to as a semi travelling wave solution connected to the equilibrium E_* and is called travelling front if

$$
\begin{aligned}
\lim_{\tau \to +\infty} \left(M_1(\tau), M_2(\tau), N(\tau)\right) = E^*, \\
\lim_{\tau \to -\infty} \left(M_1(\tau), M_2(\tau), N(\tau)\right) = E_*.
\end{aligned}
$$

Define $\omega^* = 2\sqrt{D_3 \left(w_1 \gamma \mathbb{K}_1 + w_2 \delta \mathbb{K}_2 - \rho\right)}$.

Theorem 1. *(i) System* (1) *has no travelling semi front for any* ω*, where* $0 < \omega < \omega^*$.

(ii) System (1) *has a travelling front with wave speed* ω *if and only if* $\omega \geq \omega^*$.

3. Existence of travelling semi fronts when $\omega \geq \omega^*$

In this section, we assume that $\omega > \omega^*$ and study the existence of traveling semifronts with wave speed ω.

We get the system for $(M_1(\tau), M_2(\tau), N(\tau))$ with $\tau = +\omega t$ as a result of substituting into the system (1),

$$\begin{aligned} D_1 M''_1 - \omega M'_1 + c_1 M_1 \left(1 - \frac{M_1}{\mathbb{K}_1}\right) - \alpha M_1 M_2 - \gamma M_1 N = 0, \\ D_2 M''_2 - \omega M'_2 + c_2 M_2 \left(1 - \frac{M_2}{\mathbb{K}_2}\right) - \beta M_1 M_2 - \delta M_2 N = 0, \\ D_3 N'' - \omega N' + w_1 \gamma M_1 N + w_2 \delta M_2 N - \rho N = 0. \end{aligned} \tag{3}$$

The two roots of the following equation $Z(\Lambda)$ are denoted by $\Lambda_1,\ \Lambda_2$,

$$Z(\Lambda) = D_3 \Lambda^2 - \omega \Lambda + (w_1 \gamma \mathbb{K}_1 + w_2 \delta \mathbb{K}_2 - \rho) = 0. \tag{4}$$

Then if $0 < \xi < \Lambda_2 - \Lambda_1$, it follows from (A) and $\omega > \omega^*$ that $0 < \Lambda_1 < \Lambda_2$ and $Z(\Lambda_1 + \xi) < 0$. We constructed the following upper lower solutions [16], [17].

Let

$$\bar{M}_1 = \mathbb{K}_1,\ \bar{M}_2 = \mathbb{K}_2,\ \bar{N} = e^{\Lambda_1 \tau},\ \underline{M}_1 = \max\left\{\mathbb{K}_1 - \rho_1 e^{\psi_1 \tau}, 0\right\},$$

$$\underline{M}_2 = \max\left\{\mathbb{K}_2 - \rho_2 e^{\psi_2 \tau}, 0\right\}, \underline{N} = \max\left\{e^{\psi_1 \tau}\left(1 - \rho_3 e^{\psi_1 \tau}\right), 0\right\},$$

where ρ_i and ψ_i (i=1, 2, 3) are positive constants that will be found later. In the following lemma, we will demonstrate that these are upper lower solutions.

Lemma 1. *We have*
$D_3 \bar{N}'' - \omega \bar{N}' + w_1 \gamma \bar{M}_1 \bar{N} + w_2 \delta \bar{M}_2 \bar{N} - \rho \bar{N} \leq 0$ *for any* $\tau \in \mathbb{R}$.

Proof. According to the definition of Λ_1,

$$\begin{aligned} & D_3 \bar{N}'' - \omega \bar{N}' + w_1 \gamma \bar{M}_1 \bar{N} + w_2 \delta \bar{M}_2 \bar{N} - \rho \bar{N} \\ &= e^{\Lambda_1 \tau}\left(D_3 \Lambda_1^2 - \omega \Lambda_1 + w_1 \gamma \mathbb{K}_1 + w_2 \delta \mathbb{K}_2 - \rho\right) \\ &= 0. \end{aligned}$$

Hence the lemma is proved. □

Lemma 2. *Suppose*

$$0<\psi_1<\min\left\{\Lambda_1,\frac{\omega}{D_1}\right\},\ 0<\psi_2<\min\left\{\Lambda_1,\frac{\omega}{D_1}\right\},$$
$$\rho_1>\max\left\{\mathbb{K}_1,\frac{\gamma\mathbb{K}_1}{\psi_1\left(\omega-D_1\psi_1\right)}\right\},\ \rho_2>\max\left\{\mathbb{K}_2,\frac{\delta\mathbb{K}_2}{\psi_2\left(\omega-D_2\psi_2\right)}\right\}.$$

Then,

$$\begin{aligned}
&D_1\underline{M}''_1-\omega\underline{M}'_1+c_1M_1\left(1-\frac{\underline{M}_1}{\mathbb{K}_1}\right)-\alpha\underline{M}_1\underline{M}_2-\gamma\underline{M}_1\bar{N}\\
&\qquad\geq 0,\ \forall\tau\neq\frac{1}{\psi_1}\ln\frac{\mathbb{K}_1}{\rho_1}=\tau_1,\\
&D_2\underline{M}''_2-\omega\underline{M}'_2+c_2\underline{M}_2\left(1-\frac{\underline{M}_2}{\mathbb{K}_2}\right)-\beta\underline{M}_1\underline{M}_2-\delta\underline{M}_2\bar{N}\\
&\qquad\geq 0,\ \forall\tau\neq\frac{1}{\psi_2}\ln\frac{\mathbb{K}_2}{\rho_2}=\tau_2.
\end{aligned}$$

Proof. The choice of ρ_1 and ρ_2 clearly indicates that $\tau_1<0$, $\tau_2<0$. We will begin with $\underline{M}_1$. It is obvious that the result is true for $\tau>\tau_1$. Suppose that $\tau<\tau_1<0$ as a result we obtain, $\underline{M}_1(\tau)=\mathbb{K}_1-\rho_1e^{\psi_1\tau}>0$. Hence,

$$\begin{aligned}
&D_1\underline{M}''_1-\omega\underline{M}'_1+c_1\underline{M}_1\left(1-\frac{\underline{M}_1}{\mathbb{K}_1}\right)-\alpha\underline{M}_1\underline{M}_2-\gamma\underline{M}_1\bar{N}\\
&\quad=e^{\psi_1\tau}\rho_1\psi_1\left(\omega-D_1\psi_1\right)+c_1\left(\mathbb{K}_1-\rho_1e^{\psi_1\tau}\right)\left(\frac{\rho_1e^{\psi_1\tau}}{\mathbb{K}_1}\right)\\
&\quad-\alpha\left(\mathbb{K}_1-\rho_1e^{\psi_1\tau}\right)\left(\mathbb{K}_2-\rho_2e^{\psi_2\tau}\right)-\gamma\left(\mathbb{K}_1-\rho_1e^{\psi_1\tau}\right)e^{\Lambda_1\tau}\\
&\quad\geq e^{\psi_1\tau}\rho_1\psi_1\left(\omega-D_1\psi_1\right)-\gamma\mathbb{K}_1e^{\Lambda_1\tau}\\
&\quad\geq e^{\psi_1\tau}\left(\rho_1\psi_1\left(\omega-D_1\psi_1\right)-\gamma\mathbb{K}_1\right)\\
&\quad\geq 0.
\end{aligned}$$

For the second inequality, the facts $\tau<0$ and $\Lambda_1>\psi_1>0$ are used. Similarly, let $\underline{M}_2(\tau) = \mathbb{K}_2 - \rho_2 e^{\psi_2 \tau} > 0$.

$$\begin{aligned}
D_2\underline{M}''_2 - \omega\underline{M}'_2 + c_2\underline{M}_2\left(1 - \frac{\underline{M}_2}{\mathbb{K}_2}\right) - \beta\underline{M}_1\underline{M}_2 - \delta\underline{M}_2\bar{N}& \\
= e^{\psi_2\tau}\rho_2\psi_2(\omega - D_2\psi_2) + c_2\left(\mathbb{K}_2 - \rho_2 e^{\psi_2\tau}\right)\left(\frac{\rho_2 e^{\psi_2\tau}}{\mathbb{K}_2}\right)& \\
- \alpha\left(\mathbb{K}_1 - \rho_1 e^{\psi_1\tau}\right)\left(\mathbb{K}_2 - \rho_2 e^{\psi_2\tau}\right) - \gamma\left(\mathbb{K}_2 - \rho_2 e^{\psi_2\tau}\right)e^{\Lambda_2\tau}& \\
\geq e^{\psi_2\tau}\rho_2\psi_2(\omega - D_2\psi_2) - \gamma\mathbb{K}_2 e^{\Lambda_2\tau}& \\
\geq e^{\psi_2\tau}\left(\rho_2\psi_2(\omega - D_2\psi_2) - \gamma\mathbb{K}_2\right)& \\
\geq 0.&
\end{aligned}$$

□

Lemma 3. *Let us use Lemma 2 to determine τ_1, τ_2 and suppose that,*

$$\begin{aligned}
&0 < \psi_3 < \min\{\psi_1, \psi_2, \Lambda_2 - \Lambda_1\}, \\
&\rho_3 > \max\left\{\left(\frac{\mathbb{K}_1}{\rho_1}\right)^{-\frac{\psi_3}{\psi_1}}, \left(\frac{\mathbb{K}_2}{\rho_2}\right)^{-\frac{\psi_3}{\psi_2}}, \frac{w_1\gamma\rho_1 + w_2\delta\rho_2}{-z(\Lambda_1 + \psi_3)}\right\}.
\end{aligned}$$

where $Z(\Lambda) = D_3\Lambda^2 - \omega\Lambda + (w_1\gamma\mathbb{K}_1 + w_2\delta\mathbb{K}_2 - \rho) = 0$. *Then,*

$$D_3N'' - \omega N' + w_1\gamma M_1 N + w_2\delta M_2 N - \rho N \geq 0,$$

for any $\tau \neq \ln(1/\rho_3)/\psi_3 = \tau_3$.

Proof. It is obvious that $\tau_3 < \min\{\tau_1, \tau_2\} < 0$ and $Z(\lambda_1 + \psi_3) < 0$ as a result of the choice of ρ_3. The conclusion is correct for $\tau > \tau_3$. We will assume $\tau < \tau_3$ in the following. Then,

$$\begin{aligned}
M_1(\tau) &= \mathbb{K}_1 - \rho_1 e^{\psi_1\tau}, \\
M_2(\tau) &= \mathbb{K}_2 - \rho_2 e^{\psi_2\tau}, \\
N(\tau) &= e^{\Lambda_1\tau}\left(1 - \rho_3 e^{\psi_3\tau}\right),
\end{aligned}$$

implies that $0 < 1 - \rho_3 e^{\psi_3\tau} < 1$. As a result of ρ_3 and considering that $Z(\Lambda_1) = 0$.

We get

$$\begin{aligned}
D_3\underline{N}'' &- \omega\underline{N}' + w_1\gamma\underline{M_1}\underline{N} + w_2\delta\underline{M_2}\underline{N} - \rho\underline{N} \\
&= e^{\Lambda_1\tau}Z(\Lambda_1) - \rho_3 e^{(\Lambda_1+\psi_3)\tau}Z(\Lambda_1+\psi_3) \\
&+ w_1\gamma\rho_1 e^{(\Lambda_1+\psi_1)\tau}\left(\rho_3 e^{\psi_3\tau}-1\right) + w_2\delta\rho_2 e^{(\Lambda_1+\psi_2)\tau}\left(\rho_3 e^{\psi_3\tau}-1\right) \\
&\geq -\rho_3 e^{(\Lambda_1+\psi_3)\tau}Z(\Lambda_1+\psi_3) - (w_1\gamma\rho_1 + w_2\delta\rho_2)\, e^{(\Lambda_1+\psi_3)\tau} \\
&\geq 0,
\end{aligned} \tag{5}$$

where the facts $\psi_3 < \psi_1, \psi_3 < \psi_2, \tau < 0$ were utilised in the 2^{nd} inequality. $\square$

Remark 2. ρ_i and ψ_i $(i = 1, 2, 3)$ can obviously be selected to satisfy Lemmas 1 - 3.

3.1. A truncated problem

We present a truncated problem with a finite interval in order to use Schauder's fixed-point theorem. Using the limiting procedure and the solution of the truncated problem, we may derive a solution (M_1, M_2, N) of system (3) satisfying $(M_1, M_2, N)(-\infty) = E_* = \left(\frac{c_1\mathbb{K}_1c_2-\alpha\mathbb{K}_1\mathbb{K}_2c_2}{c_1c_2-\alpha\beta\mathbb{K}_1\mathbb{K}_2}, \frac{c_1\mathbb{K}_2c_2-\beta\mathbb{K}_1\mathbb{K}_2c_1}{c_1c_2-\alpha\beta\mathbb{K}_1\mathbb{K}_2}, 0\right)$, which could be a strong choice for travelling wave solutions of system (3). Assume $l >$ -min $\tau_1, \tau_2, \tau_3 > 0$. Consider the situation below:

$$\begin{aligned}
D_1M''_1 - \omega M'_1 + c_1M_1\left(1 - \frac{M_1}{\mathbb{K}_1}\right) - \alpha M_1M_2 - \gamma M_1N = 0, \tau \in (-l, l), \\
D_2M''_2 - \omega M'_2 + c_2M_2\left(1 - \frac{M_2}{\mathbb{K}_2}\right) - \beta M_1M_2 - \delta M_2N = 0, \tau \in (-l, l), \\
D_3N'' - \omega N' + w_1\gamma M_1N + w_2\delta M_2N - \rho N = 0, \tau \in (-l, l), \\
M_1(\tau) = \underline{M}_1(\tau), M_2(\tau) = \underline{M}_2(\tau), N(\tau) = \underline{N}(\tau), \tau \in (-l, l).
\end{aligned} \tag{6}$$

Set $I_l = [-l, l], B = \mathbb{C}\left(I_l, \mathbb{R}^3\right)$ for ease of use and then define

$$\Omega = \Big\{ (M_1, M_2, N) \in B : \underline{M}_1 \leq M_1 \leq \bar{M}_1, \underline{M}_2 \leq M_2 \leq \bar{M}_2, \\ \underline{N} \leq N \leq \bar{N} \text{ in } I_l \Big\},$$

which is a closed convex set in the Banach space B with the norm $\|(M_1, M_2, N)\|_B = \|M_1\|_{\mathbb{C}(I_l)} + \|M_2\|_{\mathbb{C}(I_2)} + \|N\|_{\mathbb{C}(I_l)}$.

M_1, M_2 and N are non negative $(i.e)\, M_1 \geq 0, M_2 \geq 0, N \geq 0$ for any (M_1, M_2, N). The mapping $G : \Omega \to B$ is then defined as follows. For all $(x_0, y_0, z_0) \in \Omega$, $G(x_0, y_0, z_0) = (x, y, z)$ where (x, y, z) is the classic solution of the following boundary value problem:

$$\begin{aligned} D_1 x'' - \omega x' + c_1 x_0 - \left(\frac{c_1 x_0}{\mathbb{K}_1} + \gamma z_0\right) x - \alpha x y_0 = 0, \tau \in (-l, l)\,, \\ D_2 y'' - \omega y' + c_2 y_0 - \left(\frac{c_2 y_0}{\mathbb{K}_2} + \delta z_0\right) y - \beta x_0 y = 0, \tau \in (-l, l)\,, \\ D_3 z'' - \omega z' + w_1 \gamma x_0 z_0 + w_2 \delta y_0 z_0 - \rho z_0 = 0, \tau \in (-l, l)\,, \\ x(\tau) = \underline{M}_1(\tau)\,, y(\tau) = \underline{M}_2(\tau)\,, z(\tau) = \underline{N}(\tau)\,, \tau \in (-l, l)\,. \end{aligned} \tag{7}$$

As a result of G's specification, any fixed point of G is a solution to the problem (6). The existence and uniqueness of solutions can then be determined using [[18], Chap. 12 Theorem 3.1].

Lemma 4. $G(\Omega) \subset \Omega$.

Proof. Assume that the solution of the system (7) with $(x_0, y_0, z_0) \in \Omega$ is (x, y, z).

From the equation one of (7), we get

$$D_1 x'' - \omega x' - \left(\frac{c_1 x_0}{\mathbb{K}_1} + \gamma z_0\right) x - \alpha x y_0 = -c_1 x_0 \leq 0,\ \tau \in (-l, l)\,.$$

Since $l > -\min\{\tau_1, \tau_2, \tau_3\} > 0$, then $x(l) = 0$ and $x(-l) = \underline{M}_1(-l) > 0$. As a result, for any $\tau \in (-l, l)$, the maximum principle returns $x(\tau) > 0$. Similarly, for any $\tau \in (-l, l)$ we get $y(\tau) > 0$, $z(\tau) > 0$.

Let $\sigma_1 = \mathbb{K}_1 - x_1$, then we get

$D_1 \sigma_1'' - \omega \sigma_1' - \left(\frac{c_1 x_0}{\mathbb{K}_1} + \gamma z_0\right) \sigma_1 - \alpha \sigma_1 y_0 = -e_1 \mathbb{K}_1 z_0 \leq 0, \tau \in (-l, l)$ and $\sigma_1(\pm l) > 0$. Then as a result, the maximal principle yields $\sigma_1 \geq 0$ for all $\tau \in I_l$, it implies that $x \leq \bar{M}_1$ for all $\tau \in I_l$. We may deduce that $y \leq \bar{M}_2$ for all $\tau \in I_l$.

As a result of third equation of (7),

$$D_3 z'' - \omega z' + w_1 \gamma x_0 z_0 + w_2 \delta y_0 z_0 - \rho z_0 \geq 0, \tau \in (-l, l)\,.$$

If $\sigma_2 = \bar{N} - z$, then $\sigma_2(\pm l) > 0$ which follows from Lemma 3 that $D_3 \sigma_2'' - \omega \sigma_2' - \rho \sigma \leq 0, \tau \in (-l, l)$. As a result, the maximal principle returns $\sigma_2 \geq 0\ \forall\ \tau \in I_l$, implying that $z \geq \bar{N}$ for all $\tau \in I_l$. If $\sigma_3 = z - \underline{M}_1$, $\sigma_3(-l) = 0$ and $\sigma_3(\tau_1) = x(\tau_1) > 0$. As a result of Lemma 2, it follows that,

$$D_1 \sigma_3'' - \omega \sigma_3' - \left(\frac{c_1 x_0}{\mathbb{K}_1} + \gamma z_0\right) \sigma_3 - \alpha \sigma_1 y_0 \leq c_1 (1 - x_0) \left(1 - \frac{1}{\mathbb{K}_1}\right)$$

$$+e_{11}\left(z_0-\bar{N}\right)\leq 0,\forall\tau\in(-l,\tau_1). \tag{8}$$

As a result, the maximum principle implies $\sigma_3\geq 0\ \forall\ \tau\in[-l,\tau_1]$. Given that $\underline{M}_1=0\ \forall\ \tau>\tau_1$, it follows that $z\geq M_1$ for all $\tau\in I_l$. It is simple to derive that a $y\geq\underline{M}_2, z\geq\underline{N}\ \forall\ \tau\in I_l$ using comparable reasoning and Lemmas 2 and 3. The proof is now complete. □

3.2. A few supplementary lemmas

Here we construct some estimations for solutions to the nonhomogeneous linear equation

$$\sigma''(\tau)-P\sigma'(\tau)+h(\tau)\sigma(\tau)=f(\tau). \tag{9}$$

The following lemma is required for this.

Lemma 5. *Assume that Φ_1 and Φ_2 are the unique solutions to the 2^{nd}-order linear equation,*

$$\mathbb{L}[t]:=u''-Pu'+h(\tau)u=0,$$

where P and h are positive constant and continuous function on $[c,d]$ respectively, with

$$\varphi_1(c)=0,\varphi'_1(c)=1, \tag{10}$$

and

$$\varphi_2(d)=0,\varphi'_2(d)=-1. \tag{11}$$

As a result, we have

$$|\varphi_1(\tau)|+\left|\varphi'_1(\tau)\right|\leq e^{(X_1+P+1)(d-c)}, \tag{12}$$

and

$$|\varphi_2(\tau)|+\left|\varphi'_2(\tau)\right|\leq e^{(X_1+P+1)(d-c)}. \tag{13}$$

where $X_1=\|h\|_{\mathbb{C}[c,d]}$ for every $\tau\in(c,d)$. If $h\leq 0$, the Wronskian of φ_1 and φ_2, denoted by $\mathbb{W}(\phi_1,\phi_2)$, may be derived for all $\tau\in(c,d)$ by

$$|\mathbb{W}(\phi_1,\phi_2)(\tau)|\geq\frac{1}{P}\left(e^{P(d-c)}-1\right)>0. \tag{14}$$

Proof. To verify (11) and (12), rephrase $L[u]=0$ as a first order system

$$U'=R(\tau)U, \tag{15}$$

where

$$U = \begin{bmatrix} u \\ u' \end{bmatrix} \text{ and } R(\tau) = \begin{bmatrix} 0 & 1 \\ -h(\tau) & P \end{bmatrix}. \tag{16}$$

Let $\tau \in (c, d)$. It is clear that every U of (14) solution fulfils the integral equation

$$U(\tau) = U(c) + \int_c^\tau R(\varsigma) U(\varsigma)\, d\varsigma, \tag{17}$$

and hence

$$\|U(\tau)\| \le \|U(c)\| (X_1 + P + 1) \int_c^\tau \|U(\varsigma)\| d\varsigma,$$

where $\|\cdot\|$ is the absolute norm, and we have used the idea that $\|R(\cdot)\| = \max\{|h(\cdot)|, P+1\} \le X_1 + P + 1$. As a result, it is simple to deduce that

$$\|U(\tau)\| \le \|U(c)\| e^{(X_1+P+1)(\tau-c)}. \tag{18}$$

We also get it by replacing c by d in (15) and by similar arguments we get,

$$\|U(\tau)\| \le \|U(d)\| e^{(X_1+P+1)(d-\tau)}. \tag{19}$$

If we choose $U = \begin{bmatrix} \phi_1 & \phi_1' \end{bmatrix}^T$, we get $\|U\| = |\phi_1(\tau)| + |\phi_1{}'(\tau)|$ and $\|U(c)\| = 1$. As a result, (11) follows directly after (16). By selecting $U = \begin{bmatrix} \phi_2 & \phi_2' \end{bmatrix}^T$ in (17) in the same way, we get (12).

Now we have to prove (13). We derive

$$\mathbb{W}(\varphi_1, \varphi_2)(\tau) = -\varphi_1(d)\, e^{P(d-\tau)}, \tag{20}$$

by using Abel's formula and noticing that $\mathbb{W}(\varphi_1, \varphi_2)(d) = -\varphi_1(d)$.

We use the function to estimate $\varphi_1(b)$,

$$\varrho(\tau) = \frac{1}{P}\left(e^{P(\tau-c)} - 1\right).$$

This is the unique solution to $\varrho'' - \mathrm{P}\varrho' = 0$ on $[c, d]$ with

$$\varrho(a) = 0, \varrho'(a) = 1.$$

Noting that $\varrho \ge 0$ $h \le 0$, we can see that the function $\varpi := \varphi_1 - \varrho$ satisfies

$$\varpi'' - P\varpi' + h\varpi = -h\varrho \ge 0 \text{ and } \varpi(c) = \varpi'(d).$$

As a result of [20, Theorem 13, Chapter 1], $\varpi \geq 0$ and so $\varphi_1 \geq \varrho$ on $[c,d]$. In general,

$$\varphi_1(d) \geq \varrho(d) = \frac{1}{P}\left(e^{P(d-c)} - 1\right) > 0. \tag{21}$$

We ultimately get,

$$|\mathbb{W}(\varphi_1, \varphi_2)(\tau)| = \varphi_1(d)\, e^{P(d-\tau)} \geq \frac{1}{P}\left(e^{P(d-c)} - 1\right) > 0,$$

by combining (18) and (19). As a result, the proof of this lemma is complete. $\square$

For a priori estimates, we have the following two lemmas.

Lemma 6. *Let P be a positive constant and let h and f be continuous functions on $[c,d]$. Suppose that $\Upsilon \in \mathbb{C}([c,d]) \cap \mathbb{C}^2((c,\mathrm{d}))$ satisfies the differential equation* (9) *in (c,d) and $\Upsilon(a) = \Upsilon(b) = 0$. If*

$$-X_1 \leq h \leq 0 \quad \textit{and} \quad |f| \leq X_2 \ \textit{on}\, [c,d],$$

for some constants X_1, X_2, then there exists a positive constant X_3, depending only on P, X_1 and the length of the interval $[c,d]$, such that

$$\|\sigma\|_{\mathbb{C}([c,d])} \leq X_2 X_3. \tag{22}$$

Proof. To begin, we should note that both $\sigma'(c+) := \lim\limits_{\sigma \to c+} \sigma'(\tau)$ and $\sigma'(d-) := \lim\limits_{\sigma \to d-} \sigma'(\tau)$ exists. Let $\bar{\tau} \in (c,d)$ be fixed to see this. We derive

$$\sigma'(\tau) = \sigma'(\bar{\tau}) - \mathrm{P}(\sigma(\tau) - \sigma(\bar{\tau})) - \int_{\bar{\tau}}^{\tau} h(\varsigma)\,\sigma(\varsigma)\,d\varsigma + f(\varsigma)\,d\varsigma,$$

for any $\tau \in (c,d)$ by integrating equation (8) from $\bar{\tau}$ to τ and recombining the resulting equation.

As σ, h and f are right-continuous at point c, the right-hand limit of the above equation exists at d. As a result $\sigma'(c+)$ exists. Using the same argument as before, we can conclude that $\sigma'(d-)$ exists. Let φ_1 and φ_2 be the same as in Lemma 5. Let us consider any point such that $\tau \in (c,d)$. By multiplying (8) by φ_1, integrating the resulting inequality from c to τ, and then using part by part integration, we get

$$\varphi_1(\tau)\,\sigma'(\tau) - \left(\varphi'(\tau) - P\varphi_1(\tau)\right)\sigma(\tau) + \sigma(c) = \int_{c}^{\tau} \varphi_1(\varsigma)\, f(\varsigma)\, d\varsigma, \tag{23}$$

where we used (9), and the fact that $\mathbb{L}[\varphi_1] = 0$ and $\sigma'(c+)$ exists. Similarly, we can deduce that

$$-\varphi_2(\tau)\sigma'(\tau) - \left(\varphi'(\tau) - P\varphi_2(\tau)\right)\sigma(\tau) + \sigma(d) = \int\limits_{\tau}^{d} \varphi_2(\varsigma) f(\varsigma)\, d\varsigma, \qquad (24)$$

by multiplying (8) by φ_2, integrating the resulting inequality from τ to d, and then using integration by parts, where (10) has been used and the fact that $\mathbb{L}[\varphi_2] = 0$ and $\sigma'(d-)$ exists.

It is important to note that $\sigma(a) = \sigma(b) = 0$. We finally get,

$$\mathbb{W}(\varphi_1, \varphi_2)(\tau)\sigma(\tau) = \varphi_2(\tau)\int\limits_{c}^{\tau} \varphi_1(\varsigma) f(\varsigma)\, d\varsigma + \varphi_1(\tau)\int\limits_{\tau}^{d} \varphi_2(\varsigma) f(\varsigma)\, d\varsigma,$$

multiplying (21) and (22) by φ_2 and φ_1, respectively, and then summing, we get

$$\sigma(\tau) = \frac{\varphi_2(\tau)\int\limits_{c}^{\tau} \varphi_1(\varsigma) f(\varsigma)\, d\varsigma + \varphi_1(\tau)\int\limits_{\tau}^{d} \varphi_2(\varsigma) f(\varsigma)\, d\varsigma}{\mathbb{W}(\varphi_1, \varphi_2)(\tau)}.$$

This, in accordance with the assumption that $|f| \leq X_2$, implies that

$$|\sigma(\tau)| \leq X_2 \frac{|\varphi_2(\tau)|\int\limits_{c}^{d} |\varphi_1(\varsigma)| d\varsigma + |\varphi_1(\tau)|\int\limits_{c}^{d} |\varphi_2(\varsigma)| d\varsigma}{|\mathbb{W}(\varphi_1, \varphi_2)(\tau)|}.$$

Eventually, using (11), (12), and (13), one can easily deduce that (20) holds from the above inequality. $\square$

Lemma 7. *Let P, h and f be the same as in Lemma 6. Assume that $\sigma \in \mathbb{C}([c,d]) \cap \mathbb{C}^2((c,d))$ satisfy (8) in (c,d). For some constant X_0, if $\|\sigma\|_{\mathbb{C}([c,d])} \leq X_0$, there exists a positive constant X_4 that depends only on P, X_0, X_1, X_2 and the length of the interval $[c,d]$, such that,*

$$\left\|\sigma'\right\|_{\mathbb{C}([c,d])} \leq X_4. \qquad (25)$$

Proof. Let φ_1 and φ_2 be the same as in Lemma 5. Let us consider $\tau \in (c, \mathrm{d})$. We can deduce

$$\begin{aligned}&\mathbb{W}(\varphi_1, \varphi_2)(\tau)(\sigma'(\tau) + P\sigma(\tau)) + {\varphi'}_2(\tau)\sigma(c) + {\varphi'}_1(\tau)\sigma(d)\\ &= {\varphi'}_2(\tau)\int\limits_{c}^{\tau} \varphi_1(\varsigma) f(\varsigma)\, d\varsigma + {\varphi'}_1(\tau)\int\limits_{\tau}^{d} \varphi_2(\varsigma) f(\varsigma)\, d\varsigma\end{aligned}$$

which results in

$$\sigma'(\tau)=\frac{\varphi'_2(\tau)\left[\int\limits_c^\tau\varphi_1(\varsigma)f(\varsigma)\,d\varsigma!-\sigma(c)\right]+\varphi'_1(\tau)\left[\int\limits_\tau^d\varphi_2(\varsigma)f(\varsigma)\,d\varsigma-\sigma(d)\right]}{\mathbb{W}(\varphi_1,\varphi_2)(\tau)}.$$

This, along with the assumptions that $|f|\le X_2$ and $|\sigma|\le X_0$, leads to the conclusion that

$$\left|\sigma'(\tau)\right|\le X_2\frac{|\varphi'_2(\tau)|\left[X_2\int\limits_c^d|\varphi_1(\varsigma)|d\varsigma+X_0\right]+|\varphi_1(\tau)|\left[X_2\int\limits_c^d|\varphi_2(\varsigma)|d\varsigma+X_0\right]}{|\mathbb{W}(\varphi_1,\varphi_2)(\tau)|}.$$

The above inequality (23) can be easily deduced by using (11), (12), and (13). By using the logic of [16, Lemmas 2.6 and 2.7], it is obvious that F is continuous and precompact. □

Lemma 8. *A solution $(M_1,M_2,N)\in\Omega$ on I_l is admitted by system (6).*

Lemma 9. *If $\omega>\omega^*$, (1.1) admits a solution (M_1,M_2,N) satisfying $0<M_1(\tau)<\mathbb{K}_1, 0<M_2(\tau)<\mathbb{K}_2, N(\tau)>0\ \forall\ \tau\in\mathbb{R}$. Furthermore, as $\tau\to-\infty$, $N(\tau)=\mathrm{O}\left(e^{\Lambda_1\tau}\right)$ and $M_1{}'(-\infty)=M_2{}'(-\infty)=N'(-\infty)=0$.*

Proof. Let $\{l_n\}_{n\in N}$ be a non decreasing sequence such that

$$l_1>-\min\{\tau_1,\tau_2,\tau_3\}>0$$

and $l_n\to+\infty$. The solution of system (6) with $l=l_n$ found in Lemma 8 is denoted as $\{(M_{1_n},M_{2_n},N_n)\}_{n\ge\mathbb{N}}$. It is obvious that in $[-l_N,l_N]$, $\{(M_{1_n},M_{2_n},N_n)\}_{n\ge\mathbb{N}}$ are uniformly bounded.

From elliptic estimates, it follows that $[\{(M'_{1_n},M'_{2_n},N'_n)\}_{n\ge\mathbb{N}}$ are uniformly bounded in $[-l_N,l_N]$. As a result of the equations in (6), $[\{(M''_{1_n},M''_{2_n},N''_n)\}_{n\ge\mathbb{N}}$ are uniformly bounded in $[-l_N,l_N]$. By differentiating (6), we can see that $[\{(M'''_{1_n},M'''_{2_n},N'''_n)\}_{n\ge\mathbb{N}}$ are uniformly bounded in $[-l_N,l_N]$. As a result of Arzela–theorem, Ascoli's a subsequence $\left\{\left(M_{1_{n_j}},M_{2_{n_j}},N_{n_j}\right)\right\}$ exists such that

$$\left(M_{1_{n_j}},M_{2_{n_j}},N_{n_j}\right)\to(M_1,M_2,N)\ \text{in}\ \mathbb{C}^2(\mathbb{R})\ \text{as}\ n_j\to\infty.$$

Then it is clear that (M_1,M_2,N) is a non-negative solution of system (3) and that $\underline{M}_1\le M_1\le\bar{M}_1, M_2\le\underline{M}_2\le\bar{M}_2,\underline{N}\le N\le\bar{N}$ over $\mathbb{R}$ is satisfied. As per the definitions of upper and lower solutions we get

$$(M_1, M_2, N)(-\infty) = E_* = \left(\frac{c_1\mathbb{K}_1 c_2 + \alpha\mathbb{K}_1\mathbb{K}_2 c_2}{c_1 c_2 - \alpha\beta\mathbb{K}_1\mathbb{K}_2}, \frac{c_1\mathbb{K}_1 c_2 - \beta\mathbb{K}_1\mathbb{K}_2 c_1}{c_1 c_2 - \alpha\beta\mathbb{K}_1\mathbb{K}_2}, 0 \right)$$

and as $\tau \to -\infty$, $N(\tau) = \mathrm{O}\left(e^{\Lambda_1 \tau}\right)$.

Now we show that $0 < M_1(\tau) < \mathbb{K}_1\ \forall\ \tau \in \mathbb{R}$. We assert that $M_1(\tau) < \mathbb{K}_1\ \forall\ \tau \in \mathbb{R}$. Assume that there is an $\tau_1^* \in \mathbb{R}$ such that $M_1(\tau_1^*) = \mathbb{K}_1$ by contradiction. Thus, ${M_1}'(\tau_1^*) = 0$ and ${M_1}''(\tau_1^*) \leq 0$. Hence, ${M_1}''(\tau_1^*) = 0$ and $N(\tau_1^*) = 0$ follow from the first equation of system (3), which entails $N(\tau) \equiv 0$ when Harnack's inequality is applied to the third equation of (3), contradicting $N(\tau) \geq \underline{N}(\tau)$ for $\tau < \tau_3$. Assume, that there is contradiction $\tau_2^* \in \mathbb{R}$ for which $M_1(\tau_2^*) = 0$, resulting in $M'_1(\tau_2^*) = M''_1(\tau_2^*) = 0$. By the uniqueness of solutions, we get $M_1(\tau) \equiv 0\ \forall\ \tau \in \mathbb{R}$, which contradicts $M_1(\tau) \geq \underline{M}_1(\tau) > 0\quad \forall\ \tau < \tau_1$. We can similarly derive $0 < M_2(\tau) < \mathbb{K}_2$, $N(\tau) > 0\ \forall\ \tau \in \mathbb{R}$.

Fix $\tilde{M}_1(\tau) = \mathbb{K}_1 - M_1(\tau) > 0$. Then,

$$D_1 \tilde{M}_1'' - \omega \tilde{M}_1' - \frac{c_1 M_1}{\mathbb{K}_1}\tilde{M}_1 + \gamma M_1 N - \alpha M_1 M_2 = 0, \quad \tau \in \mathbb{R}.$$

Noting that $\tilde{M}_1(-\infty) = 0$, it implies that $\tilde{M}_1'(-\infty) = 0$ ([17], Lemma 3.7), resulting in $M'_1(-\infty) = 0$. By similar argument, we get $M'_2(-\infty) = 0$ and $N'(-\infty) = 0$. □

4. Existence of travelling fronts when $\omega > \omega^*$

Here we demonstrate the existence of travelling fronts in (1) with $w > w*$. As a result, we must demonstrate the existence of travelling semi-fronts connecting E^*. Note that Lemma 9 establishes the existence of travelling semi-fronts connecting E_* with wave speed $w > w*$. We provide prior estimations for these travelling semi-waves to ensure that they also connect E^*.

Lemma 10. *For all $w > w^*$, denote the travelling semi-front of (1) obtained in Lemma 9 with wave speed ω by $(M_1, M_2, N)(\tau)$. $N(\tau)$ is then bounded in $\mathbb{R}$.*

Proof. Assume the contrary that $N(\tau)$ is unbounded in $\mathbb{R}$. Then $\limsup_{\tau\to+\infty} N(\tau) = +\infty$ arrives from $(M_1, M_2, N)(-\infty) = E_*$. To begin, we prove that the following statements are true:

Claim 1:
If there is a sequence $\tau_\kappa \to +\infty$ such that $N(\tau_\kappa) \to +\infty$, then $M_{1_\kappa}(\tau) \to 0, M_{2_\kappa}(\tau) \to 0$, is uniform in any compact interval $\mathbb{R}$ as $\kappa \to +\infty$, where

$$M_{1_\kappa}(\tau) = M_1(\tau_\kappa + \tau), M_{2_\kappa}(\tau) = M_2(\tau_\kappa + \tau).$$

Note that $0 < M_i(\tau) < \mathbb{K}_i \, (i = 1, 2)$. Then, using Harnack's inequality [21], Theorem 2.1, to the 3^{rd} of (3), there exists a constant $\mathbb{K}_L > 0$ such that

$$N(\tau_\kappa) \leq \sup_{\tau \in \Lambda_\kappa} N(\tau) \leq \mathbb{K}_L \inf_{\tau \in \Lambda_\kappa} N(\tau),$$

where $\Lambda_\kappa = [\tau_\kappa - L, \tau_\kappa - L]$, exists for any $L > 0$. As a result

$$\inf_{\tau \in \Lambda_k} N(\tau) \to +\infty, \text{ as } k \to +\infty. \tag{26}$$

Take $\widetilde{M}_1(\cdot) \in \mathbb{C}^2([-L, L])$ such that $\widetilde{M}_1(\pm L) = \mathbb{K}_1$ and $\inf\limits_{\tau \in [-L,L]} \widetilde{M}_1(\tau) > 0$. From (26), as a result, for any $k \gg 1$,

$$D_1 \widetilde{M''}_1(\tau) - \omega \widetilde{M'}_1(\tau) + \widetilde{M}_1(\tau) \left(c_1 - \frac{c_1 \widetilde{M}_1(\tau)}{\mathbb{K}_1} - \gamma N(\tau_\kappa + \tau) \right)$$

$$\leq 0, \; \forall \tau \in [-L, L],$$

which means that for all $M_{1_\kappa}(\tau) \leq \widetilde{M}_1(\tau) \; \forall \; \tau \in [-L, L]$ and $\kappa \gg 1$. By the subjectivity of $\widetilde{M}_1$ it is evident that

$$M_{1_\kappa}(\tau) \to 0, \text{ uniformly in } \left[-\tfrac{L}{2}, \tfrac{L}{2}\right] \text{ as } \kappa \to +\infty,$$

up to a subsequence if necessary. We get a similar result for $M_{2_\kappa}(\tau)$ based on similar arguments. As a result, we can make this claim.

Claim 2: $N(\tau)$ is non decreasing for $\tau \gg 1$

Suppose that $N(\tau)$ is decreasing for $\tau \gg 1$. Since $\lim \sup\limits_{\tau \to +\infty} N(\tau) = +\infty$, there exists a sequence $\tau_\kappa \to +\infty$ that has $N'(\tau_\kappa) = 0$ and $N''(\tau_\kappa) = 0$ and has $N(\tau_\kappa) \to +\infty$. From Claim 1, we get

$$M_{1_\kappa}(\tau) = M_1(\tau_\kappa + \tau) \to 0, M_{2_\kappa}(\tau) = M_2(\tau_\kappa + \tau) \to 0$$

as $\kappa \to +\infty$ uniformly in any compact interval of $\mathbb{R}$. For large κ we have,

$$M_1(\tau_\kappa) = M_{1_\kappa}(0) < \frac{\rho}{4w_1\gamma}, M_2(\tau_\kappa) = M_{2_\kappa}(0) < \frac{\rho}{4w_2\delta}.$$

As a result of the third equation in (3),

$$0 = D_3 N''(\tau_\kappa) - \omega N'(\tau_\kappa) + (w_1\gamma M_1(\tau_\kappa) + w_2\delta M_2(\tau_\kappa) - \rho) N(\tau_\kappa) < 0,$$

which results in contradiction. As a result, Claim 2 is true.

According to Claim 2, $\tau' \in \mathbb{R}$ exists such that $N(\tau)$ increases in $[\tau', +\infty]$. For any non decreasing sequence $\tau_\kappa \to +\infty$ satisfying $\tau_\kappa > \tau' + \kappa \ \forall\ \kappa$, it implies from $\lim \sup_{\tau\to+\infty} N(\tau) = +\infty$ that $N(\tau_\kappa) \to +\infty$.

Let $\widetilde{N}_\kappa(\tau) = \frac{N(\tau_\kappa+\tau)}{N(\tau_\kappa)}, \tau \in \mathbb{R}$. Then it satisfies

$$D_3 \widetilde{N''}_\kappa(\tau) - \omega \widetilde{N'}_\kappa(\tau) + (w_1\gamma M_1(\tau_\kappa+\tau) + w_2\delta M_2(\tau_\kappa+\tau) - \rho)$$

$$\times \widetilde{N}_k(\tau) = 0, \quad \tau \in \mathbb{R}.$$

Also Harnack's inequality states that for any $L > 0$, there exists a constant $K > 0$ such that

$$\sup_{\tau\in[-L,L]} \widetilde{N}_\kappa(\tau) \le \mathbb{K} \inf_{\tau\in[-L,L]} \widetilde{N}_\kappa(\tau) \le \mathbb{K}\, \widetilde{N}_\kappa(0) = \mathbb{K}.$$

Claim 1 is based on elliptic estimates, it follows that

$$\widetilde{N}_\kappa(\tau) \to \widetilde{N}_\infty(\tau) \text{ in } \mathbb{C}^2_{loc}(R),$$

where $\widetilde{N}_\infty(\tau)$ is a non-negative solution of

$$D_3 \widetilde{N''}_\infty(\tau) - \omega \widetilde{N'}_\infty(\tau) - \rho \widetilde{N}_\infty(\tau) = 0, \ \tau \in \mathbb{R},$$

if necessary, up to a subsequence. As a result, $\widetilde{N}_\infty(\tau) = E_1 e^{\Lambda_1^*\tau} + E_2 e^{\Lambda_2^*\tau}$, where $\Lambda_1^* < 0 < \Lambda_2^*$ are the roots of the equation $D_3\Lambda^2 - \omega\Lambda - \rho = 0$. As $\widetilde{N}_\kappa(0) = 1$, we have $\widetilde{N}_\infty(0) = E_1 + E_2 = 1$. $E_1 \ge 0$, $E_2 \ge 0$ derives from the non-negativity of $\widetilde{N}_\infty(\tau)$, $\tau \in \mathbb{R}$. We know $\tilde{N}_\kappa(\tau)$ is increasing on $\mathbb{R}$ because $\widetilde{N}_k(\tau)$ is non decreasing for large $\tau \in \mathbb{R}$, yielding $E_1 = 0$ and hence $\widetilde{N}_\infty(\tau) = e^{\Lambda_2^*\tau}$. Thus,

$$\lim_{\kappa\to\infty} \frac{N'(\tau_\kappa)}{N(\tau_\kappa)} = \widetilde{N'}_\infty(0) = \Lambda_2^*.$$

The arbitrariness of $\{\tau_\kappa\}$ therefore leads to the conclusion that

$$\lim_{\kappa\to\infty} \inf \left\{ \inf_{[\tau'+\kappa,+\infty)} \frac{N'(\tau_\kappa)}{N(\tau_\kappa)} \right\} \geq \Lambda_2^*.$$

Since $\Lambda_2^* > \Lambda_1 > 0$, for $\kappa \gg 1$, we have

$$\inf_{[\tau'+\kappa,+\infty)} \frac{N'(\tau_\kappa)}{N(\tau_\kappa)} \geq \frac{\Lambda_1 + \Lambda_2^*}{2}.$$

This results in $\frac{N'(\tau_\kappa)}{N(\tau_\kappa)} \geq \frac{\Lambda_1+\Lambda_2^*}{2}$ for $\tau \geq \tau' + \kappa$. Thus, for $\tau \geq \tau' + \kappa$, we have

$$N(\tau_\kappa) \geq N'(\tau_\kappa).$$

Since $(\Lambda_1 + \Lambda_2^*)/2 > \Lambda_1, \exists\ \tau_1^* > \tau'_1 + \kappa$ such that

$$N(\tau) > e^{\Lambda_1 \tau}, \forall \tau \geq \tau',$$

which contradicts $N(\tau) \leq e^{\Lambda_1 \tau}, \forall \tau \in \mathbb{R}$. Hence the proof follows. □

Lemma 11. *For every $w > w*$, system (1) has a traveling front $(M_1, M_2, N)(+\omega t)$.*

Proof. We only need to prove that $(M_1, M_2, N)(+\infty) = E^*$, using Lemma 9, where (M_1, M_2, N) is the solution derived in Lemma 9. Lemma 1 demonstrates that $N(\tau)$ is bounded on $\tau \in \mathbb{R}$. Then [[19], Lemma 3.7] shows that there exist $M_1(\omega) > 0$, $M_2(\omega) > 0$ and $M_3(\omega) > 0$ such that

$$\left|\frac{M'_1(\tau)}{M_1(\tau)}\right| \leq M_1(\omega), \left|\frac{M'_2(\tau)}{M_2(\tau)}\right| \leq M_2(\omega), \left|\frac{N'(\tau)}{N(\tau)}\right| \leq M_3(\omega), \forall \omega \in \mathbb{R}.$$

Let $q_1 = M'_1, q_2 = M'_2, q_3 = N'$, system (3) can be written in the following form:

$$\begin{cases} M'_1 = q_1, \\ D_1 q'_1 = \omega q_1 - r_1(M_1, M_2, N), \\ M'_2 = q_2, \\ D_2 q'_2 = \omega q_2 - r_2(M_1, M_2, N), \\ N' = q_3, \\ D_3 q'_3 = \omega q_3 - r_3(M_1, M_2, N), \end{cases} \tag{27}$$

where

$$r_1\left(M_1,M_2,N\right)=c_1M_1\left(1-\tfrac{M_1}{\mathbb{K}_1}\right)-\gamma M_1N-\alpha M_1M_2,$$
$$r_2\left(M_1,M_2,N\right)=c_2M_2\left(1-\tfrac{M_2}{\mathbb{K}_2}\right)-\delta M_1N-\beta M_1M_2,$$
$$r_3\left(M_1,M_2,N\right)=w_1\gamma M_1N+w_2\delta M_2N-\rho N.$$

Define

$$Y_1=\omega M_1-D_1q_1+\frac{D_1q_1m_1^*}{M_1}-\omega m_1^*\int_{m_1^*}^{M_1}\frac{1}{q}dq,$$

$$Y_2=\omega M_2-D_2q_2+\frac{D_2q_2m_2^*}{M_2}-\omega m_2^*\int_{m_2^*}^{M_2}\frac{1}{q}dq,$$

$$Y_3=\omega N-D_3q_3+\frac{D_3q_3m_3^*}{N}-\omega n^*\int_{n^*}^{N}\frac{1}{q}dq.$$

It is obvious that

$$\begin{aligned}Y'_1|_{(3.2)}&=\frac{r_1\left(M_1,M_2,N\right)}{M_1}\left(M_1-m_1^*\right)-\frac{D_1m_1^*q_1^2}{M_1^2}\\&=-\frac{r_1}{M_1}(M_1-m_1^*)^2+\gamma\left(n^*-N\right)\left(M_1-m_1^*\right)-\frac{D_1m_1^*q_1^2}{M_1^2}.\end{aligned}\tag{28}$$

By similar arguments, we get

$$\begin{aligned}Y'_2|_{(3.2)}&=-\frac{r_2}{M_2}(M_2-m_2^*)^2+\gamma\left(n^*-N\right)\left(M_2-m_2^*\right)-\frac{D_2m_2^*q_2^2}{M_2^2},\\Y'_3|_{(3.2)}&=w_1\gamma\left(M_1-m_1^*\right)\left(n^*-N\right)+w_2\delta\left(M_2-m_2^*\right)\left(n^*-N\right)-\\&\quad\frac{D_3n^*q_3^2}{N^2}.\end{aligned}\tag{29}$$

Let $\mathbb{L}=w_1L_1+w_2L_2+L_3$. Along the traveling wave solution, the derivative of $\mathbb{L}$ satisfies

$$\begin{aligned}Y'|_{(3.2)}=&-\frac{c_1w_1}{K_1}(M_1-m_1^*)^2-\frac{c_2w_2}{K_2}(M_2-m_2^*)^2-\frac{w_1D_1m_1^*q_1^2}{M_1^2}\\&-\frac{w_2D_2m_2^*q_2^2}{M_2^2}-\frac{D_3n^*q_3^2}{N^2}.\end{aligned}\tag{30}$$

It is obvious that $Y'|_{(4)}=0$ if and only if

$$(M_1,q_1,M_2,q_2,N,q_3)=({m_1}^*,0,{m_2}^*,0,N,0)$$

and the maximal invariant set of

$$\{(M_1, q_1, M_2, q_2, N, q_3) : L' = 0]$$

includes only the equilibrium. $\mathbb{Y}$ is obviously continuous and bounded below. As a result, the proof is completed by La Salle's invariance principle [20], Theorem 6.4. □

5. Existence of travelling fronts when $\omega = \omega^*$

We prove the presence of a travelling front with wave speed $\omega = \omega*$ in this section.

Lemma 12. *A travelling front $(M_1, M_2, N)(+\omega^* t)$ is present in system (1).*

Proof. Let $\omega_n = \omega^* + 1/n$, $n \in \mathbb{N}_+$. System (1) has a travelling front $(M_{1_n}, M_{2_n}, N_n)(+\omega_n t)$ such that $M_{1_n}(\tau) < \mathbb{K}_1, M_{2_n}(\tau) < \mathbb{K}_2$ and $N_n(\tau)$ is bounded in $\tau \in \mathbb{R}$ for any n, as shown by Lemmas 9, 10, and 11. Furthermore, $M'_{1_n}(-\infty) = M'_{2_n}(-\infty) = N'_n(-\infty) = 0$.

N_n is uniformly bounded with regard to n on $\tau \in \mathbb{R}$, as we prove now. Assume that there exists a sequence $\{\tau_n\}$ such that $N_n\{\tau_n\} \to +\infty$ as $n \to +\infty$ up to a subsequence if necessary, by contradiction. As $(M_{1_n}, M_{2_n}, N_n)(\tau)$ satisfies (2), $\{\tau_n\}$ can be chosen so that

$N_n(\tau_n) = \sup_{\tau \in R} N_n(\tau) = \max_{\tau \in R} N_n(\tau)$, implies $N_n''\{\tau_n\} \leq 0$ for large n. Let us define

$$\widehat{N}_n(\tau_n) = \frac{N_n(\tau_n + \tau)}{N_n(\tau_n)}, \tau \in \mathbb{R}.$$

We can affirm that, up to and including a subsequence if necessary, using arguments similar to those in Lemma 1, $\widehat{N}_n(\tau) \to \widehat{N}_\infty(\tau)$ in $\mathbb{C}^2_{loc}(\mathbb{R})$, where,

$$\widehat{N}_\infty(\tau) = E_1 e^{\Lambda_1{}^* \tau} + E_2 e^{\Lambda_2{}^* \tau}, E_1 \geq 0, E_2 \geq 0, E_1 + E_2 = 1.$$

As $\widehat{N}''_n(0) \leq 0$, we have

$$0 \geq \widehat{N}''_\infty(0) = E_1 \Lambda_1^{*^2} + E_2 \Lambda_2^{*^2} > 0.$$

This is a contradiction. As a result, on $\tau \in \mathbb{R}$, $\{(M_{1_n}, M_{2_n}, N_n)\}$ is uniformly bounded with respect to n.

As $(M_{1_n}, M_{2_n}, N_n)(-\infty) = (\mathbb{K}_1, \mathbb{K}_2, 0)$, we may conclude that

$\mathbb{K}_1 - \tau \leq M_{1_n}(\tau) \leq \mathbb{K}_1, \mathbb{K}_2 - \tau \leq M_{2_n}(\tau) \leq \mathbb{K}_2, N_n(\tau) \leq \varepsilon, \forall \tau \leq 0,$

holds for any $0 < \varepsilon < 1/2 \min\{\mathbb{K}_1 - m_1^*, \mathbb{K}_2 - m_2^*, n^*\}$ and one of the following holds: $M_{1_n}(0) = \mathbb{K}_1 - \varepsilon, M_{2_n}(0) = \mathbb{K}_2 - \varepsilon, N_n(0) = \varepsilon$.

Since $\{(M_{1_n}, M_{2_n}, N_n)\}$ is uniformly bounded, [4, Lemma 3.7] implies that $\{(M'_{1_n}, M'_{2_n}, N'_n)\}$ is similarly uniformly bounded. It is evident that $\{(M''_{1_n}, M''_{2_n}, N''_n)\}$ is uniformly bounded, and that $\{(M'''_{1_n}, M'''_{2_n}, N'''_n)\}$ is uniformly bounded as well.

As a result of the Arzela–Ascoli theorem, there exists a subsequence $\{(M_{1_{n_j}}, M_{2_{n_j}}, N_{n_j})\}$ in which

$$\left(M_{1_{n_j}}, M_{2_{n_j}}, N_{n_j}\right) \to (M_1, M_2, N) \text{ in } \mathbb{C}^2_{loc}(\mathbb{R}) \text{ as } n_j \to +\infty,$$

where (M_1, M_2, N) is a non-negative solution of (3) with $\omega = \omega^*$ and $0 \leq M_1 \leq \mathbb{K}_1, 0 \leq M_2 \leq \mathbb{K}_2, N(\tau)$ is a bounded solution. Furthermore, we have

$$\mathbb{K}_1 - \tau \leq M_1(\tau) \leq \mathbb{K}_1, \mathbb{K}_2 - \tau \leq M_2(\tau) \leq \mathbb{K}_2, N(\tau) \leq e, \forall \tau \leq 0,$$

and at least one of the following is true:

$$M_1(0) = \mathbb{K}_1 - \varepsilon, M_2(0) = \mathbb{K}_2 - \varepsilon, N(0) = \varepsilon.$$

It is simple to demonstrate that $(\mathbb{K}_1, 0, \mathbb{K}_2, 0, 0, 0)$ is a hyperbolic system equilibrium (4). Hartman–Grobman theorem states that $(M_1, M_2, N)(-\infty) = E_*$ if the setting ε is small enough.

We now prove that (M_1, M_2, N) is positive. Since additional cases can be examined identically, we only consider the case $M_1(0) = \mathbb{K}_1 - \varepsilon$. First, we show that $N(\tau) \geq 0$ for all $\tau \in \mathbb{R}$. Assume, on the other hand, that there is an τ_1^* such that $N(\tau_1^*) = 0$. $N(\tau) \equiv 0 \ \forall \ \tau \in R$ when Harnack's inequality is applied to the third equation of (3). As a result, $M_1(\tau)$ satisfies,

$$D_1 M_1'' - \omega M_1' + c_1 M_1 \left(1 - \frac{M_1}{\mathbb{K}_1}\right), \tau \in \mathbb{R},$$

and $M_1(0) = \mathbb{K}_1 - e, 0 \leq M_1 \leq \mathbb{K}_1$. Simple phase-plane analysis yields $M_1(+\infty) = -\infty$, a contradiction, since $M_1(-\infty) = \mathbb{K}_1$. As a result, $N(\tau) > 0 \ \forall \ \tau \in \mathbb{R}$. It is important to note that $N(\tau)$ is bounded. Similarly, $M_1(\tau) > 0$ and $M_2(\tau) > 0$ for every $\tau \in \mathbb{R}$ derived from $(M_1, M_2, N)(-\infty) = E_*(\mathbb{K}_1, \mathbb{K}_2, 0)$ using reasons similar to those in Lemma 2. The proof is now complete. $\square$

Lemma 13. *There is no travelling semi-front with wave speed ω for any $0 < \omega < \omega^*$ system (1).*

Remark 3. Theorem 1 can now be proved by combining Lemmas 11, 12, and 13.

6. Discussion

Many researchers have been interested in the existence of travelling wave solution in mathematical models. The presence of travelling waves with minimal wave speed ω for a diffusive prey predator model was investigated in this study. By constructing a set of upper–lower solutions, we demonstrated the existence of travelling wave solutions. The existence of a travelling semi front with wave speed w has been demonstrated using the auxiliary truncated problem. For the case $\omega > \omega^*$, LaSalle invariance principle is used to connect two equilibria. We have shown the nonexistence of travelling semi fronts for the case $\omega < \omega^*$. Our method describes the existence of travelling waves for a spatio temporal model.

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