EXISTENCE AND UNIQUENESS FOR ONE-PHASE SPHERICAL STEFAN PROBLEM WITH NONLINEAR THERMAL COEFFICIENTS AND HEAT FLUX CONDITION

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Abstract: We study a non-classical one-phase Stefan problem for heat transfer in spherical domain of electrical contact materials when heating process on electrical contact surface arises. Mathematical model involves non-linear thermal coefficients and heat flux condition at a known free boundary. Solution of the problem based on similarity principle. Moreover, we determine the temperature distribution in melted zone and the free boundary on melting interface whether direct Stefan problem is considered. The existence and uniqueness of similarity solution to the problem is established. Solutions for constant and linear thermal coefficients and existence of uniqueness for particular cases are provided.

1. Introduction

Stefan problems are considered mostly in phase change problems arising in

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physical, mechanical and chemical processes like freezing, melting, molecular diffusion and etc. It has a wide range in engineering and industries applications. In eighteenth century, Lame and Clapeyron studied Stefan problems related to the solidification process on planet Earth [1]. In classical one-phase Stefan problem we describe the temperature in material and the location of melting or freezing interfaces which separates two phases. The non-classical Stefan problem involves nonclassical heat equation with thermal coefficients which are depended on temperature. The one-dimensional non-classical Stefan problems with given temperature and Diriclet, Neumann, Robin conditions on fixed face of semi-infinite material are considered and widely studied in [2]-[6].

Recently, Briozzo, Natale and Tarzia also discussed an inverse nonclassical Stefan problem of determining unknown thermal coefficients of a semi-infinite material of Storm's type through a phase-change process with an overspecified condition on the fixed face [7]. The inverse Stefan problem for finding the time-dependent thermal conductivity and the transient temperature satisfying the heat is considered succesfully considered on the base of similarity principle, [8].

In Stefan problem with temperature-dependent thermal coefficients to determine heat process between on melting isotherm is an important to give attention to temperature dependence of specific heat and thermal conductivity. Similarity principle is very useful method to solve these kind of problems that enable us to reduce free boundary partial differential problem to ordinary differential equation with fixed boundary.

In this article we consider one-phase spherical Stefan problem with two free boundaries which one of them is given. This problem encounters in electrical contact phenomena, when heat flux enters to material through electrical contact spot which takes the form ideal hemisphere and heat distributed in spherical domain. We assume that contact surface of material with spot at the given radius $r = \alpha(t)$ initial time takes melting temperature ($\theta(\alpha(t), 0) = \theta_m$) and melted liquid phase domain lies between two moving boundaries. The aim of the article, finding temperature solution for liquid phase and the second free boundary on melting interface. The Stefan problems arising in electrical contact processes widely studied in [9]-[14]. In first section of the recent work we represent mathematical model and similarity solution of the Stefan problem. The existence and uniqueness of the solution by using fixed point Banach theorem is provided in the second section. The last section represents the solution of the problems with particular types of nonlinear thermal coefficients, existence and uniqueness for particular cases also discussed and proved.

2. Mathematical model and its solution

The mathematical model describing the process of the interaction of the electrical arc with electrodes and the dynamics of their melting is based on the spherical model introduced by R. Holm [15]. The mentioned problem can be modelled as:

$$c(\theta)\gamma(\theta)\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda(\theta)r^2\frac{\partial \theta}{\partial r}\right], \quad \alpha(t) < r < \beta(t), \ t > 0, \tag{1}$$

$$-\lambda(\theta(\alpha(t),t))\frac{\partial \theta}{\partial r}(\alpha(t),t) = \frac{P_0 e^{-\frac{\alpha_0^2}{a}}}{\sqrt{\pi t}}, \quad t > 0, \tag{2}$$

$$\theta(\beta(t),t) = \theta_m, \quad t > 0, \tag{3}$$

$$-\lambda(\theta(\beta(t),t))\frac{\partial \theta}{\partial r}(\beta(t),t) = L\gamma\beta'(t), \quad t > 0, \tag{4}$$

$$\beta(0) = 0, \tag{5}$$

where $c(\theta)$, $\gamma(\theta)$ and $\lambda(\theta)$ are the heat capacity, mass density and thermal conductivity of the electrical contact material that depend of temperature $\theta(r,t)$ in liquid phase which has to be determined, L is the latent heat of melting and γ is the density of the material, θ_m - melting temperature, $P_0 e^{-\alpha_0^2/a}/(\sqrt{\pi}t)$ represents heat flux entering in electrical contact spot at free boundary $r = \alpha(t)$ and $P_0 > 0$ is the given constant. We suppose that the left free boundary $\alpha(t)$ is known and $\beta(t)$ denotes the location of the moving melting interface which has to be determined.

If we use the following dimensionless transformation

$$T(r,t) = \frac{\theta(r,t) - \theta_m}{\theta_m}, \tag{6}$$

the problem (1)-(5) becomes

$$\tilde{N}(T)\frac{\partial T}{\partial t} = a\frac{1}{r^2}\frac{\partial}{\partial r}\left[\tilde{L}(T)r^2\frac{\partial T}{\partial r}\right], \quad \alpha(t) < r < \beta(t), \ t > 0, \tag{7}$$

$$\tilde{L}(T(\alpha(t),t))\frac{\partial T}{\partial r}(\alpha(t),t) = -\frac{P_0 e^{-\frac{\alpha_0^2}{a}}}{\lambda_0\sqrt{\pi t}\theta_m}, \quad t > 0, \tag{8}$$

$$T(\beta(t),t) = 0, \quad t > 0, \tag{9}$$

$$\tilde{L}(T(\beta(t),t))\frac{\partial T}{\partial r}(\beta(t),t) = -\frac{L\gamma\beta'(t)}{\lambda_0\theta_m}, \quad t > 0, \tag{10}$$

$$\beta(0) = 0, \tag{11}$$

where c_0, γ_0, λ_0 and $a = \lambda_0/(c_0\gamma_0)$ are heat capacity, density, thermal conductivity and thermal diffusivity of the material and

$$\tilde{L}(T) = \frac{\lambda(\theta_m(T+1))}{\lambda_0}, \quad \tilde{N}(T) = \frac{c(\theta_m(T+1))\gamma(\theta_m(T+1))}{c_0\gamma_0}.$$

To solve the problem (7)-(11) we use the following similarity substitution

$$T(r,t) = u(\xi), \tag{12}$$

where $\xi = \dfrac{r}{2\sqrt{at}}$. From (8)-(11) and (12), it can be supposed that given and unknown free boundaries must be proportional to $\sqrt{at}$ and can be presented as follows:

$$\alpha(t) = 2\alpha_0\sqrt{at}, \quad \beta(t) = 2\mu\sqrt{at}, \tag{13}$$

where α_0 is given positive constant and μ is an unknown constant to be found.

With the help of (12), the problem (7)-(11) becomes

$$[L^*(u)\xi^2 u']' + 2\xi^3 N^*(u)u' = 0, \quad \alpha_0 < \xi < \mu, \tag{14}$$

$$L^*(u(\alpha_0))u'(\alpha_0) = -p_*, \tag{15}$$

$$u(\mu) = 0, \tag{16}$$

$$u'(\mu) = -K\mu, \tag{17}$$

where $p^* = \dfrac{2P_0\sqrt{a}e^{-\frac{\alpha_0^2}{a}}}{\lambda_0\sqrt{\pi}\theta_m}$, $K = \dfrac{2aL\gamma}{\theta_m\lambda(\theta_m)}$ and

$$L^*(u) = \frac{\lambda(\theta_m(u+1))}{\lambda_0}, \quad N^*(u) = \frac{c(\theta_m(u+1))\gamma(\theta_m(u+1))}{c_0\gamma_0}. \tag{18}$$

We can deduce that $(\xi, u(\xi))$ is the solution of the problem (14),(15),(16) and (17) if and only if $(\xi, u(\xi))$ satisfy the integral equation

$$u(\xi) = p^*\left(F[\mu, u(\mu) - F[\xi, u(\xi)]]\right) \tag{19}$$

where

$$F[\xi, u(\xi)] = \int_{\alpha_0}^{\xi} \frac{E[s, u(s)]}{s^2 L^*(u(s))}ds, \tag{20}$$

$$E[s, u(s)] = \exp\left(-2\int_{\alpha_0}^{s} t\frac{N^*(u(t))}{L^*(u(t))}dt\right),$$
(21)

together with condition (17) which becomes

$$p^*\frac{E[\mu, u(\mu)]}{K\lambda(\theta_m)} = \mu^3.$$
(22)

From (22) we can determine unknown constant μ for free boundary $\beta(t)$.

3. Existence and uniqueness of the similarity solution of the problem

To analyze existence of solution (19) we assume that $\mu > 0$ is given constant. At first we consider the space $C^0[\alpha_0, \mu]$ of continuous real valued functions defined on interval $[\alpha_0, \mu]$ endowed with supremum norm

$$||u|| = \max_{\xi \in [\alpha_0, \mu]} |u(\xi)|.$$
(23)

To prove that we use the fixed point Banach theorem as $(C^0[\alpha_0, \mu], || \cdot ||)$ is a Banach space. Let we define the operator V on $C^0[\alpha_0, \mu]$ that is

$$V(u)(\xi) = p^*\big[F[\mu, u(\mu)] - F[\xi, u(\xi)]\big]$$
(24)

By using the fixed point Banach we have to that for each $\mu > 0$ there exists a unique function u such that

$$V(u)(\xi) = u(\xi), \quad \forall \xi \in [\alpha_0, \mu],$$
(25)

which is the solution to (19).

We assume that L^*, N^* are bounded and satisfy Lipschitz inequalities as follows:

a) There exists $L_m = \dfrac{\lambda_m}{\lambda_0} > 0$ and $L_M = \dfrac{\lambda_M}{\lambda_0} > 0$ such that

$$L_m \le L^*(u) \le L_M, \quad \forall u \in C^0(\mathbb{R}_0^+) \cup L^\infty(\mathbb{R}_0^+).$$
(26)

There exists $\bar{L} = \dfrac{\bar{\lambda}(\theta_m + 1)}{\lambda_0} > 0$ such that

$$||L^*(u_1) - L^*(u_2)|| \le \bar{L}||u_1 - u_2||, \quad \forall u_1, u_2 \in C^0(\mathbb{R}_0^+) \cup L^\infty(\mathbb{R}_0^+).$$
(27)

b) There exists $N_m = \dfrac{\sigma_m}{c_0, \gamma_0} > 0$ and $N_M = \dfrac{\sigma_M}{c_0 \gamma_0} > 0$ such that

$$N_m \leq N^*(u) \leq N_M, \quad \forall u \in C^0(\mathbb{R}_0^+) \cup L^\infty(\mathbb{R}_0^+). \tag{28}$$

There exists $\bar{N} = \dfrac{\bar{\sigma}(\theta_m + 1)}{c_0 \gamma_0} > 0$ such that

$$||N^*(u_1) - N^*(u_2)|| \leq \bar{N}||u_1 - u_2||, \quad \forall u_1, u_2 \in C^0(\mathbb{R}_0^+) \cup L^\infty(\mathbb{R}_0^+). \tag{29}$$

Now we have to obtain some preliminary results to prove the existence and uniqueness of the solution to the equation (25).

Lemma 1. *For all $z \in [\alpha_0, \mu]$ the following inequalities hold:*

$$\exp\left(-\frac{N_m}{L_M}(z^2 - \alpha_0^2)\right) \leq E[z, u(z)] \leq \exp\left(-\frac{N_M}{L_m}(z^2 - \alpha_0^2)\right), \tag{30}$$

$$
\begin{aligned}
&\exp\left(\frac{N_m}{L_M}\alpha_0^2\right)\sqrt{\frac{N_m}{L_M}}\left[\frac{\sqrt{L_M}}{\alpha_0\sqrt{N_m}}\exp\left(-\frac{\alpha_0 N_m}{L_M}\right)\right.\\
&+ \sqrt{\pi}\,\mathrm{erf}\left(\alpha_0\sqrt{\frac{N_m}{L_M}}\right) - \frac{\sqrt{L_M}}{z\sqrt{N_m}}\exp\left(-\frac{z N_m}{L_M}\right)\\
&\left.- \sqrt{\pi}\,\mathrm{erf}\left(z\sqrt{\frac{N_m}{L_M}}\right)\right] \leq F[z, u(z)] \leq \exp\left(\frac{N_M}{L_m}\alpha_0^2\right)\sqrt{\frac{N_M}{L_m}}\\
&\times \left[\frac{\sqrt{L_m}}{\alpha_0\sqrt{N_M}}\exp\left(-\frac{\alpha_0 N_M}{L_m}\right) + \sqrt{\pi}\,\mathrm{erf}\left(\alpha_0\sqrt{\frac{N_M}{L_m}}\right)\right.\\
&\left.- \frac{\sqrt{L_m}}{z\sqrt{N_M}}\exp\left(-\frac{z N_M}{L_m}\right) - \sqrt{\pi}\,\mathrm{erf}\left(z\sqrt{\frac{N_M}{L_m}}\right)\right].
\end{aligned}
\tag{31}
$$

Proof. They can be proved analogously by using definition of (20)-(21) and assumptions (26)-(29). $\square$

Lemma 2. *Let be given $\mu > 0$ and for all $z \in [\alpha_0, \mu]$ and $u_1, u_2 \in C^0[\alpha_0, \mu]$ the following inequalities hold:*

$$|E[z, u_1] - E[z, u_2]| \leq \tilde{E}(z)||u_1 - u_2||, \tag{32}$$

$$|F[z, u_1] - F[z, u_2]| \leq \tilde{F}(z)||u_1 - u_2||, \tag{33}$$

where

$$
\tilde{E}(z) = \left(\frac{\bar{N}}{L_m} + \frac{N_M \bar{L}}{L_m^2} \right)(z^2 - \alpha_0^2),
$$

$$
\tilde{F}(z) = \left[\frac{\tilde{E}(z)}{L_m} + \frac{\bar{L}}{L_m^2} \exp\left(-\frac{N_M}{L_m}(z^2 - \alpha_0^2) \right) \right]\left(\frac{1}{\alpha_0} - \frac{1}{z} \right). \tag{34}
$$

Proof. Taking into account assumptions (29)-(31) and inequality $|\exp(x) - \exp(y)| \le |x - y|$, we have

$$
E[z, u_1(s)] - E[z, u_2(s)]| = \left| \exp\left(-2 \int_{\alpha_0}^{z} s \frac{N^*(u_1(s))}{L^*(u_1(s))} ds \right) \right.
$$

$$
\left. - \exp\left(-2 \int_{\alpha_0}^{z} s \frac{N^*(u_2(s))}{L^*(u_2(s))} ds \right) \right| \le 2 \left| \int_{\alpha_0}^{z} s \frac{N^*(u_1(s))}{L^*(u_1(s))} ds \right.
$$

$$
\left. - \int_{\alpha_0}^{z} s \frac{N^*(u_2(s))}{L^*(u_2(s))} ds \right| \le 2 \int_{\alpha_0}^{z} \left| \frac{N^*(u_1(s))}{L^*(u_1(s))} - \frac{N^*(u_2(s))}{L^*(u_2(s))} \right| s ds
$$

$$
\le 2 \int_{\alpha_0}^{z} \left| \frac{N^*(u_1(s))}{L^*(u_1(s))} - \frac{N^*(u_2(s))}{L^*(u_1(s))} + \frac{N^*(u_2(s))}{L^*(u_1(s))} - \frac{N^*(u_2(s))}{L^*(u_2(s))} \right| s ds
$$

$$
\le 2 \int_{\alpha_0}^{z} \left(\frac{|N^*(u_1(s)) - N^*(u_2(s))|}{|L^*(u_1(s))|} \right.
$$

$$
+ |N^*(u_2(s))| \frac{|L^*(u_2(s)) - L^*(u_1(s))|}{|L^*(u_1(s))||L^*(u_2)|} \right) s ds \le 2 \left(\frac{\bar{N}}{L_m} + \frac{N_M \bar{L}}{L_m^2} \right)
$$

$$
\times ||u_1 - u_2|| \int_{\alpha_0}^{z} s ds \le \left(\frac{\bar{N}}{L_m} + \frac{N_M \bar{L}}{L_m^2} \right)(z^2 - \alpha_0^2)||u_1 - u_2||
$$

$$
= \tilde{E}(z)||u_1 - u_2||.
$$

In a similar way, we obtain the next result

$$|F[z, u_1(s)] - F[z, u_2(s)]| \le \left| \int_{\alpha_0}^{z} \frac{E[s, u_1(s)]}{s^2 L^*(u_1(s))} ds - \int_{\alpha_0}^{z} \frac{E[s, u_2(s)]}{s^2 L^*(u_2(s))} ds \right|$$

$$\le \int_{\alpha_0}^{z} \frac{|E[s, u_1(s)] - E[s, u_2(s)]|}{|L^*(u_1(s))|} \frac{ds}{s^2} + \int_{\alpha_0}^{z} \left| \frac{1}{L^*(u_1(s))} - \frac{1}{L^*(u_2(s))} \right| \frac{1}{s^2}$$

$$\times \ tE[s, u_2(s)]|ds \equiv G_1(z) + G_2(z),$$

where

$$G_1(z) \equiv \int_{\alpha_0}^{z} \frac{|E[s, u_1(s)] - E[s, u_2(s)]|}{|L^*(u_1(s))|} \frac{ds}{s^2} \le \frac{\tilde{E}(z)}{L_m} ||u_1 - u_2|| \int_{\alpha_0}^{z} \frac{ds}{s^2}$$

$$= \frac{\tilde{E}(z)}{L_m} \left(\frac{1}{\alpha_0} - \frac{1}{z} \right) ||u_1 - u_2||,$$

$$G_2(z) \equiv \int_{\alpha_0}^{z} \left| \frac{1}{L^*(u_1(s))} - \frac{1}{L^*(u_2(s))} \right| |E[s, u_2(s)]| \frac{1}{s^2} ds$$

$$\le \exp \left(-\frac{N_M}{L_m} (z^2 - \alpha_0^2) \right) \int_{\alpha_0}^{z} \frac{|L^*(u_2(s)) - L^*(u_1(s))|}{|L^*(u_1(s)||L^*(u_2(s))|} \frac{ds}{s^2}$$

$$\le \exp \left(-\frac{N_M}{L_m} (z^2 - \alpha_0^2) \right) \frac{\bar{L}}{L_m^2} ||u_1 - u_2|| \int_{\alpha_0}^{z} \frac{ds}{s^2}$$

$$\le \exp \left(-\frac{N_M}{L_m} (z^2 - \alpha_0^2) \right) \frac{\bar{L}}{L_m^2} \left(\frac{1}{\alpha_0} - \frac{1}{z} \right) ||u_1 - u_2||.$$

Then totally we obtain

$$G_1(z) + G_2(z) \equiv \frac{\tilde{E}(z)}{L_m} \left(\frac{1}{\alpha_0} - \frac{1}{z} \right) ||u_1 - u_2|| + \exp \left(-\frac{N_M}{L_m} (z^2 - \alpha_0^2) \right)$$

$$\times \ \frac{\bar{L}}{L_m^2} \left(\frac{1}{\alpha_0} - \frac{1}{z} \right) ||u_1 - u_2|| = \left[\frac{\tilde{E}(z)}{L_m} + \frac{\bar{L}}{L_m^2} \exp \left(-\frac{N_M}{L_m} (z^2 - \alpha_0^2) \right) \right]$$

$$\times \ \left(\frac{1}{\alpha_0} - \frac{1}{z} \right) ||u_1 - u_2|| = \tilde{F}(z) ||u_1 - u_2||.$$

$\square$

Now we are able to prove the following theorem.

Theorem 3. *Suppose that L^* and N^* satisfy the conditions (26)-(29). If $\alpha_0 < \mu < \mu^*$ where $\mu^* > 0$ is defined as unique solution to $\epsilon(z) = 1$ with*

$$\epsilon(z) := 2p^* \tilde{F}(z) \tag{35}$$

where $\tilde{F}(z)$ is given by (34), then there exists a unique solution $u \in C^0[\alpha_0, \mu]$ for the integral equation (19).

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Proof. We have to show that operator $V : C^0[\alpha_0, \mu] \to C^0[\alpha_0, \mu]$ defined by (25) is contracting operator.

Let us have $u_1, u_2 \in C^0[\alpha_0, \mu]$ and using Lemma 2, we have the following

$$
\begin{aligned}
|V(u_1)(\xi) - V(u_2)(\xi)| &\leq |p^*(F[\mu, u_1(\mu)] - F[\xi, u_1(\xi)]) \\
&- p^*(F[\mu, u_2(\mu)] - F[\xi, u_2(\xi)])p^*(|F[\mu, u_1(\mu)] - F[\mu, u_2(\mu)]| \\
&+ |F[\xi, u_1(\xi)] - F[\xi, u_2(\xi)]|) \leq 2p^* \tilde{F}(\mu)||u_1 - u_2||.
\end{aligned}
$$

It follows that

$$V(u_1) - V(u_2))| \leq \epsilon(\mu)||u_1 - u_2||,$$

where ϵ is defined by (36). Notice that

$$\epsilon(\alpha_0) < 1, \quad \epsilon(+\infty) = +\infty, \quad \epsilon'(z) > 0, \quad \forall z > 0.$$

Then we can make conclusion that ϵ is increasing function and thus there exists a unique $\mu^* > 0$ such that $\epsilon(\mu^*) = 1$ so the operator V becomes a contraction operator of mapping. By the fixed point Banach theorem there must exist a unique solution $u \in C^0[\alpha_0, \mu]$ to integral equation (19). $\square$

Now let us analyze the existence of unique solution for (22). We have to show that equation

$$\nu(\mu) = \mu^3, \tag{37}$$

where
$$\nu(\mu) = \nu(u(\mu), \mu) := \frac{p^* E[\mu, u(\mu)]}{KL^*(u(\mu))}$$

has a unique solution $\mu \in [\alpha_0, \mu^*]$.

Lemma 5. *Suppose (26)-(29) hold, then for all $\mu \in [\alpha_0, \mu^*]$ we have that*

$$\nu_1(\mu) < \nu(\mu) < \nu_2(\mu), \tag{38}$$

where $\nu_1(\mu)$ and $\nu_2(\mu)$ are functions defined by

$$
\begin{aligned}
\nu_1(\mu) &= \frac{p^*}{KL_M} \exp\left(-\frac{N_m}{L_M}(z^2 - \alpha_0^2) \right), \quad \mu > \alpha_0, \\
\nu_2(\mu) &= \frac{p^*}{KL_m} \exp\left(\frac{N_m}{L_M}(\mu^2 - \alpha_0^2) - \frac{N_M}{L_m}(\mu^2 - \alpha_0^2) \right), \quad \mu > \alpha_0
\end{aligned}
\tag{39}
$$

satisfying the following properties:

$$
\begin{aligned}
\nu_1(\alpha_0) &= \frac{p^*}{KL_M} > 0, \quad \nu_1(+\infty) = 0, \quad \nu_1'(\mu) < 0, \quad \forall \mu > \alpha_0, \\
\nu_2(\alpha_0) &= \frac{p^*}{KL_M} > 0, \quad \nu_2(+\infty) = 0, \quad \nu_2'(\mu) < 0, \quad \forall \mu > \alpha_0.
\end{aligned}
\tag{40}
$$

Proof. Inequality references directly to bound (30) and using straightforward definition (39) and (30) we can easily obtain properties ν_1 and ν_2. □

Lemma 6. *There exists a unique solution μ_1 to the equation*

$$\nu_1(\mu) = \mu^3, \quad \mu > \alpha_0 \tag{41}$$

and there exists a unique solution $\mu_2 > \mu_1$ to the equation

$$\nu_2(\mu) = \mu^3, \quad \mu > \alpha_0. \tag{42}$$

Proof. We can prove by using properties of ν_1 and ν_2 shown in Lemma 5. □

Theorem 7. *Suppose (26)-(29) hold. Consider μ_1 and μ_2 determined from (41) and (42). If $\epsilon(\mu_2) < 1$, where ϵ is defined by (36), then there exists at least one solution $\bar{\mu} \in (\mu_1, \mu_2)$ to the equation (37).*

Proof. By hypothesis of Lemma 5 if $\epsilon(\mu_2) < 1$ then we have that the inequality (38) holds for each $\mu_1 \leq \mu \leq \mu_2 \leq \mu^*$ and $\epsilon(\mu) < 1$. As function ν is continuous decreasing function we obtain that there exists at least one solution $\bar{\mu} \in [\mu_1, \mu_2]$ to the equation (37). $\qquad\square$

Now we can make conclusion by following main theorem.

Theorem 8. *Assume that (26)-(29) hold and $\epsilon(\mu_2) < 1$ where ϵ defined by (36) and μ_2 defined from (42) then there exist at least one solution to the problem (1)-(5) where unknown free boundary is given by*

$$\beta(t) = 2\bar{\mu}\sqrt{at}, \ \ t > 0 \tag{43}$$

where $\bar{\mu}$ defined from Theorem 7 and temperature is given by

$$\theta(r,t) = \theta_m(u_{\bar{\mu}}(\xi) + 1), \ \ \alpha_0 \leq \xi \leq \bar{\mu} \tag{44}$$

where $\xi = \dfrac{r}{2\sqrt{at}}$ being similarity substitution and $u_{\bar{\mu}}$ is the unique solution of the integral equation (19) which was established in Theorem 4.

4. Particular cases for non-linear thermal coefficients

In this section we are going to represent solution forms of the problem (1)-(5) considering types of non-linear thermal coefficients that analysed in [16]. The existence of each solution will be proved.

4.1. Constant thermal coefficients

If we take $c(\theta)$, $\gamma(\theta)$ and $\lambda(\theta)$ as follows

$$c(\theta) = c_0, \ \ \gamma(\theta) = \gamma_0, \ \ \lambda(\theta) = \lambda_0, \tag{45}$$

then solution of the problem $(\xi, u(\xi))$, taking into account that $L^* = N^* = 1$ defined by (18), must satisfy the following function

$$u(\xi) = p^* \exp(\alpha_0)\left[\frac{1}{\xi}\exp(-\xi^2) - \frac{1}{\mu}\exp(-\mu^2) + \sqrt{\pi}\operatorname{erf}(\xi) - \sqrt{\pi}\operatorname{erf}(\mu)\right] \tag{46}$$

and condition

$$\frac{p^*\exp(\alpha_0^2)\exp(-\mu^2)}{L\mu^2} = \mu, \tag{47}$$

where $p^* = \dfrac{2P_0\sqrt{a}e^{-\frac{\alpha_0^2}{a}}}{\lambda_0\sqrt{\pi}\theta_m}$, $K = \dfrac{2aL\gamma}{\theta\lambda(\theta_m)}$. To show the existence and uniqueness of solution, it is sufficient to show that exists unique value μ which satisfy the equation (47) that can be rewritten as

$$f(\mu) = \mu, \tag{48}$$

where

$$f(\mu) = \frac{p^* \exp(\alpha_0^2)\exp(-\mu^2)}{L\mu^2}.$$

We can easily check that function $f(\mu)$ is always decreasing function on interval $(0,+\infty)$ as $\lim\limits_{\mu\to 0} f(\mu) = \infty$, $\lim\limits_{\mu\to+\infty} f(\mu) = 0$ and $f'(\mu) < 0$ for all positive parameter. It implies that equation (48) has unique solution.

4.2. Linear thermal coefficients

We assume that density $\gamma(\theta)$ of the material is unchanged and specific heat $c(\theta)$ and thermal conductivity $\lambda(\theta)$ are linear as follows

$$\gamma(\theta) = \gamma_0, \quad c(\theta) = c_0\left(1 + \alpha\frac{\theta - \theta_m}{\theta_m}\right), \quad \lambda(\theta) = \lambda_0\left(1 + \beta\frac{\theta - \theta_m}{\theta_m}\right) \tag{49}$$

where $\alpha \geq 0$ and $\beta \geq 0$.

From (18) we get

$$L^*(u) = 1 + \beta u, \quad N^*(u) = 1 + \alpha u.$$

Let we have $u \in C^0[\alpha_0, \mu]$ and if we assume that $\alpha_0 = 1$, $\mu = 2$ and from consumptions (26) and (17) taking into account $L_m = 1 + \beta$, $L_M = 1+2\beta$, $N_m = 1+\alpha$ and $N_M = 1+2\alpha$ then we obtain that L^*, N^* must satisfy the following inequality

$$1 + \beta \leq L^*(u) \leq 1 + 2\beta, \quad 1 + \alpha \leq N^*(u) \leq 1 + 2\alpha. \tag{50}$$

Then solution (19) and condition (22) becomes

$$u(\xi) = \frac{p^*}{1+2\beta}\left[\frac{1}{\xi} - \frac{1}{\mu} + \sqrt{\frac{\pi(1+2\alpha)}{1+2\beta}}\exp\left(\alpha_0^2\frac{1+2\alpha}{1+2\beta}\right)\right.$$

$$\times \operatorname{erf}\left(\xi\sqrt{\frac{1+2\alpha}{1+2\beta}}\right) - \sqrt{\frac{\pi(1+2\alpha)}{1+2\beta}}\exp\left(\alpha_0^2\frac{1+2\alpha}{1+2\beta}\right) \tag{51}$$

$$\left.\times \operatorname{erf}\left(\mu\sqrt{\frac{1+2\alpha}{1+2\beta}}\right)\right],$$

$$\frac{p^* \left[\dfrac{1}{\xi^2} - \sqrt{\pi} \dfrac{1+2\alpha}{1+2\beta} \exp\left(\alpha_0^2 \dfrac{1+2\alpha}{1+2\beta} \right) \exp\left(-\xi^2 \dfrac{1+2\alpha}{1+2\beta} \right) \right]}{K(1+2\beta)} = \mu. \qquad (52)$$

To prove existence and uniqueness of the solution, it is enough to show that there is unique solution of the equation

$$g(\mu) = \mu \qquad (53)$$

where

$$g(\mu) = \frac{p^* \left[\dfrac{1}{\xi^2} - \sqrt{\pi} \dfrac{1+2\alpha}{1+2\beta} \exp\left(\alpha_0^2 \dfrac{1+2\alpha}{1+2\beta} \right) \exp\left(-\xi^2 \dfrac{1+2\alpha}{1+2\beta} \right) \right]}{K(1+2\beta)}.$$

It is easy to check that $g(\mu)$ is decreasing function on interval $(0, +\infty)$ such that $\lim\limits_{\mu \to 0} g(\nu) = \infty$, $\lim\limits_{\mu \to +\infty} g(\mu) = 0$ and $g'(\mu) < 0$. It follows that equation (53) has unique solution.

Conclusion

We have studied one-phase spherical Stefan problem with heat flux entering to electrical contact material through electrical arc and temperature in liquid metal zone and free boundary on melting interface are determined. Existence and uniqueness of the similarity solution imposing Neumann condition at the given left free boundary is proved. We have represented solution forms to the particular cases when nonlinear thermal coefficients are constant or linear and for each cases existence and uniqueness of the solution is proved.

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