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ON THE MAGNETIC RADIAL SCHRÖDINGER-HARTREE EQUATION

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Abstract: We prove, in any space dimension $d \geq 4$, the decay in the energy space for the defocusing magnetic Schrödinger-Hartree equation with radial initial data in $H^1(\mathbb{R}^d)$. We will exhibit also new Morawetz inequalities and localized correlation estimates.

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1. Introduction

Consider the Cauchy problem for the nonlinear defocusing magnetic Schrödinger equation with Hartree-type nonlinearity (mSH), for $d \geq 4$:

$$\begin{cases} i\partial_t u + \Delta_x^A u - [|\cdot|^{d-\gamma} * |u|^2]u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x) \in H^1(\mathbb{R}^d), \end{cases} \tag{1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\nabla_x^A = \nabla - iA$, $A = \left(A^1, \ldots, A^d\right) \in \mathbb{C}^1_{loc}(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$, so that $\operatorname{div} A = 0$ and $-\Delta_x^A = -\nabla_x^A \cdot \nabla_x^A$ is self-adjoint on $L^2(\mathbb{R}^d)$. We shall assume also

$$|A|^2 - 2iA \cdot \nabla \in L^{\frac{d}{2},\infty}(\mathbb{R}^d), \quad A \in L^{d,\infty}(\mathbb{R}^d). \tag{2}$$

Moreover

$$\||x|xB\|^2_{L^\infty(\mathbb{R}^d)} \leq \frac{2}{3}(d-1)(d-3), \tag{3}$$

where $B : \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{R})$ is defined by $B := DA - (DA)^t$, with $(DA)_{ij} = \partial_i A^j, (DA)^t_{ij} = (DA)_{ji}$. We will impose, from now on, also that $d-2 \leq \gamma \leq d$.

The main result of this paper is the decay of the solutions to (1) in the energy space. More specifically,

Theorem 1. *Assume $d \geq 4$ and let $u \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ be a global solution to* (1) *with radial initial data $f \in H^1(\mathbb{R}^d)$ such that* (2) *and* (3) *are satisfied. Then, for any $2 < r < \frac{2d}{d-2}$, one has*

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^r(\mathbb{R}^d)} = 0. \tag{4}$$

The equation (1) is important in many models of the mathematical physics. For instance, it was introduced in quantum mechanics in order to analyse the behaviour of the Bose-Einstein condensates, by considering the self-interactions of the charged particles, as it can be seen in [6], [11], [12] and the references therein. This enhanced several works treating the Schrödinger-Hartree equation (SH). We cite [9] where the asymptotic completeness and the existence of the wave operators are shown for both the nonlinear Schrödinger equation with $L^2 - H^1$ intercritical nonlinearity and for the SH equation. Regarding the latter equation, the previous results were successively improved in [14]. Moreover, in [8] and [15] the pseudo-conformal transform is utilized to study the scattering for the solutions to the SH equation in spaces with higher regularity than H^1. In the critical case, [13] established scattering for general data with $d \geq 5$. Scattering in the focusing case was achieved in [1] and [2] for small data and radial data, respectively. We refer also to [16] and [18] for the NLS in a general setting. One of the fundamental tools used to study the dynamics of solutions to (1) is the Morawetz multiplier technique and the associated estimates. In our recent work [17], we successfully developed a method combining the Morawetz inequalities, a localization step, and interpolation together with a contradiction argument accomplishing the decay of the solutions to the SH. This strong property plays a crucial role in the theory of the scattering, as underlined in [3], [17] and [18] (see also references therein). Moved by that we present here a generalization of such a method to the case of the mSH. We emphasize that our result is new in the literature and minimal assumptions are made on the magnetic function $A(x)$. In addition, we mention that our strategy eases the one exhibited in [4].

2. Preliminaries

We indicate $L^r(\mathbb{R}^d) = L^r_x$, for $1 \le r \le \infty$. We denote also

$$H^{1,r}(\mathbb{R}^d) = (1-\Delta_x)^{-\frac{1}{2}} L^r(\mathbb{R}^d), \quad H^{1,r}(\mathbb{R}^d) = H^{1,r}_x,$$

and $H^{1,2}(\mathbb{R}^d) = H^1(\mathbb{R}^d) = H^1_x$. Given any Banach space X, we define

$$\|f\|_{L^\infty_t X} = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|f(x)\|_X .$$

We adopt the notation $L^\infty_T X$ when one restricts $t \in (-T, T)$, for $T > 0$. The following results are also useful (see [1] and [5], respectively).

Lemma 2. *Let f be a radial function in H^1_x. Then*

$$\left\| |x|^{\frac{d-1}{2}} f \right\|^2_{L^\infty_x} \lesssim \|f\|_{L^2_x} \|\nabla_x f\|_{L^2_x}. \tag{5}$$

Proposition 3. *Let A be as in (2) and (3). For any $1 < r < d$, one gets*

$$\left\| (-\Delta^A_x)^{\frac{1}{2}} f \right\|_{L^r_x} \lesssim \left\| (-\Delta_x)^{\frac{1}{2}} f \right\|_{L^r_x} \tag{6}$$

and

$$\left\| (-\Delta_x)^{\frac{1}{2}} f \right\|_{L^2_x} \lesssim \left\| (-\Delta^A_x)^{\frac{1}{2}} f \right\|_{L^2_x} . \tag{7}$$

We need also the next maximal estimate (see [10]) which is a consequence of the Hardy inequality.

Proposition 4. *Let $0 < \gamma < d$, we have*

$$\left\| [|\cdot|^{d-\gamma} * |u|^2] \right\|_{L^\infty_x} \le C(d,\gamma) \|u\|^2_{\dot{H}_x^{\frac{d-\gamma}{2}}}.$$

Recall that the solution u to (1) satisfies two conservation laws

$$\|u(t)\|_{L^2_x} = \|f\|_{L^2_x}, \quad E(u(t)) = E(f), \tag{8}$$

where

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x^A u(t,x)|^2\, dx + \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^{d-\gamma}}\, dxdy. \tag{9}$$

We address [4] for more details on the argument.

3. Well-posedness

In this section, we establish the following global existence result, which is mandatory for the proof of (4). More precisely we achieve

Proposition 5. *Let be $d \ge 4$, assume that (2) and (3) are satisfied. Then for all $f \in H^1_x$ there exists a unique global solution $u \in \mathcal{C}(\mathbb{R}; H^1_x)$ to (1) and such that*

$$\|u\|_{L^\infty_t H^1_x} \lesssim \|f\|_{H^1_x}.$$

Proof. We will perform a contraction argument. Namely, let the integral operator associated to (1) be defined for all $f \in H^1_x$ as

$$\mathcal{T}_f(u) = e^{t\Delta_x^A} f + k\int_0^t e^{(t-\tau)\Delta_x^A}\Big([|\cdot|^{d-\gamma} * |u|^2]u\Big)(\tau)\, d\tau.$$

One has to show that there exist a $T = T\big(\|f\|_{H^1_x}\big) > 0$ and a unique

$$u(t,x) \in L^\infty_T H^1_x$$

satisfying the property

$$\mathcal{T}_f(u(t)) = u(t), \tag{10}$$

for any $t \in (-T, T)$. For the aim of simplicity, we shall divide the proof into four different steps.

Step One: For any $u \in H^1_x$, *there exist* $T = T\big(\|f\|_{H^1_x}\big) > 0$ *and* $R = R\big(\|f\|_{H^1_x}\big) > 0$ *such that*

$$\mathcal{T}_f(B_{L^\infty_{T'} H^1_x}(0, R)) \subset B_{L^\infty_{T'} H^1_x}(0, R),$$

for any $T' < T$.

By the classical Hardy-Littlewood-Sobolev inequality in combination with (6) and (7), we have

$$\begin{aligned}
&\|\mathcal{T}_f(u)\|_{L^\infty_T L^2_x} + \|\nabla_x \mathcal{T}_f(u)\|_{L^\infty_T L^2_x} \\
&\lesssim \|\mathcal{T}_f(u)\|_{L^\infty_T L^2_x} + \|\nabla^A_x \mathcal{T}_f(u)\|_{L^\infty_T L^2_x} \\
&\lesssim \|f\|_{H^1_x} + T\|[|\cdot|^{d-\gamma} * |u|^2]u\|_{L^\infty_T H^1_x} \\
&\lesssim \|f\|_{H^1_x} + T\left\|[|\cdot|^{d-\gamma} * |u|^2]\right\|_{L^\infty_T L^\infty_x} \|u\|_{L^\infty_T H^1_x} \\
&\quad + T\left\|[|\cdot|^{d-\gamma} * |u|^2]\right\|_{L^\infty_T H_x^{1,\frac{2d}{d-\gamma}}} \|u\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}} \\
&\lesssim \|f\|_{H^1_x} + T\|u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}} \|u\|_{L^\infty_T H^1_x} + T\left\||u|^2\right\|_{L^\infty_T H_x^{1,\frac{2d}{d+\gamma}}} \|u\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}} \\
&\lesssim \|f\|_{H^1_x} + T\|u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}} \|u\|_{L^\infty_T H^1_x} + T\|u\|_{L^\infty_T H^1_x} \|u\|^2_{L^\infty_T L_x^{\frac{2d}{\gamma}}} \\
&\lesssim \|f\|_{H^1_x} + T\|u\|^2_{L^\infty_T \dot{H}_x^{\frac{d-\gamma}{2}}} \|u\|_{L^\infty_T H^1_x} \lesssim \|f\|_{H^1_x} + TR^3.
\end{aligned}$$

Once we choose R and T such that

$$\|f\|_{H^1_x} = \frac{R}{2}, \quad CTR^3 \leq \frac{R}{2},$$

we complete the proof of the step.

Step Two: Let $T, R > 0$ *be as in the previous step, then there exists* $\overline{T} = \overline{T}\big(\|f\|_{H^1_x}\big) < T$ *such that* $\mathcal{T}_f$ *is a contraction on* $B_{L^\infty_{\overline{T}} H^1_x}(0, R)$, *equipped with the norm* $\|.\|_{L^\infty_{\overline{T}} L^2_x}$.

Given any $v_1, v_2 \in B_{L^\infty_{\overline{T}} H^1_x}(0, R)$ we get the bounds

$$\begin{aligned}
&\|\mathcal{T}_f v_1 - \mathcal{T}_f v_2\|_{L^\infty_T L^2_x} \\
&\lesssim T\left\|[|\cdot|^{d-\gamma} * |v_1|^2]v_1 - [|\cdot|^{d-\gamma} * |v_2|^2]v_2\right\|_{L^\infty_T L^2_x}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim T\left\|[|\cdot|^{d-\gamma}*|v_1|^2](v_1-v_2)\right\|_{L^\infty_T L^2_x}\\
&+T\left\|[|\cdot|^{d-\gamma}*(|v_1|^2-|v_2|^2)]v_2\right\|_{L^\infty_T L^2_x}\\
&\lesssim T\|v_1\|^2_{L^\infty_T H_x^{\frac{d-\gamma}{2}}}\|v_1-v_2\|_{L^\infty_T L^2_x}\\
&+T\left\|[|\cdot|^{d-\gamma}*(|v_1|^2-|v_2|^2)]\right\|_{L^\infty_T L_x^{\frac{2d}{d-\gamma}}}\|v_2\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\\
&\lesssim T\left(R^2\|v_1-v_2\|_{L^\infty_T L^2_x}+R\|v_1+v_2\|_{L^\infty_T L_x^{\frac{2d}{\gamma}}}\|v_1-v_2\|_{L^\infty_T L^2_x}\right)\\
&\lesssim TR^2\|v_1-v_2\|_{L^\infty_T L^2_x}.
\end{aligned}
$$

Then we arrive at

$$\|\mathcal{T}_f v_1-\mathcal{T}_f v_2\|_{L^\infty_T L^2_x}\lesssim TR^2\|v_1-v_2\|_{L^\infty_T L^2_x}.$$

The upper inequality says that $\mathcal{T}_f$ is a contraction on $B_{L^\infty_T H^1_x}(0,R)$ if T is suitably small.

Step Three: The solution exists and is unique in $L^\infty_{\overline{T}} H^1_x$, where $\overline{T}$ is as in the above step.

We are in a position to show existence and uniqueness of the solution by using the contraction principle to the map $\mathcal{T}_f$ defined on the complete metric space $B_{L^\infty_{\overline{T}} H^1_x}(0,R)$, equipped with the topology induced by $\|.\|_{L^\infty_{\overline{T}} L^2_x}$.

Step Four: The solution can be extended globally.

It is easy to see that the conservation laws (8) lead then to the global well-posedness for (1).

□

4. Morawetz identities and inequalities

We introduce also some further notations. Given a function $f\in H^1(\mathbb{R}^d;\mathbb{C})$, we denote by

$$m_f(t,x):=|f(t,x)|^2,\qquad j_f^A(t,x):=\operatorname{Im}\left[\overline{f}(t,x)\nabla_x^A f(t,x)\right],\tag{11}$$

the mass density and the momentum density, respectively. Our first contribution reads then as;

Lemma 6. *Let be $d \geq 1$ and $u \in \mathcal{C}(\mathbb{R}; H^1_x)$ be a global solution to* (1) *with radial initial data $f \in H^1_x$ such that* (2) *and* (3) *are satisfied. Moreover, let be $\phi = \phi(x) : \mathbb{R}^d \to \mathbb{R}$ a sufficiently regular and decaying function, and denote by*

$$V(t) := \int_{\mathbb{R}^d} \phi(x)\, m_u(t,x)\, dx.$$

Then the following identities hold:

$$\dot{V}(t) = \int_{\mathbb{R}^d} \phi(x) \dot{m}_u(t,x)\, dx = 2 \int_{\mathbb{R}^d} j^A_u(t,x) \cdot \nabla_x \phi(x)\, dx \tag{12}$$

and

$$\begin{aligned} \ddot{V}(t) = \int_{\mathbb{R}^d} \phi(x) \ddot{m}_u(t,x)\, dx = & -\int_{\mathbb{R}^d} \Delta^2_x \phi(x) |u(t,x)|^2\, dx \\ & +4 \int_{\mathbb{R}^d} \nabla^A_x u(t,x) D^2_x \phi(x) \cdot \overline{\nabla^A_x u(t,x)}\, dx \\ & -4\mathrm{Im} \int_{\mathbb{R}^d} \nabla_x \phi(x) \cdot B(x) \overline{\nabla^A_x u(t,x)}\, dx \\ -2 \int_{\mathbb{R}^d} \nabla_x \phi(x) \cdot \nabla_x & \left[|x|^{-(d-\gamma)} * |u(t,x)|^2 \right] |u(t,x)|^2\, dx, \end{aligned} \tag{13}$$

where $D^2_x \phi \in \mathcal{M}_{d \times d}(\mathbb{R})$ is the Hessian matrix of ϕ and $\Delta^2_x \phi = \Delta_x(\Delta_x \phi)$ the Bi-Laplacian operator.

Proof. We prove the identities for a smooth rapidly decreasing solution $u = u(t,x)$, letting the general case $u \in \mathcal{C}(\mathbb{R}; H^1_x)$ to a density argument. The proof of (12) can be found in [7]. We give details for obtaining (13) and drop the variable t for simplicity. An integration by parts and (1) give

$$\begin{aligned} & 2\partial_t \int_{\mathbb{R}^d} j^A_u(x) \cdot \nabla_x \phi(x)\, dx \\ &= 2\mathrm{Re} \int_{\mathbb{R}^d} i\partial_t u(x) \left(\Delta_x \phi(x) \bar{u}(x) + 2 \nabla_x \phi(x) \cdot \overline{\nabla^A_x u(x)} \right) dx \\ &= 2\mathrm{Re} \int_{\mathbb{R}^d} \left(-\Delta^A_x u(x) \right) \left(\Delta_x \phi(x) \bar{u}(x) + 2 \nabla_x \phi(x) \cdot \overline{\nabla^A_x u(x)} \right) dx \\ & \quad +2\mathrm{Re} \int_{\mathbb{R}^d} [|x|^{-(d-\gamma)} * |u(x)|^2] u(x) \Delta_x \phi(x) \bar{u}(x)\, dx \\ & \quad +4\mathrm{Re} \int_{\mathbb{R}^d} [|x|^{-(d-\gamma)} * |u(x)|^2] u(x) \nabla_x \phi(x) \cdot \overline{\nabla^A_x u(x)}\, dx \end{aligned} \tag{14}$$

with $\tilde{\gamma} = d-\gamma$. We have the following identity for the linear terms (see Theorem 1.2 in [7]),

$$\begin{aligned} &2\mathrm{Re}\int_{\mathbb{R}^d}\left(-\Delta_x^A u(x)\right)\left(\Delta_x\phi(x)\bar{u}(x)+2\nabla_x\phi(x)\cdot\overline{\nabla_x^A u(x)}\right)dx \\ &= -\int_{\mathbb{R}^d}\Delta_x^2\phi(x)\,|u(x)|^2\,dx - 4\mathrm{Im}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot B(x)\overline{\nabla_x^A u(x)}\,dx \\ &\quad +4\int_{\mathbb{R}^d}\nabla_x^A u(x)D_x^2\phi(x)\overline{\nabla_x^A u(x)}\,dx. \end{aligned} \tag{15}$$

Moreover, one gets for the nonlinear terms

$$\begin{aligned} &2\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]|u(x)|^2\Delta_x\phi(x)\,dx \\ &+4\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]u(x)\nabla_x\phi(x)\cdot\overline{\nabla_x^A u(x)}\,dx \\ &= 2\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]|u(x)|^2\Delta_x\phi(x)\,dx \\ &+4\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]u(x)\nabla_x\phi(x)\cdot\nabla_x\overline{u}(x)\,dx \\ &= 2\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]|u(x)|^2\Delta_x\phi(x)\,dx \\ &+2\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]\nabla_x\phi(x)\cdot\nabla_x|u(x)|^2\,dx. \end{aligned}$$

Then, by an integration by parts of the second term in the last line above, one retrieves

$$\begin{aligned} &2\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]|u(x)|^2\Delta_x\phi(x)\,dx \\ &+4\mathrm{Re}\int_{\mathbb{R}^d}[|x|^{-(d-\gamma)}*|u(x)|^2]u(x)\nabla_x\phi(x)\cdot\overline{\nabla_x^A u(x)}\,dx \\ &= -2\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot\nabla_x\left[|x|^{-(d-\gamma)}*|u(x)|^2\right]|u(x)|^2\,dx. \end{aligned} \tag{16}$$

Combining now identities (15) and (16), we arrive finally at (13). □

4.1. A localized Morawetz inequality

At this point we can prove the following

Lemma 7. *Assume $d \geq 4$ and let $u \in \mathcal{C}(\mathbb{R}; H_x^1)$ be a global solution to* (1) *with radial initial data $f \in H_x^1$ such that* (2) *and* (3) *are satisfied. Then it holds that*

$$\int_{\mathbb{R}^d} \frac{1}{|x|^3} |u(t,x)|^2 \, dx \lesssim \dot{V}(t). \tag{17}$$

Proof. We choose $\psi = \psi(x) = |x|$. This gives

$$\nabla_x \psi = \frac{x}{|x|}, \quad \Delta_x \psi = \frac{d-1}{|x|}, \quad \Delta_x^2 \psi = -\frac{(d-1)(d-3)}{|x|^3}, \tag{18}$$

if $d \geq 4$. By the Morawetz identity one obtains then the following

$$\begin{aligned} 2\partial_t \int_{\mathbb{R}^d} j_u^A(t,x) \cdot \nabla_x \phi(x) \, dx = -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x^2 \phi(x) m_u(t,x) \, dx \\ -4\mathrm{Im} \int_{\mathbb{R}^d} \nabla_x \phi(x) \cdot B(x) \overline{\nabla_x^A u(t,x)} \, dx \\ +4 \int_{\mathbb{R}^d} \nabla_x^A u(t,x) D_x^2 \phi(x) \overline{\nabla_x^A u(t,x)} \, dx \\ +(d-\gamma) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{d-\gamma+2}} |u(t,x)|^2 |u(t,z)|^2 K(x,z) \, dx dz, \end{aligned} \tag{19}$$

with

$$K(x,z) = (x-z) \cdot \left(\frac{x}{|x|} - \frac{z}{|z|} \right) = (|x||y| - x \cdot y) \left(\frac{1}{|x|} + \frac{1}{|y|} \right) \geq 0. \tag{20}$$

We can discard the last term on the l.h.s of (19). Let us focus now on the linear terms in (19), for which we will follow the approach of [7]. Notice that, with this choice of the multiplier $\phi(x)$, we have

$$\nabla_x^A u(t,x) D^2 \phi(x) \overline{\nabla_x^A u(t,x)} = \frac{|\nabla_A^\tau u(t,x)|^2}{|x|}, \tag{21}$$

(see identity (3.9) in [7]) where ∇_A^τ is defined as

$$\nabla_A^\tau u(t,x) = \nabla_x^A u(t,x) - \left(\nabla_x^A u(t,x) \cdot \frac{x}{|x|} \right) \frac{x}{|x|}.$$

Thus, having in mind (18), we get the following identity

$$\begin{aligned} &-2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x^2\phi(x)m_u(t,x)\,dx \\ &-4\mathrm{Im}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot B(x)\overline{\nabla_x^A u(t,x)}\,dx \\ &+4\int_{\mathbb{R}^d}\nabla_x^A u(t,x)D_x^2\phi(x)\overline{\nabla_x^A u(t,x)}\,dx \\ &= 4\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx+(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx \\ &+4\mathrm{Im}\int_{\mathbb{R}^d}u(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A u(t,x)}\,dx. \end{aligned} \tag{22}$$

We have the estimate

$$\begin{aligned} &-\left|\mathrm{Im}\int_{\mathbb{R}^d}u(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A u(t,x)}\,dx\right| \\ &\geq-\left(\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}|x|^2\,|xB(x)|^2\,|\nabla_A^\tau u(t,x)|^2\,dx\right)^{\frac12} \\ &\geq -C^*\left(\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx\right)^{\frac12}, \end{aligned} \tag{23}$$

with

$$C^*=\sqrt{\frac{2}{3}(d-1)(d-3)}.$$

As a result, the r.h.s. of (22) can be bounded as

$$\begin{aligned} &4\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx+(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx \\ &+4\mathrm{Im}\int_{\mathbb{R}^d}u(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A u(t,x)}\,dx \\ &\geq 4\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx+(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx \\ &-4\widetilde{C}\left(\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx\right)^{\frac12}>0. \end{aligned} \tag{24}$$

Notice also that the previous inequality guarantees that

$$4\int_{\mathbb{R}^d}\frac{|\nabla_A^\tau u(t,x)|^2}{|x|}\,dx+(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx \\ +4\mathrm{Im}\int_{\mathbb{R}^d}u(t,x)\frac{x}{|x|}B(x)\cdot\overline{\nabla_x^A u(t,x)}\,dx \\ >\eta(d-1)(d-3)\int_{\mathbb{R}^d}\frac{|u(t,x)|^2}{|x|^3}\,dx, \tag{25}$$

for some $\eta>0$. The above bound in combination with (19), (20) and (22) gives the proof of (17). □

We have the following corollary, that is a consequence of (17).

Corollary 8. *Let $u\in\mathcal{C}(\mathbb{R};H_x^1)$ be a global solution to (1) with radial initial data $f\in H_x^1$ such that (2) and (3) are satisfied. Moreover, let be $\mathcal{Q}_{\tilde{x}}^d(r)=\tilde{x}+[-r,r]^d$, with $r>0$ and $\tilde{x}\in\mathbb{R}^d$. Hence one gets,*

$$\int_{\mathbb{R}}\int_{\mathcal{Q}_{\tilde{x}}^d(r)}\frac{1}{|x|^3}|u(t,x)|^2\,dxdt<\infty. \tag{26}$$

Proof. By integrating (17) with $\psi(x)$ as in (18) w.r.t. the time variable on the interval $J=[t_1,t_2]$, with $t_1,t_2\in\mathbb{R}$, one arrives at

$$2\left[\int_{\mathbb{R}^d}j_u^A(t,x)\cdot\nabla_x\psi(x)\,dx\right]_{t=t_1}^{t=t_2} \\ +\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\frac{1}{|x|^3}|u(t,x)|^2\,dxdt\gtrsim\int_{t_1}^{t_2}\int_{\mathcal{Q}_{\tilde{x}}^d(r)}\frac{1}{|x|^3}|u(t,x)|^2\,dxdydt.$$

By applying the Cauchy-Schwartz inequality and the Proposition 3, we have also

$$\left[\int_{\mathbb{R}^d}j_u^A(t,x)\cdot\nabla_x^A\psi(x,y)\,dx\right]_{t=t_1}^{t=t_2}\lesssim\|f\|_{H_x^1}^2<\infty, \tag{27}$$

since the H_x^1-norm is a conserved quantity. In the end, we get (26) once $t_1\to-\infty, t_2\to\infty$. □

5. The decay of solutions

In this section we prove the main Theorem 1.

Proof. It is sufficient to prove the property (4) for a suitable $2 < q < \frac{2d}{d-2}$, because the thesis for the general case follows by the conservation laws (8) and interpolation. More precisely it is enough to show that

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L_x^{2+\frac{4}{d}}} = 0. \tag{28}$$

Then the property (4) follows for all $2 < q < \frac{2d}{d-2}$ by combining (28) with

$$\sup_{t\in\mathbb{R}} \|u(t,x)\|_{H_x^1} < \infty. \tag{29}$$

We recall the following localized Gagliardo-Nirenberg inequality (see [17])

$$\|\varphi\|^{\frac{2d+4}{d}}_{L_x^{\frac{2d+4}{d}}} \le C\left(\sup_{x\in\mathbb{R}^d} \|\varphi\|_{L^2(\mathcal{Q}_x(1))}\right)^{\frac{4}{d}} \|\varphi\|^2_{H_x^1}, \tag{30}$$

where $\mathcal{Q}_x^d(r) = x+[-1,1]^d$. Next, assume by contradiction that (28) is not true, then by (29) and by (30) we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ with $|t_n| \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|u(t_n,x)\|^2_{L^2(\mathcal{Q}_{x_n}(1))} = \epsilon_0^2. \tag{31}$$

For simplicity we can assume that $t_n \to \infty$ because the case $t_n \to -\infty$ can be treated by a similar argument. Notice that by (12) in conjunction with (29) we get

$$\sup_{n,t}\left|\frac{d}{dt}\int \chi(x-x_n)|u(t,x)|^2\,dx\right| < \infty,$$

where $\chi(x)$ is a smooth and non-negative cut-off function, s.t. $\chi(x) = 1$ for $x \in \mathcal{Q}_0(1) = [-1,1]^d$ and $\chi(x) = 0$ for $x \notin \mathcal{Q}_0(2) = [-2,2]^d$. Consequently, by the Fundamental Theorem of calculus we infer

$$\left|\int_{\mathbb{R}^d} \chi(x-x_n)|u(\sigma,x)|^2\,dx - \int_{\mathbb{R}^d} \chi(x-x_n)|u(t,x)|^2\,dx\right| \le \widetilde{C}|t-\sigma|, \tag{32}$$

for a $\widetilde{C} > 0$ which is independent form n. By combining this fact with (31) and the geometric properties of χ, we get the existence of $T > 0$ such that

$$\inf_n \big(\inf_{t\in(t_n,t_n+T)} \|u(t,x)\|^2_{L^2(\mathcal{Q}_{x_n}(2))}\big) \gtrsim \epsilon_1^2, \tag{33}$$

for some $\epsilon_1 > 0$, which implies, by Hölder inequality,

$$\inf_n \Big(\inf_{t\in(t_n,t_n+T)} \|u(t,x)\|^2_{L^\infty(\mathcal{Q}_{x_n}(2))}\Big) \gtrsim \epsilon_1^2. \tag{34}$$

Observe that the previous bound (34) guarantees, in combination with the radial inequality (5), that $|x_n| \lesssim 1$ for all n. Observe also that since $t_n \to \infty$ we can assume (via subsequence) that the intervals $(t_n, t_n + T)$ are disjoint. In particular we have, for $d \geq 4$,

$$\sum_n T\epsilon_1^2 \lesssim \sum_n \int_{t_n}^{t_n+T} \int_{\mathcal{Q}^d_{x_n}(2)} |u(t,x)|^2 \, dxdt \tag{35}$$

$$\lesssim \int_{\mathbb{R}} \sup_{\tilde{x}\in\mathbb{R}^d} \int_{\mathcal{Q}^d_{\tilde{x}}(2)} \frac{1}{|x|^3} |u(t,x)|^2 \, dxdt, \tag{36}$$

consequently we get a contradiction because the left hand side is divergent and the right hand side is bounded by (26). □

Remark 9. We underline once again the fact that the assumptions (2), (3) made on the operator ∇_x^A and the function $A(x)$ are less stringent than the one imposed in [7], [5] and [4]. This is due to the fact that our well-posedness analysis is based only on the energy estimate for (1). We are not using any Strichartz estimates here, forcing to the constraint $d-2 \leq \gamma < d$. The obstacle here is that the multipliers utilized in the aforementioned papers are not well suited to handle a non-local nonlinearity, because their application can not guarantee the non-negativity of the last term in (19). Such aspect determines also the radial assumption we made on the initial data. A second problem is the equivalence of norm result of the Proposition 3, valid only in the L^2 framework. We confide to overcome all this issues and shed light on the lower regularity scenario $d - 4 \leq \gamma < d - 2$ in a forthcoming paper.

Remark 10. Note that by (9), Young inequality and Gronwall's inequality, one can acquire the following

$$\|u(t)\|_{H^1_x} \lesssim \|f\|_{H^1_x} e^{Kt}.$$

with $K > 0$ depending on the L^2 size of f and $E(f)$. Then, the Sobolev embedding and interpolation with the conservation of mass in (8) lead to

$$\|u(t)\|_{L^r_x} \lesssim e^{\theta Kt},$$

with $0 < \theta < 1$ and $2 < q \leq \frac{2d}{d-2}$, which is not enough to guarantee a behavior like the one unveiled by (4) in Theorem 1.

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