

A SURVEY FOR A GENERALIZED LIÉNARD
EQUATION WITHOUT THE MASSERA CONDITION

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Abstract: Massera's method [13] is powerful for the criterion of the unique existence of limit cycles of a classical Liénard equation as is seen in the Van der Pol equation. However, by the reason of which the condition of $f(x)$ is strong (it is the condition that there does not exist the solutions satisfying the equation $f'(x) = 0$ except $x = 0$), we cannot apply it to many examples. Our purpose is to give several theorems such as can be applied to the generalized Liénard equation without Massera's condition. As an application, the unique existence of limit cycles of the Duff-Levinson system which is well-known in [3] or [15] is discussed as an improvement of [7].

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1. Introduction

Many results for the qualitative properties of limit cycles of a classical Liénard equation

$$\ddot{x} + f(x)\dot{x} + x = 0 \tag{1}$$

have been shown in [1], [6], [8], [14] and [15] et al. They have been accomplished widely by mathematicians, physicians and the other researchers, the topics as is seen in one of the famous Hilbert 23-th problem have been studied until now.

The Massera method ([13], 1954) is one of significant tools to give these results. Though it is classical, it is a powerful and an important method such as are seen in Examples 1 and 2. It is stated as follows.

Proposition 1. ([13]) *Assume that the conditions:*

(C1) *$f(x)$ is a continuous function,*

(C2) *there exist a and b ($a < 0 < b$) such that $f(x) < 0$ ($a < x < b$) and $f(x) > 0$ ($x \leq a, x \geq b$),*

(C3) *$f(x)$ is nondecreasing as $|x|$ increase*

are satisfied. Then equation (1) has a unique limit cycle.

As an important example of Proposition 1, for instance, we can show the following.

Example 1. Consider equation (1) with $f(x) = \epsilon(x^2 - 1)$. This is the well-known as "the Van der Pol equation". We shall remark that the condition (C3) holds in this equation. Thus, the system has a unique limit cycle for all $\epsilon > 0$.

Equation (1) is generalized to

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (2)$$

and it is transformed to the following system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \quad (3)$$

where $F(x) = \int_0^x f(\xi)d\xi$.

Example 2. Llibre-Ordóñez-Ponce [6] gave the unique existence of the limit cycle of system (3) in the form

$$F(x) = \begin{cases} t_R(x - 1) + t_C & (x \geq 1) \\ t_C x & (|x| \leq 1) \\ t_L(x + 1) - t_C & (x \leq -1), \end{cases}$$

$$g(x) = \begin{cases} d_R(x - 1) + d_C & (x \geq 1) \\ d_C x & (|x| \leq 1) \\ d_L(x + 1) - d_C & (x \leq -1). \end{cases}$$

This system is called a continuous piecewise linear Liénard system. Since $F'(x) = f(x)$, the system (3) with the above form do not satisfy Massera's condition (C1) in Proposition 1 except $t_R = t_C = t_L$. They proved the existence of limit cycles by using the original method in the phase analysis, and gave its uniqueness by the similar discussion to the idea in the proof of Proposition 1. In this system also, remark that the condition (C3) is assumed.

Though the Massera method can be applied to several equations as are shown in the above examples, for instance, we cannot apply Proposition 1 to the equation with $f(x) = 3x^2 - 4x - 3$ or $g(x) = x^{2n-1}$ ($n \in \mathbb{N}$) in system (2). In facts, this function $f(x)$ can't be satisfied the condition (C3). This example is resolved by Corollary 2.

Ciambellotii [1] gave a generalization of Massera's theorem for the particular cases of the generalized Liénard equation $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$. He proved that the limit cycle is star-shaped without using a transformation in polar coordinats. However, in this paper also, we note that the conditions (C2) for $a = b$ and (C3) is proposed for the uniqueness of the limit cycle.

Our aim is to give the uniqueness of the limit cycles of equation (2) without Massera's condition (C3).

In next section, our main results and the proofs are given. In §3, an example proposed by Duff-Levinson is discussed and confirm the power of our method.

2. Main results and proofs

First, we consider the generalized equation of (1) with one parameter:

$$\ddot{x} + \lambda f(x)\dot{x} + g(x) = 0, \quad (4)$$

where λ is a positive real number. For instance, see [7]–[10].

Equation (4) is equivalent to the following Liénard system:

$$\dot{x} = y - \lambda F(x), \quad \dot{y} = -g(x). \quad (5)$$

Through this paper, we assume the conditions:

(C4) $f(x)$ and $g(x)$ are locally Lipschitz continuous functions and $g(x)/x > 0$,

(C5) there exist α and β with $\alpha < 0 < \beta$ such that $x(x - \alpha)(x - \beta)F(x) > 0$.

It is well-known under the above assumptions that the uniqueness of solutions of system (5) for initial value problems is guaranteed and the only equilibrium point $(0, 0)$ is unstable. For instance, see [5].

Let $G(x) = \int_0^x g(\xi)d\xi$ and $G(\alpha) \geq G(\beta)$ without loss of generality. Then there exists $\beta^* \in [\alpha, 0)$ such that $G(\beta) = G(\beta^*)$.

The following (see [2] or [4]) is a useful result for the existence of the limit cycle.

Lemma 2. *Assume that the conditions (C4), (C5) and besides*

$$(C6) \quad \limsup_{x \rightarrow \pm\infty} \{G(x) \pm F(x)\} = +\infty$$

are satisfied, then system (5) has at least one limit cycle.

The following ([7]) is one of the key to state our result. Let Ω be the region surrounded by a closed curve $V(x, y) = (\lambda/2)y^2 + G(x) = G(\beta)$. It surrounds the only one equilibrium point $(0, 0)$ of system (5) under the conditions (C4) and (C5).

Lemma 3. *Assume the conditions (C4), (C5) and (C6). All limit cycles of system (5) exist outside Ω and intersect the lines $x = \beta^*$ and $x = \beta$.*

Consider the case in which there exists $r \neq 0$ such that $F''(r) = f'(r) = 0$. Then remark that Massera's condition (C3) is not satisfied.

Let

$$p = \min\{p_i \in [\alpha, 0) \mid F'(p_i) = 0, F''(p_i) \neq 0, i \in \mathbb{N}\} < 0$$

and $r = \min\{r_i < 0 \mid F''(r_i) = 0, i \in \mathbb{N}\}.$

For the function $\Phi_\lambda(x) = \lambda F(x)^2 + 2G(x)$ and some $\lambda = \lambda_1 > 0$, we set

$$a_{\lambda_1} = \max \left\{ x \in [0, \beta] \mid \Phi_{\lambda_1}(x) = \max_{\xi \in [0, \beta]} (\lambda_1 F(\xi)^2 + 2G(\xi)) \right\}.$$

Then, let

$$M_{\lambda_1} = \min \left\{ -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1})}, -\sqrt{2G(\beta)} \right\} < 0$$

and $y_A(x)$ be the solution orbit starting from the point $A(0, M_{\lambda_1})$.

Lemma 4. *Under the conditions (C4) and (C5), if there exists a negative number $x_1 = x_1(\lambda_1) \in (\alpha, 0]$ such that $y_A(x) < 0$ for all $x \in [x_1, 0]$, then*

$$y_A(x_1) < M_{\lambda_1} + \int_0^{x_1} \frac{g(x)}{\lambda_1 F(x)} dx.$$

In fact, we have

$$y_A(x_1) = y_A(0) + \int_0^{x_1} \frac{-g(\xi)}{y_A(\xi) - \lambda_1 F(\xi)} d\xi < M_{\lambda_1} + \int_0^{x_1} \frac{g(\xi)}{\lambda_1 F(\xi)} d\xi.$$

So we can take x_1 such that the equation

$$M_{\lambda_1} + \int_0^{x_1} \frac{g(x)}{\lambda_1 F(x)} dx = 0 \tag{6}$$

is satisfied.

Lemma 5. *In the case of $M_{\lambda_1} = -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1})}$, the solution orbits $(x(t), y(t))$ cannot across the following curve $C_1 \cup C_2$ from the right side to the left side:*

$$\begin{aligned} C_1 & : \quad y = \lambda_1 F(x) \quad \text{if } a_{\lambda_1} \leq x \leq \beta, \\ C_2 & : \quad V(x, y) = \frac{\lambda_1}{2} F(a_{\lambda_1})^2 + G(a_{\lambda_1}) \quad \text{and } y \leq 0 \quad \text{if } 0 \leq x \leq a_{\lambda_1}. \end{aligned}$$

In fact, we have $\dot{x}(t) = 0$ and $\dot{y}(t) < 0$ on C_1 , and also $\dot{V}(x(t), y(t)) = -\lambda_1 g(x(t))F(x(t)) > 0$ on C_2 . See Figure 1 below.

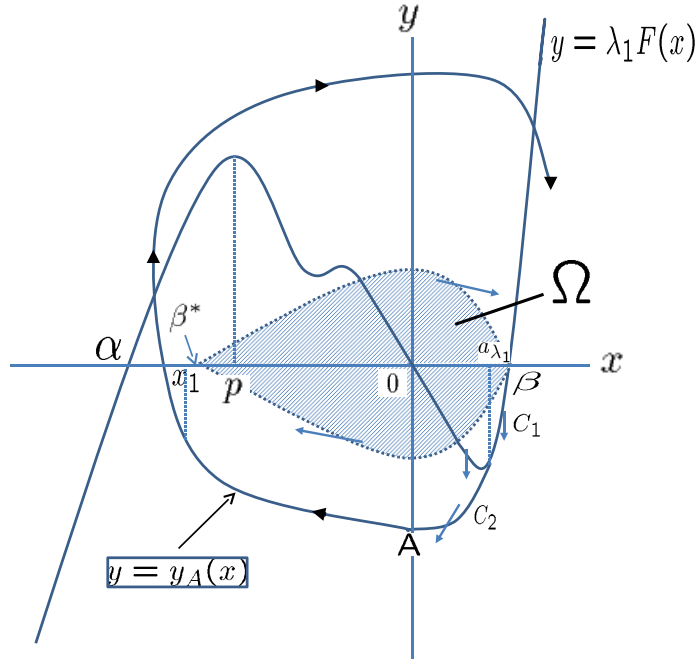


Figure 1 $\left(M_{\lambda_1} = -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1})} \right)$

In the case of $x_1 \leq p$, the following has been known by using the result in [8] or [11].

Theorem 6. *Let $x_1 \leq p$. Under the conditions (C4), (C5), (C6) and besides,*

(C7) $F(x)$ is nondecreasing for $x \leq x_1$ and $x \geq \beta$,

then system (5) has a unique limit cycle for all $\lambda > 0$. It intersects both the lines $x = x_1$ and $x = \beta$, is stable and hyperbolic.

Note that the above result is independent to the existence of r .

If $\beta^* \leq p$, we do not need to calculate the above value x_1 . In fact, the following result has been given in [9].

Proposition 7. *Let $\beta^* \leq p$. Under the conditions (C4), (C5), (C6) and besides,*

(C8) *$F(x)$ is nondecreasing for $x \leq \beta^*$ and $x \geq \beta$,*

then system (5) has a unique limit cycle for all $\lambda > 0$. It intersects both the lines $x = \beta^$ and $x = \beta$, is stable and hyperbolic.*

Example 3. (the case of $\beta^* < p < r$)

Proposition 7 is applied to system (5) with $F(x) = x^3 + \frac{x^2}{2} - \frac{3}{2}x$ and $g(x) = x^{2n-1}$. In facts, the system has $\beta^* = -\beta = -1 < p = -\frac{1 + \sqrt{19}}{6} < r = -\frac{1}{6}$.

Example 4. (the case of $r \leq \beta^* \leq p$)

Consider system (5) with $F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^4 + \frac{8}{3}x^3 - 3x^2 - 9x$ and $g(x) = x^{2n-1}$. The system has $r = -3 < \beta^* = p = -1$.

Thus, we see from Proposition 7 that the above systems have a unique limit cycle without the calculation of x_1 .

Remark. In [9], we gave the results as $M_\lambda = -\sqrt{2G(\beta)}$. When λ is large, we have $M_\lambda = -\sqrt{\lambda F(a_\lambda)^2 + 2G(a_\lambda)}$. In this case, we have an effective result than [9] for λ .

We consider the case of $p < x_1$ and give the suppliment function

$$L(x; s) = \sqrt{\frac{1}{F(x)} \int_s^x \frac{g(\xi)}{F(\xi)} d\xi}$$

for some constant s . This is the function defined in [9].

Theorem 8. *Let $p < x_1$. Assume the conditions (C4), (C5), (C6) and besides,*

(C9) *$F(x)$ is nondecreasing for $x \leq p$ and $x \geq \beta$.*

Then system (5) has a unique limit cycle for all $\lambda \geq \lambda_1$ if

$$\lambda_1 \geq \lambda^* = \max_{x \in [p, x_1]} L(x; x_1)$$

It intersects both the lines $x = p$ and $x = \beta$, is stable and hyperbolic.

We shall prove the above theorem.

Let $y_A(x)$ be the solution orbit starting from the point $A(0, M_{\lambda_1})$. By solving the equation (6) for Lemma 4, there exists $x_1 \in (\alpha, 0]$ such that $y_A(x) \leq 0$ for all $x \in [x_1, 0]$.

Let $y_B(x)$ be the solution orbit starting from the point $B(x_1, 0)$. If $x_1 \leq p$, then $y_B(x)$ must intersect the curve $y = \lambda_1 F(x)$ on $x < x_1$. See Figure 1.

If $x_1 > p$ and $\lambda_1 \geq \lambda^*$, then we see from the Theorem in [10] that the orbit $y_B(x)$ must intersect the half segment $l = \{(x, y) \mid x = p, y \leq \lambda_1 F(x)\}$. Thus, we see from Lemma 3, Lemma 5 and the uniqueness of solution orbits that the orbit $y_B(x)$ must round clockwise outside Ω and cannot stay in the domain $\{(x, y) \mid p < x, y \in \mathbb{R}\}$. See Figure 2.

Therefore, we conclude from [8] or [11] that system (5) has at most one limit cycle, and the limit cycle exists unique from Lemma 2. Also, see [9] or [12].

Note that the above theorems is also independent to the existence of r ($p < r < \beta^*$ or $p < \beta^* \leq r$). Namely, Massera's condition (C3) is not assumed.

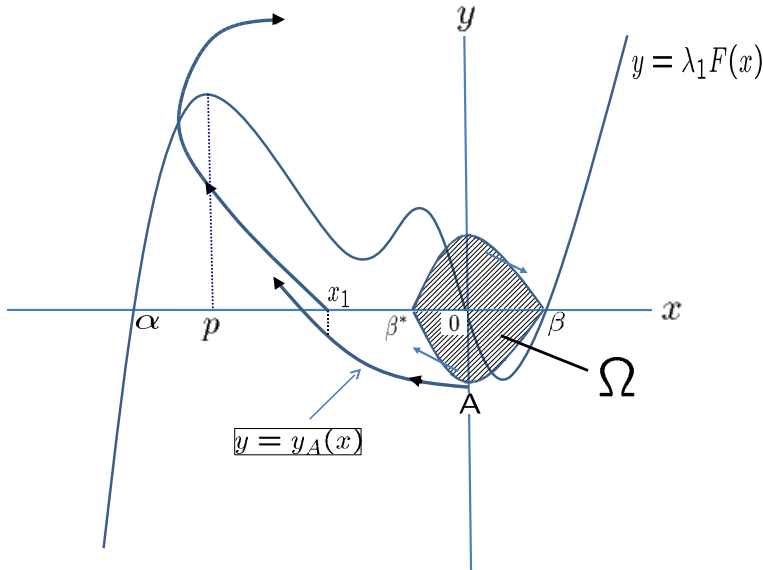


Figure 2 ($M_{\lambda_1} = -\sqrt{2G(\beta)}$)

Example 5. (the case of $p < r \leq \beta^*$)

Consider system (5) with $F(x) = x^3 + 4x^2 - 5x$ and $g(x) = x^3$. Since $F'(x) = 3x^2 + 8x - 5$ and $F''(x) = 6x + 8$, the system has $\alpha = -5$ and $p = -\frac{4 + \sqrt{31}}{3} < r = -\frac{4}{3} < \beta^* = -\beta = -1$.

We give several values $a_{\lambda_1}, M_{\lambda_1}, x_1$ and λ^* by the Computer Algebra System Maple.

Let $\lambda_1 = 1$. We have $a_1 \doteq 0.5324$ and

$$M_1 = -\sqrt{F(a_1)^2 + 2G(a_1)} \doteq -1.9371 < -\sqrt{2G(\beta)} = -\frac{\sqrt{2}}{2}.$$

Then, by solving the equation (6)

$$-1.9371 + \int_0^{x_1} \frac{x^2}{x^2 + 4x - 5} dx = 0,$$

we get $\alpha < x_1 \doteq -3.2684 < p$. Thus, by Theorem 6, the system has a unique limit cycle for all $\lambda \geq 1$ and it intersects both the lines $x = x_1$ and $x = \beta$.

Let $\lambda_1 = 0.3$. We have $a_{0.3} \doteq 0.5607$ and

$$\begin{aligned} M_{0.3} &= -\sqrt{0.3F(a_{0.3})^2 + 2G(a_{0.3})} \doteq -\sqrt{0.61223} \\ &< -\sqrt{2G(\beta)} = -\frac{\sqrt{2}}{2}. \end{aligned}$$

Similarly, solving the equation (6), we get $p < x_1 \doteq -1.8099 < \beta^*$ and

$$\lambda^* = \max_{x \in [p, x_1]} \sqrt{\frac{1}{F(x)} \int_0^x \frac{30\xi^2}{6\xi^4 + 45\xi^3 + 80\xi^2 - 90\xi - 270} d\xi} \doteq 0.2080$$

$< \lambda_1 = 0.3$.

Thus, we conclude from Theorem 8 that this system has a unique limit cycle for all $\lambda \geq 0.3$.

We consider the case of $G(\alpha) < G(\beta)$. Then there exists $\alpha^* \in (0, \beta]$ such that $G(\alpha) = G(\alpha^*)$. For some $\lambda = \lambda_2 > 0$, we set

$$q = \max\{q_j \in (0, \beta) \mid F'(q_j) = 0, F''(q_j) \neq 0, j \in \mathbb{N}\} > 0,$$

$$r^* = \max\{r_i > 0 \mid F''(r_i) = 0, i \in \mathbb{N}\}$$

$$\text{and } b_{\lambda_2} = \min \left\{ x \in [\alpha, 0] \mid \Phi_{\lambda_2}(x) = \max_{\xi \in [\alpha, 0]} (\lambda_2 F(\xi)^2 + 2G(\xi)) \right\}.$$

Then, let

$$M_{\lambda_2} = \max\{\sqrt{\lambda_2 F(b_{\lambda_2})^2 + 2G(b_{\lambda_2})}, \sqrt{2G(\alpha)}\} > 0$$

and $y_C(x)$ be the solution orbit starting from the point $C(0, M_{\lambda_2})$.

Lemma 9. *Under the conditions (C4) and (C5), if there exists a positive number $x_2 = x_2(\lambda_2) \in [0, \beta)$ such that $y_C(x) > 0$ for all $x \in [0, x_2]$, then*

$$y_C(x_2) > M_{\lambda_2} + \int_0^{x_2} \frac{g(x)}{\lambda_2 F(x)} dx.$$

We can take x_2 such that the equation

$$M_{\lambda_2} + \int_0^{x_2} \frac{g(x)}{\lambda_2 F(x)} dx = 0 \tag{7}$$

is satisfied.

Corollary 10. *Let $x_2 \geq q$. Under the conditions (C4), (C5), (C6) and besides,*

(C10) $F(x)$ is nondecreasing for $x \leq \alpha$ and $x \geq x_2$,

then system (5) has a unique limit cycle for all $\lambda > 0$. It intersects both lines $x = \alpha$ and $x = x_2$, is stable and hyperbolic.

Corollary 11. *Let $x_2 < q$. Assume the conditions (C4), (C5), (C6) and besides*

(C11) $F(x)$ is nondecreasing for $x \leq \alpha$ and $x \geq q$.

Then system (5) has a unique limit cycle for all $\lambda \geq \lambda_2$ if

$$\lambda_2 \geq \lambda^{**} = \max_{x \in [x_2, q]} L(x; x_2)$$

It intersects both lines $x = \alpha$ and $x = q$, is stable and hyperbolic.

3. An application

In 1964, Duff and Levinson ([3]) proposed a system in the following form:

$$\begin{cases} \dot{x} = y - \lambda \left(\frac{64}{35\pi}x^7 - \frac{112}{5\pi}x^5 + \frac{196}{3\pi}x^3 - \frac{C}{2}x^2 - \frac{36}{\pi}x \right) \\ \dot{y} = -x \end{cases} . \quad (\text{DL})$$

It is well-known that system (DL) has at least three limit cycles if C is large enough and λ is small(also see [15, pp.227-228]). The author([7]) in 1997 gave a sufficient condition for λ in order that system (DL) has exactly one limit cycle the following

Proposition 12. *System (DL) with $C = 45$ has exactly one limit cycle for all $\lambda \doteq 2.86896$.*

Applying our result to the Computer system Maple, we see easily that system (DL) with $C = 45$ has $\alpha \doteq -0.38144$, $\beta \doteq 3.1798$, $r^* \doteq 2.2300$, $q \doteq 2.694$ and satisfies the conditions (C4), (C5), (C6) and (C10).

Let $\lambda(= \lambda_2) = 0.15$. We have $b_{0.15} \doteq -0.21735$ and

$$M_{0.15} = \sqrt{0.15F(b_{0.15})^2 + 2G(b_{0.15})} \doteq \sqrt{0.2696} > \sqrt{2G(\alpha)} = -\alpha = 0.3814,$$

where $F(x) = \frac{64}{35\pi}x^7 - \frac{112}{5\pi}x^5 + \frac{196}{3\pi}x^3 - \frac{45}{2}x^2 - \frac{36}{\pi}x$ and $g(x) = x$.

Then, solving the equation (7)

$$\sqrt{0.2696} + \int_0^{x_2} \frac{x}{F(x)} dx = 0,$$

we get $\alpha^* < x_2 \doteq 1.3995 < q$ and $\lambda^{**} \doteq 0.03478 < \lambda = 0.15$. Thus, we conclude from Corollary 11 the following

Theorem 13. *System (DL) with $C = 45$ has exactly one limit cycle for all $\lambda \geq 0.15$.*

This is an essential improvement of the result in Proposition 12.

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