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SCALING PROPERTY OF THE GLIMM FUNCTIONAL

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Abstract: In this paper we study a new property for the Glimm potential introduced by L. Caravenna [3]. This new property enable us to study scalar conservation laws with a particular source term called linear damping. By the operator splitting method joined with the polygonal approximation method introduced by C. Dafermos [4] we shown the well-posedness of the Cauchy problem for scalar conservation laws with linear damping and finally we show that the solution exponentially decays.

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1. Introduction

We consider the Cauchy problem for the following scalar conservation law with linear damping

$$\begin{cases} u_t + f(u)_x = \alpha u, \\ u = u_0(x) \in BV(\mathbb{R}), \end{cases} \tag{1}$$

where α is a positive real number. It is well known that the presence of a source term in scalar conservation laws has smoothing effect over the solutions. In this paper we show the well-posedness of the Cuachy problem (1). For that we use the operator splitting method, developed in [1] and [5], which produces a sequence of approximate solutions in the space of the bounded variation functions $BV(\mathbb{R})$, we will show the convergence of the approximate sequence to the unique weak entropy solution of (1). Moreover we study the behavior of this solution through the time.

In this section we recall some basic definitions about scalar conservation laws.

Definition 1. A function u defined in $\Omega = \mathbb{R} \times (0, \infty]$ is a weak solution of the Cauchy problem (1) if it satisfies

$$\int_0^\infty \int_{\mathbb{R}} \big(u(t,x)\phi_t(x,t) + f(u(x,t))\phi_x(x,t)\big)dxdt - \int_{\mathbb{R}} u_0(x)\phi(0,x)dx$$
$$= 0, \text{ for all}, \phi \in C_0^\infty(\mathbb{R} \times (0, \infty)). \tag{2}$$

Definition 2. Let (η, q) be a pair of function in $C^2(\mathbb{R})$. The pair (η, q) is called a entropy-entropy flux pair if

$$\eta'(u)f'(u) = q'(u), u \in \mathbb{R}. \tag{3}$$

If the function η is a strictly convex then the pair (η, q) is called **strictly convex entropy-entropy flux pair**.

Definition 3. A weak solution to the problem of Cauchy (1) with $\alpha = 0$ is called entropy solution if satisfies the following condition

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\phi_t + q(u)\phi_x)\, dxdt \geq 0 \tag{4}$$

for any entropy-entropy flux pair (η, q) and for any non-negative test function.

2. Preliminaries

In this section, we let us to borrow some definitions and properties of the Glimm potential from [3], in particular the functional is lower semicontinuous and decreases along piecewise constants approximation obtained by the front-tracking approach. We present some basic definitions, taken from [3] in order to show the main results of this paper.

Definition 4. Given a smooth function $F : \mathbb{R} \to \mathbb{R}$, we defined

1. The convex envelope of F on the interval $[a, b]$ is defined by

$$F_*^{[a,b]} = \sup \{g | g : [a, b] \to \mathbb{R} \text{ convex}, \ g \leq F\}. \tag{5}$$

2. The concave envelope of F on the interval $[a, b]$ is defined by

$$F^*_{[a,b]} = \inf \{g | g : [a, b] \to \mathbb{R} \ \text{ concave}, \ g \geq F\}. \tag{6}$$

Definition 5. Given a smooth function $F : \mathbb{R} \to \mathbb{R}$ the area functional is defined by

$$\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^+, \tag{7}$$
$$\mathcal{A}(a, b) = \chi_{\{a<b\}} \int_a^b (F - F_*^{[a,b]})(x)dx + \chi_{\{b<a\}} \int_b^a (F^*_{[a,b]} - F)(x)dx.$$

Notice that:

a) If $a \leq b$ then $\mathcal{A}(a, b)$ is the area between f and its convex envelope in $[a, b]$.

b) If $a > b$ then $\mathcal{A}(a, b)$ is the area between f and its concave envelope in $[b, a]$.

Let X be the disjunct countable union of intervals not necessarily open or bounded.

Definition 6. Let $u \in BV(X)$, with points of discontinuity $\{x_i\}_{i\in\mathbb{N}}$ in X and $\{u_i^-, u_i^+\}$ the left and right limits of the function in each jump point. Define the functional

$$\mathcal{D}(u) = \sum_i \mathcal{A}(u(x_i^-), u(x_i^+)). \tag{8}$$

Here let us borrow from [3], [6] a nice relation between (4) and (8). For that we take a function $\phi \in C_c^1((t, t+h) \times \mathbb{R}; \mathbb{R})$ and the Kruzkov entropy-entropy flux pairs defined by

$$\eta(u) = \frac{u^2}{2} \ \text{ y } q(u) = uf(u) - \int^u f(v)dv. \tag{9}$$

If moreover u is a weak entropy solution taken the constant value u_i in the regions

$$\Omega_i = \Big\{(s, x); s \in [t, t+h], \gamma_i(s) \leq x \leq \gamma_{i+1}(s)\Big\},$$

where γ_i are continuous functions, then we obtain the following relation

$$\begin{aligned}
&\int\int_{(t,t+h)\times\mathbb{R}} (\eta(u))\phi_t + q(u)\phi_x)\, dxdt \\
&\equiv \sum_i \int\int_{\Omega_i} (\eta(u)\phi_t + q(u)\phi_x)\, dxdt.
\end{aligned}$$

Since u is a function constant in Ω_i, we define the vector $V := (\phi\eta(u), \phi q(u))$. By an application of the divergence theorem in the region Ω_i we obtain

$$\begin{aligned}
&\int\int_{\Omega_i} \operatorname{div}(V) \\
&= \int_t^{t+h} \left[(v(t, \gamma_{i+1}^-(t))) \cdot \begin{pmatrix} -\gamma_{i+1}'(t) \\ 1 \end{pmatrix} - (v(t, \gamma_i^+(t))) \cdot \begin{pmatrix} -\gamma_i'(t) \\ 1 \end{pmatrix}\right] dt
\end{aligned}$$

by the Rankine-Hugoniot condition γ_i satisfies

$$\begin{aligned}
\gamma_i'(t) &= \frac{f(u(t, \gamma_i(t)^+)) - f(u(t, \gamma_i(t)^-))}{u(t, \gamma_i(t)^+) - u(t, \gamma_i(t)^-)} \\
&= \frac{f(u_i^+) - f(u_i^-)}{u_i^+ - u_i^+}.
\end{aligned}$$

By using the entropy-entropy flux pair given in (9) we obtain that

$$\begin{aligned}
&\gamma_i'(t)[\eta(u_i^+) - \eta(u_i^-)] - [q(u_i^+) - q(u_i^-)] \\
&= \gamma_i' \frac{(u_i^+ + u_i^-)(u_i^+ - u_i^-)}{2} \\
&\qquad - \left(u_i^+ f(u_i^+) - \int^{u_i^+} f(v)dv - u_i^- f(u_i^-) + \int^{u_i^-} f(v)dv \right) \\
&= \frac{1}{2}[f(u_i^+) - f(u_i^-)](u_i^+ - u_i^-) \\
&\qquad - \left[u_i^+ f(u_i^+) - \int_{u_i^-}^{u_i^+} f(v)dv - u_i^- f(u_i^-) \right] \\
&= \mathcal{A}(u_i^-, u_i^+).
\end{aligned} \tag{10}$$

Here, $f_*^{[u_i^-, u_i^+]}(u) = y(u) = f(u_i^-) + \frac{f(u_i^+) - f(u_i^-)}{u_i^+ - u_i^-}(u - u^-)$ and $y(u_i^-) = f(u_i^-)$, is the area between f and the secant line that joins the points $(u_i^-, f(u_i^-))$ y $(u_i^+, f(u_i^+))$, over the interval $[u_i^-, u_i^+]$. By summing sum over all the regions Ω_i we obtain $\mathcal{D}$. Therefore,

$$\int\int_{(t,t+h)\times\mathbb{R}} (\eta(u))\phi_t + q(u)\phi_x)\, dxdt = h \sum_i \mathcal{A}(u_i^-, u_i^+). \tag{11}$$

Finally following [2] and in [3] we introduce the Glimm potential **for homogeneous case** as follows

$$\mathcal{E} := TV - k^{-1}\mathcal{D}(u), \tag{12}$$

where $k = \|f''\|_\infty$. The functional $\mathcal{E}$ satisfies the following properties, see [3] for a detailed description of these properties:

1) $\mathcal{E}$ is a non-negative functional in $BV(J)$

2) Let $J \subset \mathbb{R}$ and $u : J \to \mathbb{R}$ be a piecewise constant function with $u \in BV$. Consider the function $\tilde{u} : J \to \mathbb{R}$ with a single jump corresponding to the first and last values of u. Then $\mathcal{E}(\tilde{u}) \le \mathcal{E}(u)$.

3) Let $\{u_n\}$, be a sequence and u in $BV(X)$. If $u_n \to u$ in $L^1_{loc}(X)$, then

$$\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u_n).$$

   Where X is the disjunct countable union of intervals. That is to say, semi-continuous inferiorly.

4) Let J be an interval and $\{\epsilon_n\}_n$ a sequence decreasing to zero and $u \in BV(J)$. Then there exists a sequence $\{u_n\}_n$ such that:

   a) $u_n \in BV(J)$ is piecewise constant and $\|u - u_n\|_\infty \leq \epsilon_n$. Also if the limits of u eat the end points of J are in $\epsilon_n\mathbb{Z}$ those of u_n can be taken equal to them.

   b) $TV(u_n) \leq TV(u)$, $TV(u_n) \to TV(u)$ y $\mathcal{D}(u_n) \to \mathcal{D}(u)$.

3. Scaling property of the Glimm functional

In this section we show a new property of the functional $\mathcal{E}$, in order to achieve this result first we note some dilatation property of the functional $\mathcal{D}$.

Lemma 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous homogeneous function of degree r, then*

$$\mathcal{A}(\lambda a, \lambda b) = \lambda^{r+1}\mathcal{A}(a, b).$$

Proof. Let f be a homogeneous function of degree r defined on the interval $[\lambda a, \lambda b]$ in this case the concave envelope is given by

$$f^*_{[\lambda a,\lambda b]}(x) = \lambda^r \left\{ \frac{f(b) - f(a)}{b - a}(\frac{x}{\lambda} - a) + f(a) \right\} = \lambda^r f^*_{[a,b]}\left(\frac{x}{\lambda}\right). \tag{13}$$

By using (7) and the dilatation property for integrals we obtain

$$\begin{aligned}
\mathcal{A}(\lambda a, \lambda b) &= \int\limits_{[\lambda a,\lambda b]} (f^*_{[\lambda a,\lambda b]}(x) - f(x))dx = \int\limits_{[\lambda a,\lambda b]} (\lambda^r f^*_{[a,b]}(\frac{x}{\lambda}) - f(x))dx \\
&= \lambda^{r+1} \int\limits_{[a,b]} f^*_{[a,b]}(x)dx - \int\limits_{[\lambda a,\lambda b]} f(x)dx = \lambda^{r+1} \int\limits_{[a,b]} (f^*_{[a,b]}(x) - f(x))dx \\
&= \lambda^{r+1}\mathcal{A}(a, b).
\end{aligned}$$

□

Proposition 8. (*Dilation property*) *Under the hypothesis of Lemma 7 the functional $\mathcal{E}$ defined in* (12) *satisfies*

$$\mathcal{E}(\lambda a, \lambda b) = \lambda\mathcal{E}(a, b) + \frac{\lambda(1 - \lambda^r)}{k}\mathcal{A}(a, b).$$

Proof. Using the definition of $\mathcal{E}$ we have that

$$\begin{aligned}\mathcal{E}(\lambda a,\lambda b) &= TV(\lambda a,\lambda b)-\frac{1}{k}\mathcal{D}(\lambda a,\lambda b) = TV(\lambda a,\lambda b)-\frac{1}{k}\mathcal{A}(\lambda a,\lambda b)\\ &= \lambda(b-a)-\frac{\lambda^{r+1}}{k}\mathcal{A}(a,b) = \lambda(b-a)-\frac{\lambda^{r+1}}{k}\mathcal{D}(a,b)\\ &= \lambda(b-a)-\frac{\lambda}{k}\mathcal{D}(a,b)+\frac{\lambda}{k}\mathcal{D}(a,b)+\frac{\lambda^{r+1}}{k}\mathcal{D}(a,b)\\ &= \lambda\mathcal{E}(a,b)+\frac{\lambda(1-\lambda^r)}{k}\mathcal{A}(a,b).\end{aligned}$$

□

4. Application of scalar conservation laws with source term

In this section we study the Cauchy problem (1). Following [1], [4], we take any value $T>0$ and consider a sequence Δt^ν such that

$$\Delta t^\nu \to 0, \text{ when } \nu\to\infty. \tag{14}$$

Now we construct a sequence of approximate solutions of (1) as follows:

a) For each ν solve the Cauchy problem

$$\begin{cases} u_t+f(u)_x=0, \ \ (t,x)\in(0,\Delta t^\nu),\\ u=u_0(x)\ \in BV(\mathbb{R}).\end{cases} \tag{15}$$

We use the polygonal approximation of Dafermos [4]. First, define Cauchy the class D_T which satisfies the following conditions:

1) The function $u(t,x)$ is weak entropy solution of (1) with $\alpha=0$ on $\mathbb{R}\times(0;T]$.
2) For any $t<T$ is a step function with finite jump points and

$$TV(u(\cdot,t))\le TV(u_0(\cdot)). \tag{16}$$

3) For any t, t' we have

$$\int_{\mathbb{R}}|u(x,t)-u(x,t')|dx\le K|t-t'|TV(u_0). \tag{17}$$

By Theorem 3.1 in [4], there exists a unique function u_ν^1 solution of (15) which belongs to the class D_{Δ^ν}.

b) Now we update the value of u at time $t_1 = \Delta t^v$, as follows

$$u_\nu(x, t_1) = (1 - \alpha \Delta t^v) u_\nu^1(x, t_1-).$$

The function $u_\nu \in D_{\Delta^\nu}$, in fact,

$$TV(u_\nu(t, \cdot)) = (1 - \alpha \Delta t^v) TV(u_\nu^1(\cdot, t)) \leq K(1 - \alpha \Delta t^v) TV(u_0),$$

and

$$\int_{\mathbb{R}} |u_\nu(x,t) - u_\nu(x, t_1)| dx = \int_{\mathbb{R}} |u_\nu^1(x,t) - (1 - \alpha \Delta t^v) u_\nu^1(t_1, x)| dx$$
$$\leq K \alpha \Delta t^v |t - t_1| TV(u_0).$$

Now we consider the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, \ (t,x) \in (\Delta t^\nu, 2\Delta t^\nu) \\ u(t_1, x) = u_\nu(t_1-, x) \ \in BV(\mathbb{R}). \end{cases} \tag{18}$$

An application of Theorem 3.1 in [4] produces a function u_ν^2 with belongs to the class $D_{\Delta t^\nu}$.

Given a partition $[t_n, t_{n+1})$ with $t_n = n\Delta t^\nu$ of the interval $[0, T]$, we obtain a sequence u_ν. Now we study the convergence of this sequence.

Theorem 9. *Let $u_0(x) \in BV(\mathbb{R})$ be a continuous bounded*

$$m \leq \bar{u} \leq M,$$

and $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz continuous function, convex and homogeneous of degree r. Then there exists a unique weak entropy solution of (1) *and the Glimm potential* (12) *exponentially decays trough these solutions.*

Proof. Notice that the sequence u_ν satisfies

$$TV(u_\nu(\cdot, t_{n+1})) \leq K(1 - \alpha \Delta t^v) TV(u_\nu(\cdot, t_n)) \leq (K(1 - \alpha \Delta t^v))^n TV(u_0),$$

and

$$\int_{\mathbb{R}} |u_\nu(x,t) - u_\nu(x,t')| dx \leq \sum_k \int_{\mathbb{R}} |u_\nu(x, t_k) - u_\nu(x, t_{k-1})| dx$$

$$\leq \sum_k K\alpha\Delta t^v |t_k - t_{k-1}| TV(u_0) \leq K\alpha\Delta t^v |t_k - t_{k-1}| TV(u_0).$$

Then, by Helly's theorem there exist a function $u \in BV(\mathbb{R})$ and a sequence still labeled u_ν such that pointwise converges to u. By construction the function u_ν satisfies

$$\int_{t_k}^{t_{k+1}} \int_{\mathbb{R}} (\phi_t u_\nu + \phi_x f(u)) dx dt$$
$$= \int_{\mathbb{R}} \phi(t_{k+1}) u_\nu(t_{k+1}-, x) dx - \int_{\mathbb{R}} \phi(t_k) u_\nu(t_k, x) dx,$$

we have

$$\int_0^T \int_{\mathbb{R}} (\phi_t u_\nu + \phi_x f(u_\nu)) dx dt = \sum_{k=1}^N \int_{t_k}^{t_{k+1}} (\phi_t u_\nu + \phi_x f(u_\nu)) dx dt$$
$$= \sum \int_{\mathbb{R}} \phi(t_{k+1}) u_\nu(t_{k+1}-, x) dx - \int_{\mathbb{R}} (1 - \alpha\Delta t^\nu) u_\nu(t_k -, x) dx.$$

□

The right hand side in the latter integral converges to $\int_0^T \int_{\mathbb{R}} \phi(x,t)\alpha u(x,t)$ and we have the desired result. In order to proof the second part of Theorem 4.1, we consider the following lemmas.

Lemma 10. *Consider the Cauchy problem* (1) *with* $\alpha = 0$. *Then the Glimm potential* $\mathcal{E}$ *decreases along poligonal approximation method.*

Proof. The existence of solutions is given by the polygonal approximation method. We let us to remind that the polygonal approximate solutions $u(t, x)$ of the notice that (9) implies that the sequence $u_n(x) = u(x, t_n)$ with t in T is a Cauchy sequence then $U(x; T-)$ there exists and satisfies

$$TV(u(\cdot, t)) \leq TV(u_0(\cdot)).$$

Then we have to show that

$$\mathcal{D}(u(T-) \leq \mathcal{D}(u(t)),$$

for any $t < T$. But by the remark in [4], page 38, D_T is constructed by superpositions of solutions in $D(T_1), D(T_2)$; ... where T_1 is the first time of the shock interaction, then the number of jumps in $u(x;t)$ for $t > t_1$ is smaller than the number of the jumps of $u_0(x)$, by Lemma 3.1 in [3] we have that

$$\mathcal{E}(u(t_1)) \leq \mathcal{E}(u_0),$$

then by induction in the number of shock interaction we have that

$$\mathcal{E}(u(T-)) \leq \mathcal{E}(u_0).$$

□

Now we are in position to give a proof of the main theorem.

Proof. (Proof of second part of Theorem 9) First at all we need the following observation, the functional $\mathcal{D}$ satisfies the following estimate

$$\mathcal{D} \leq \frac{k}{12}(TV(u))^3,$$

this is by the trapezoidal rule, and for $\lambda < 1$, then $0 < 1 - \lambda^r$, then we have that

$$\frac{\lambda(1-\lambda^r)}{k}\mathcal{D}(u) \leq \frac{\lambda(1-\lambda^r)}{12}(TV(u))^3. \tag{19}$$

By the previous Lemma 10, we now that there exists a solution into the interval $[0, t_1]$ in the $D(t_1)$ set.

By using the polygonal-splitting formulation we update the value $u(x, t_1)$

$$u(t_1, x) = (1 - \alpha\Delta t^v)u(t_1-, x),$$

and by Proposition 8, Lemma 10 and equation (19), we have

$$\begin{aligned}\mathcal{E}(u(x,t_1)) &\leq \lambda\mathcal{E}(u(x,t_1-)) + \frac{\lambda(1-\lambda^r)}{12}(TV(u(t_1-)))^3\\ &\leq \lambda\mathcal{E}(u_0) + \frac{\lambda(1-\lambda^r)}{12}(TV(u_0))^3\end{aligned}$$

with $\lambda = 1 - \alpha\Delta t^v$. Now we apply the polygonal approximation to the Cauchy problem

$$u_t + f(u)_x = 0, \quad u(x,0) = u(x,t_1) \tag{20}$$

into the interval $[t_1, t_2]$. By the above step we have that

$$\mathcal{E}(u(x,t_2)) \leq \lambda^2\mathcal{E}(u(0)) + (\lambda^2+\lambda)(1-\lambda^r)(TV(u(0)))^3.$$

Iterating this process until time $t_n = T$, we have

$$\mathcal{E}(u(x,t_n)) \le \left(1 - \frac{\alpha T}{n}\right)^n \mathcal{E}(u_0) + \frac{\lambda - \lambda^{n+1}}{1-\lambda}(1-\lambda^r)(TV(u(0)))^3.$$

If we make $u_n(x) = u(x,t_n)$ and $\Delta t^v \to 0$, then $n \to \infty$ and $\lambda \to 1$ by using the lower semi-continuity of the functional Theorem 3.3 in [3] we have

$$\mathcal{E}(u(x,T)) \le e^{-\alpha T}\mathcal{E}(u_0).$$

$\square$

References

[1] D. Amadori, G. Guerra, Global BV solutions and relaxation limit for a system of conservation laws, *Proc. of the Royal Society of Edinburgh, Sect. A: Mathematics* **131**, No 1 (2001), 1–26.

[2] L. Caravenna, Master's Thesis, Universita degli Studi di Trieste Facolta di Scienze Matematiche Fisiche e Naturalil, Corso di Laurea Magistrale in Matematica, 2005.

[3] L. Caravenna, An entropy based Glimm-type functional. *J. of Hyperbolic Differential Equations* **5**, No 3 (2008).

[4] C.M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law. *J. Math. Anal. and Appl.* **38**, No 1 (1972), 33–41.

[5] P. Goatin, N. Laurent-Brouty, The zero relaxation limit for the Aw–Rascle–Zhang traffic flow model. *Zeitschrift für angew. Math. und Phys.* **70**, No 1 (2019), 31.

[6] S.A. Modena, "Forward-in-time" quadratic potential for systems of conservation laws. *Nonlinear Diff. Equations and Appl. NoDEA* **24**, No 5 (2017), 1–23.