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DIRICHLET-NEUMANN SPECTRAL PROBLEMS FOR THREE-DIMENSIONAL ELLIPTIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract: In this paper, in domains consisting of parts of a sphere, the Dirichlet-Neumann spectral problems are formulated for elliptic type equations with two and three singular coefficients. The region of values of the parameter where there are no eigenvalues of the problem and a countable number of eigenvalues of the problem are found and eigenfunctions corresponding to the found eigenvalues are constructed.

AMS Subject Classification: 35J75
Key Words: singular coefficient; Heun equation; elliptic equation; eigenvalues; eigenfunction

1. Introduction. A general solution of the degenerate Heun equation

Linear ordinary differential equations of the second order of the Heun class are generated by the Heun equation, a fux equation with four singular points. The

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Heun equation was first investigated by K. Heun [1]. Heun equations with four regular singular points and has the form

$$T''(t) + \left(\frac{a_3}{t-a_0} + \frac{a_4}{t} + \frac{a_5}{t-1}\right) T'(t) + \frac{a_1 a_2 t - q}{t(t-1)(t-a_0)} T(t) = 0, \qquad (1)$$

where $q,\ a_j,\ j = \overline{0,5}$ are number parameters, and the parameters $a_j,\ j = \overline{1,5}$ satisfy the Fuchs condition $1+a_1+a_2 = a_3+a_4+a_5$ which ensures the regularity of an infinitely distant singular point.

Equation (1) represents the most general linear differential equation of the second order of Fuchs class with four singular points: any such equation with four singular points can be reduced to Heun equation by appropriate transformation of the independent and dependent variables. A complete classification of differential equations of Heun class is considered in [2], and studies of problems on eigenvalues for such classes of equations are published in [3], [4].

From equation (1), passing to the limit at $a_0 \to 0$ we get

$$t(1-t)T''(t) + [a_3 + a_4 - (a_3 + a_4 + a_5)t]\,T'(t) - \left(a_1 a_2 - \frac{q}{t}\right) T(t) = 0. \qquad (2)$$

Equation (2) can be called a degenerate Goyne equation. Let us find its general solution.

Due to the presence of an additional regular singular point, equation (2) is a natural generalization of the hypergeometric Gauss equation. Taking this into account and assuming $a_3 + a_4 \notin Z$, we look for a partial solution of equation (2) in the form

$$X(t) = \sum_{k=0}^{\infty} A_k X_k(t) = \sum_{k=0}^{\infty} A_k F(a_1, a_2; a_3 + a_4 + k; t), \qquad (3)$$

where A_k are yet unknown coefficients, and the hypergeometric Gaussian function $F(a_1, a_2; a_3 + a_4 + k; t)$, satisfies the following equation

$$t(1-t)X_k''(t) + [(a_3 + a_4 + k) - (1 + a_1 + a_2)t]\,X_k'(t) - a_1 a_2 X_k(t) = 0. \qquad (4)$$

Substituting (3) into equation (2) and considering equality (4), we obtain

$$\sum_{k=0}^{\infty} A_k \left[-ktX_k'(t) + qX_k(t)\right] = 0. \qquad (5)$$

By virtue of the well-known equality [5]

$$z\frac{dF(b_1,b_2;b_3;z)}{dz} = (b_3-1)\left[F(b_1,b_2;b_3-1;z) - F(b_1,b_2;b_3;z)\right]$$

we verify the equality

$$tX_k'(t) = (a_3+a_4+k-1)\left[X_{k-1}(t) - X_k(t)\right]. \tag{6}$$

Taking into account (6), equation (5) can be written as

$$\sum_{k=0}^{\infty} A_k \left[-k(a_3+a_4+k-1)X_{k-1}(t)\right.$$

$$\left.+\left(k(a_3+a_4+k-1)+q\right)X_k(t)\right] = 0. \tag{7}$$

From (7) we uniquely find the expansion coefficients (3) in the form

$$A_0 = 1, \quad A_{k+1} = \frac{k(a_3+a_4+k-1)+q}{(k+1)(a_3+a_4+k)}A_k, \quad k = 0,1,2,\ldots\,.$$

Now, find the second partial solution of equation (2). After substituting $T(t) = t^{1-a_3-a_4}Y(t)$ of equation (2), it turns into equation:

$$t(1-t)Y''(t) + \left[2-a_3-a_4-(2+a_5-a_4-a_3)t\right]Y'(t)$$

$$-\left[a_1a_2 + a_5(1-a_3-a_4) - \frac{q}{t}\right]Y(t) = 0.$$

Using the above method, find the second partial solution of equation (2):

$$Y(t) = t^{1-a_3-a_4} \tag{8}$$

$$\times\sum_{k=0}^{\infty} A_k F(a_1-a_3-a_4+1, a_2-a_3-a_4+1; 2-a_3-a_4+k; t).$$

Based on (3) and (8), the general solution of equation (10) can be represented as

$$T(t) = b_1\sum_{k=0}^{\infty} A_k F(a_1,a_2;a_3+a_4+k;t) + b_2 t^{1-a_3-a_4} \tag{9}$$

$$\times\sum_{k=0}^{\infty} A_k F(a_1-a_3-a_4+1, a_2-a_3-a_4+1; 2-a_3-a_4+k; t),$$

where b_1 and b_2 are arbitrary constants.

Thus, we have constructed a series expansion of the solutions to equation (2) by hypergeometric functions of the form
$F(a_1, a_2; a_3 + a_4 + k; t)$. There are several methods of decomposition of solutions of Heun equation by hypergeometric functions. For example, in [6], [7], for this purpose a function of the form $F(a_1, a_2; a_4 - k; t)$ and in the works of Svartholm [8], [9], [10] and Schmidt [11], a function of the form $F(a_1 + k, a_2 - k; a_4, t)$. These decompositions differ also from the Jacobi polynomial decompositions constructed by Kalnins and Miller, whose functions can be expressed in terms of functions of the form $F(a_1 + k, a_2 - k; a_4 + 2k; t)$ [12]. The above decompositions of solutions for equation (2) are not suitable. Therefore in this paper we used decomposition of the form (3).

2. Main part. Investigation of the spectral problem

It is known that recently spectral problems for partial differential equations of different types have been intensively studied. Research works carried out on the spectral theory can be divided into two directions. The first is the proof of theorems on the uniqueness of solutions of boundary value problems for equations with spectral parameters and the second is the finding of eigenvalues and eigenfunctions of boundary value problems under consideration. Research on the second direction is currently being intensively continued and developed. Many studies have been devoted to finding eigenvalues and eigenfunctions of boundary value problems for various equations of elliptic and mixed types in the plane, among which the works [13], [14], and others should be mentioned. Problems of this type for three-dimensional elliptic and mixed equations are also well studied, for example, in works [15], [16], [17], [18], [19], [20], [21].

This paper is a continuation of [15], where the spectral Dirichlet problem for a three-dimensional elliptic equation with singular coefficients was studied.

In this paper we investigate problems on eigenvalues for elliptic equations with singular coefficients in three-dimensional space. We identify the region of values of the parameter where there are no eigenvalues of the problem, and we find a countable number of eigenvalues of the problem and construct eigenfunctions corresponding to the found eigenvalues.

Let Ω be a three-dimensional region bounded by part of the sphere

$$S_0 = \left\{(x, y, z) : x^2 + y^2 + z^2 = 1,\ y > 0,\ z > 0\right\}$$

and two semicircles,

$$S_1 = \left\{(x, y, z) :\ x^2 + z^2 < 1,\ y = 0,\ z > 0\right\},$$

$$S_2 = \left\{(x,y,z):\, x^2+y^2<1,\, y>0,\, z=0\right\}.$$

In the domain Ω consider an elliptic equation of the form

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z+\lambda u=0, \qquad (10)$$

where $u=u\,(x,y,z)$ is the unknown function, λ is the parameter, and $\beta,\ \gamma \in R$ and $\beta,\gamma \in (0,1/2)$.

Let us investigate the following problem for eigenvalues:

Problem $\mathbf{DN}_\lambda^{\beta\gamma}$. Find the values of the parameter and corresponding non-trivial functions $u\,(x,y,z) \in C\left(\bar{\Omega}\right) \cap C^2\,(\Omega)$, satisfying the equation (10) in the domain Ω and the boundary condition

$$u\,(x,y,z)=0, \quad (x,y,z)\in \bar{S}_0, \qquad (11)$$

$$\lim_{y\to 0} y^{2\beta}u_y\,(x,y,z)=0,\ \ (x,y,z)\in S_1, \qquad (12)$$

$$\lim_{z\to 0} z^{2\beta}u_z\,(x,y,z)=0,\ \ (x,y,z)\in S_2. \qquad (13)$$

Theorem 1. *If $\lambda \le 0$, then the problem $DN_\lambda^{\beta\gamma}$ has only a trivial solution.*

Proof. In the domain Ω the identity

$$y^{2\beta}z^{2\gamma}u\left(u_{xx}+u_{yy}+u_{zz}+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z+\lambda u\right)$$
$$=\left(y^{2\beta}z^{2\gamma}uu_x\right)_x+\left(y^{2\beta}z^{2\gamma}uu_y\right)_y+\left(y^{2\beta}z^{2\gamma}uu_z\right)_z$$
$$-y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2-\lambda u^2\right)=0.$$

Let us integrate this identity over the domain $\Omega^{\varepsilon}_{\delta_1,\delta_2}\subset\Omega$, bounded at $z\ge \delta_1,\ y\ge\delta_2$ by the part of the sphere

$$\tilde{S}_0=\left\{(x,y,z):\, x^2+y^2+z^2=(1-\varepsilon)^2,\, z\ge\delta_1,\ y\ge\delta_2\right\}$$

and at $z=\delta_1,\, y=\delta_2$ circles

$$\tilde{S}_1=\left\{(x,y,z):\, x^2+z^2<(1-\varepsilon)^2,\, y=\delta_2, z\ge\delta_1\right\},$$

$$\tilde{S}_2=\left\{(x,y,z):\, x^2+y^2<(1-\varepsilon)^2,\, y\ge\delta_2,\, z=\delta_1\right\},$$

where ε, δ_1 and δ_2 are sufficiently small positive numbers. As a result, we have

$$\iiint\limits_{\Omega^{\varepsilon}_{\delta_1,\delta_2}} \left[\left(y^{2\beta}z^{2\gamma}uu_x\right)_x + \left(y^{2\beta}z^{2\gamma}uu_y\right)_y + \left(y^{2\beta}z^{2\gamma}uu_z\right)_z\right] dxdydz$$

$$= \iiint\limits_{\Omega^{\varepsilon}_{\delta_1,\delta_2}} \left[y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2-\lambda u^2\right)\right] dxdydz.$$

Applying the Gauss-Ostrogradsky formula [22] to the integral on the left side of the last equality, we obtain

$$\iint\limits_{\tilde{S}_0} y^{2\beta}z^{2\gamma}u\frac{\partial u}{\partial n}ds - \iint\limits_{\tilde{S}_1} \delta_2^{2\beta}z^{2\gamma}u\left(x,\delta_2,z\right)u_z\left(x,\delta_2,z\right)dxdz$$

$$- \iint\limits_{\tilde{S}_2} y^{2\beta}\delta_1^{2\gamma}u\left(x,y,\delta_1\right)u_z\left(x,y,\delta_1\right)dxdy$$

$$= \iiint\limits_{\Omega^{\varepsilon}_{\delta_1,\delta_2}} \left[y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2-\lambda u^2\right)\right] dxdydz, \tag{14}$$

where n is the external normal to the $\tilde{S}_0$.

Hence, we go to the limit at $\varepsilon,\ \delta_1,\delta_2 \to 0$. Then $\Omega^{\varepsilon}_{\delta_1,\delta_2} \to \Omega$ and considering the boundary condition (11)-(13), and $u,\ u_x,\ u_y,\ u_z \in C\left(\Omega\right)$ we obtain

$$\iiint\limits_{\Omega} \left[y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2-\lambda u^2\right)\right] dxdydz = 0.$$

By virtue of $\lambda \le 0$ from this equality, it follows that $u_x \equiv u_y \equiv u_z \equiv 0$ in Ω. Hence $u\left(x,y,z\right) \equiv 0,\ \left(x,y,z\right) \in \Omega$. Sinse $u\left(x,y,z\right) \in C\left(\bar{\Omega}\right)$ and $u(x,y,z)|_{\bar{S}_0\cup\bar{S}_1\cup\bar{S}_2} = 0$, then $u\left(x,y,z\right) \equiv 0,\ \left(x,y,z\right) \in \bar{\Omega}$. From this follows the statement of the theorem. □

Now let us investigate the problem $\mathrm{DN}_{\lambda}^{\beta\gamma}$ at $\lambda > 0$. In the domain Ω we introduce spherical coordinates $\left(r,\theta,\varphi\right)$, related to the Cartesian coordinates $\left(x,y,z\right)$ according to the formulas $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, where $r = \sqrt{x^2+y^2+z^2}$, θ is the angle between vector $\overrightarrow{OM}$ and the axis z; φ is the angle between the vector $\overrightarrow{OM'}$ and the axis x where $O = O\left(0,0,0\right)$, $M = M\left(x,y,z\right)$, $M' = M'\left(x,y,0\right)$.

In coordinates (r,θ,φ) equation (10) takes the form

$$u_{rr}+\frac{2(1+\beta+\gamma)}{r}u_r+\frac{1}{r^2}\{u_{\theta\theta}+[(1+2\beta)\,ctg\theta-2\gamma tg\theta]\,u_\theta$$

$$+\left(1/\sin^2\theta\right)u_{\varphi\varphi}+\left(2\beta ctg\varphi/\sin^2\theta\right)u_\varphi\}+\lambda u=0. \qquad (15)$$

For equation (15), let us apply the method of separation of variables. First, represent the unknown function as $u(r,\theta,\varphi)=R(r)\,Q(\theta,\varphi)$ and substitute it into equation (15). Then, introducing the separation constant χ we obtain two differential equations

$$r^2R''(r)+2(1+\beta+\gamma)\,rR'(r)+\left(\lambda r^2-\chi\right)R(r)=0,\quad 0<r<1;$$

$$Q_{\theta\theta}+[(1+2\beta)\,ctg\theta-2\gamma tg\theta]\,Q_\theta$$

$$+\frac{1}{\sin^2\theta}Q_{\varphi\varphi}+\frac{2\beta ctg\varphi}{\sin^2\theta}Q_\varphi+\chi Q=0,\ 0<\theta<\pi/2,\ 0<\varphi<\pi. \qquad (16)$$

Now assuming $Q(\theta,\varphi)=T(\theta)\,S(\varphi)$, we obtain from (16)

$$\frac{\sin^2\theta}{T(\theta)}\left[T''(\theta)+[(1+2\beta)\,ctg\theta-2\gamma tg\theta]\,T'(\theta)\right]+\chi\sin^2\theta$$

$$=-\frac{S''(\varphi)+2\beta ctg\varphi S'(\varphi)}{S(\varphi)}. \qquad (17)$$

Introducing another separation constant we obtain two ordinary differential equations from (17):

$$S''(\varphi)+2\beta ctg\varphi S'(\varphi)+\mu\,S(\varphi)=0,\ \ 0<\varphi<\pi,$$

$$\sin^2\theta\left\{T''(\theta)+[(1+2\beta)\,ctg\theta-2\gamma tg\theta]\,T'(\theta)\right\}$$

$$+(\chi\sin^2\theta-\mu)T(\theta)=0,\ \ 0<\theta<\pi/2.$$

Boundary conditions (11)-(13) lead to boundary conditions for the function : $R(r):R(1)=0$ and $|R(0)|<+\infty$. For fixed variables r and φ from conditions (11)-(13) and $u(x,y,z)\in C\left(\bar{\Omega}\right)$, we obtain the condition for the function $T(\theta):|T(0)|<+\infty,\ \lim\limits_{\theta\to\pi/2}(\cos\theta)^{2\gamma}T'(\theta)=0$. From conditions (11)-(13) for the function $S(\varphi)$ we obtain the following conditions $\lim\limits_{\varphi\to 0}(\sin\varphi)^{2\beta}S'(\varphi)=0$ $\lim\limits_{\varphi\to\pi}(\sin\varphi)^{2\beta}S'(\varphi)=0$.

As a result, the original three-dimensional problem is decomposed into three one-dimensional eigenvalue problems:

$$r^2 R''(r) + 2(1+\beta+\gamma) r R'(r) + \left(\lambda r^2 - \chi\right) R(r) = 0, \tag{18}$$

$$|R(0)| < +\infty, \quad R(1) = 0; \tag{19}$$

$$T''(\theta) + [(1+2\beta) ctg\theta - 2\gamma tg\theta] T'(\theta) + \left(\chi - \frac{\mu}{\sin^2\theta}\right) T(\theta) = 0, \tag{20}$$

$$|T(0)| < +\infty, \quad \lim_{\theta\to\pi/2} (\cos\theta)^{2\gamma} T'(\theta) = 0; \tag{21}$$

$$S''(\varphi) + 2\beta ctg\varphi S'(\varphi)(\varphi) + \mu S(\varphi) = 0, \ \ 0 < \varphi < \pi, \tag{22}$$

$$\lim_{\varphi\to 0} (\sin\varphi)^{2\beta} S'(\varphi) = 0, \quad \lim_{\varphi\to\pi} (\sin\varphi)^{2\beta} S'(\varphi) = 0. \tag{23}$$

Let us first investigate problems {(22),(23)}. Let us find a general solution to equation (22). To this end, by substituting $t = \sin^2(\varphi/2)$ in equation (22), we obtain a hypergeometric equation

$$t(1-t)\tilde{S}''(t) + [(\beta+1/2) - (1+2\beta)t]\tilde{S}'(t) + \mu\tilde{S}(t) = 0,$$

where $\tilde{S}(t) = S\left(2\arcsin\sqrt{t}\right)$.

Using the general solution of this equation [5], we find the general solution of equation (22) in the form

$$S(\varphi) = b_7 F\left(\beta - \omega/2, \beta + \omega/2; \beta + 1/2; \sin^2(\varphi/2)\right) \tag{24}$$

$$+ b_8 \left(\sin^2(\varphi/2)\right)^{1/2-\beta} F((1-\omega)/2, (1+\omega)/2; 3/2-\beta; \sin^2(\varphi/2)),$$

where b_7 and b_8 are arbitrary constants, $\omega = 2\sqrt{\mu+\beta^2}$, $\mu > -\beta^2$.

Substituting (24) into the first condition (23), we obtain $d_8 = 0$. Hence, the solution of equation (22) satisfying the first condition (23) is defined as

$$S(\varphi) = b_7 F\left(\beta - \omega/2, \beta + \omega/2; \beta + 1/2; \sin^2(\varphi/2)\right). \tag{25}$$

Substituting (25) into the second condition (23), we obtain

$$2b_7 \frac{\beta^2 - \omega^2/4}{\beta + 1/2} F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}; \frac{3}{2} + \beta; 1\right)$$

$$= 2^{1+2\beta} b_7 \frac{\beta^2 - \omega^2/4}{\beta + 1/2} \frac{\Gamma(3/2+\beta)\Gamma(1/2+\beta)}{\Gamma(1+\beta-\omega/2)\Gamma(1+\beta+\omega/2)} = 0.$$

Multiplying the numerator and denominator of the resulting fraction by $\Gamma\left[1-(1+\beta-\omega/2)\right]\neq 0$ and taking into account the formulas [5] $\Gamma(a)\Gamma(1-a)=\left[\pi/\sin(a\pi)\right]$, from the last equality we obtain

$$2b_7\frac{\beta^2-\omega^2/4}{\beta+1/2}\frac{\Gamma(3/2+\beta)\Gamma(1/2+\beta)\sin\left[\pi(1+\beta-\omega/2)\right]}{\pi\Gamma(1+\beta+\omega/2)\Gamma^{-1}(\omega/2-\beta)}=0.$$

We demand that $b_7\neq 0,\ \omega\neq 2\beta+2-2n,\ \omega\neq -2n-2\beta,\ n\in N$. Solving the equation $\sin\left[\pi(1+\beta-\omega/2)\right]=0$, we obtain

$$\omega_n=2n+2\beta,\ \ n\in N. \tag{26}$$

Hence, the eigenvalues of the problem $\{(22),(23)\}$ are $\mu_n=(\omega_n/2)^2-\beta^2$ where is defined by formula (26).

The eigenfunctions of the problem $\{(22),(23)\}$, corresponding to the eigenvalues μ_n, has the form

$$S_n(\varphi)=b_{7n}F(-n,n+2\beta;\beta+\frac{1}{2};\sin^2\frac{\varphi}{2})=\frac{b_{7n}n!}{(2\beta)_n}C_n^{\beta}(\cos\varphi), \tag{27}$$

where $C_n^{\lambda}(z)$ are the Gegenbauer polynomials, [23].

Now we turn to the problem $\{(18),(19)\}$. By substituting

$$R(r)=\left(\rho/\sqrt{\lambda}\right)^{-(1/2+\beta+\gamma)}\tilde{R}(\rho),$$

we obtain the Bessel equation from (18) in the following form [5]:

$$\rho^2\tilde{R}''(\rho)+\rho\tilde{R}'(\rho)+\left(\rho^2-v^2\right)\tilde{R}(\rho)=0, \tag{28}$$

where $\rho=\sqrt{\lambda}r,\ \ v=\sqrt{(1/2+\beta+\gamma)^2+\chi}$.

Taking into account the form of the general solution of equation (28) [5] and the introduced notations, we obtain a general solution of equation (18) in the form

$$R(r)=b_3r^{-(1/2+\beta+\gamma)}J_v\left(\sqrt{\lambda}r\right)+b_4r^{-(1/2+\beta+\gamma)}Y_v\left(\sqrt{\lambda}r\right), \tag{29}$$

where $r\in(0,1)$, b_3 and b_4 are arbitrary constants.

It follows from (29) that a solution of equation (18) satisfying the first condition (19) exists at $\mathrm{Re}v\geq 1/2+\beta+\gamma$ and it is defined by equality

$$R(r)=b_3r^{-(1/2+\beta+\gamma)}J_v\left(\sqrt{\lambda}r\right). \tag{30}$$

To find the value of the parameter λ, it is necessary to determine the values of the parameter v, i.e. the values of the parameter χ, which is found from the solution of the problem $\{(20),(21)\}$. Therefore, let us investigate this problem.

Turning to the new variable $\xi = \sin^2\theta$ we obtain from equation (20)

$$\xi(1-\xi)\tilde{T}''(\xi) + \left[(1+\beta) - \left(\frac{3}{2}+\beta+\gamma\right)\xi\right]\tilde{T}'(\xi) + \left(\frac{\chi}{4} - \frac{\mu_n}{4\xi}\right)\tilde{T}(\xi) = 0, \qquad (31)$$

where $\tilde{T}(\xi) = T\left(\arcsin\sqrt{\xi}\right)$.

The ordinary differential equation (31) is a degenerate Goyne equation (see formula (2)).

Based on formulas (9), let us find a general solution of equation (31) and, taking into account the introduced notations, we obtain a general solution of equation (20) in the form

$$T(\theta) = b_5 \sum_{k=0}^{\infty} A_k F\left(\frac{1}{4} + \frac{\gamma+\beta+v}{2}, \frac{1}{4} + \frac{\gamma+\beta-v}{2}; 1+\beta+k; \sin^2\theta\right) + b_6 (\sin\theta)^{-2\beta} \times \sum_{k=0}^{\infty} A_k F\left(\frac{1}{4} + \frac{\gamma-\beta+v}{2}, \frac{1}{4} + \frac{\gamma-\beta-v}{2}; 1-\beta+k; \sin^2\theta\right),$$

where b_5, b_6 are arbitrary constants, and the coefficients A_k are defined as follows:

$$A_0 = 1, A_{k+1} = \frac{k(\beta+k) - (\mu_n/4)}{(k+1)(1+\beta+k)} A_k, \quad k = 0, 1, 2, \ldots .$$

The solution of equation (20) satisfying the first condition (21), at $b_5 = 1$, has the form

$$T(\theta) = \sum_{k=0}^{\infty} \frac{(\beta+n/2)_k (-n/2)_k}{k!(1+\beta)_k} \times F\left(\frac{1}{4} + \frac{\beta+\gamma+\nu}{2}, \frac{1}{4} + \frac{\beta+\gamma-\nu}{2}; 1+\beta+k; \sin^2\theta\right). \qquad (32)$$

By Raabe's criterium we can prove the convergence of a number series

$$T(0) = \sum_{k=0}^{\infty} \frac{(\beta+n/2)_k (-n/2)_k}{k!(1+\beta)_k},$$

and based on the expansion of the hypergeometric function, its sum equals

$$F(\beta+n/2,-n/2;1+\beta;1)=\begin{cases}\dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta+n/2)\,\Gamma(1-n/2)}, & n=1,3,5,...,\\[2ex] 0,\ \ n=2,4,6,...\ . & \end{cases}$$

Consequently, function (32) satisfies the first condition (21).

Now let us show that function (32) satisfies the second condition (21). To this end, consistently using the following formulas for the hypergeometric function [5]

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x),$$

$$F(a,b,c;1)=[\Gamma(c)\,\Gamma(c-a-b)]\,/\,[\Gamma(c-a)\,\Gamma(c-b)]\,,c-a-b>0,$$

have

$$\lim_{\theta\to 0}(\cos\theta)^{2\gamma}T'(\theta)=2\left[\left(\frac{1}{4}+\frac{\gamma+\beta}{2}\right)^2-\frac{v^2}{4}\right]\times$$

$$\sum_{k=0}^{\infty}\frac{(\beta+n/2)_k\,(-n/2)_k}{k!\,(1+\beta)_k}\frac{\Gamma(2+\beta+k)\,\Gamma(1/2+\gamma-k)}{\Gamma\left(\frac{5}{4}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{5}{4}+\frac{\beta}{2}+\frac{\gamma}{2}-\frac{\nu}{2}\right)}=0.$$

By multiplying the numerator and denominator of the resulting fraction by $\Gamma[1-(5/4+\beta/2+\gamma/2-v/2)]\neq 0$ (since $\mathrm{Re}v\geq 1/2+\beta+\gamma$) and considering formulae $\Gamma(a)\,\Gamma(1-a)=[\pi/\sin(a\pi)]$ we obtain from the last equation

$$\lim_{\theta\to 0}(\cos\theta)^{2\gamma}T'(\theta)=2\left[\left(\frac{1}{4}+\frac{\gamma+\beta}{2}\right)^2-\frac{v^2}{4}\right]$$

$$\times\sum_{k=0}^{\infty}\frac{(\beta+n/2)_k\,(-n/2)_k}{k!\,(1+\beta)_k}$$

$$\times\frac{\Gamma(2+\beta+k)\,\Gamma(1/2+\gamma-k)\sin[\pi(5/2+\beta+\gamma-v)/2]}{\pi\Gamma\left(\frac{5}{4}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\nu}{2}\right)\Gamma^{-1}\left[1-\left(\frac{5}{4}+\frac{\beta}{2}+\frac{\gamma}{2}-\frac{\nu}{2}\right)\right]}=0.$$

This equality is fulfilled, for example, at $\sin[\pi(5/2+\beta+\gamma-v)/2]=0$. This trigonometric equation has only real roots.

Using the formula that gives the solution of this equation, and the inequality $v\geq 1/2+\beta+\gamma$ and the relations $5/4+\beta/2+\gamma/2+v/2\neq 0,-1,-2,...$, $-1/4-\beta/2-\gamma/2+v/2\neq 0,-1,-2,...$, let us find

$$v=v_l=2l+1/2+\beta+\gamma,\ \ l\in N. \tag{33}$$

By the equality $v=\sqrt{(1/2+\beta+\gamma)^2+\chi}$, we get $\chi_l=v_l^2-(1/2+\beta+\gamma)^2$, $l\in N$, where v_l- are the numbers defined by equality (33). Hence, χ_l are the eigenvalues of the problem {(20), (21)}.

Assuming in (32) $v=v_l,\ \ l\in N$, we obtain the eigenfunctions of the problem {(20),(21)} corresponding to eigenvalues χ_l :

$$T_{nl}(\theta)=\sum_{k=0}^{\infty}\frac{(\beta+n/2)_k\,(-n/2)_k}{k!\,(1+\beta)_k} \tag{34}$$

$$\times F\left(l+1/2+\beta+\gamma,-l;1+\beta+k;\sin^2\theta\right),\ \theta\in[0,\pi/2],\ n,l\in N.$$

For each value l,k and $\theta\in[0,\pi/2)$, the function $F(l+1/2+\beta+\gamma,-l;1+\beta+k;\sin^2\theta)$ is bounded. Hence, series (34) with $\theta\in[0,\pi/2)$ converges absolutely and uniformly, the function $T_{nl}(\theta)$ at $\theta\to(\pi/2)$ is bounded. On the basis of the above, we can conclude that series (34) converges absolutely and uniformly in $[0,\pi/2]$.

By virtue of formula (3), p. 378 of book [24], the function $T_{nl}(\theta)$ can be written in the form

$$T_{nl}(\theta)=F_3(1/2+\beta+\gamma+l,\beta+n/2,-l,-n/2,1+\beta;\sin^2\theta,1), \tag{35}$$

where $n,l\in N$; $F_3(a,a',b,b';c;w,z)$, $(|w|,|z|<1)$ is the hypergeometric Appel function.

Now taking into account that v_l, $l\in N-$ known numbers defined by equations (33), we find the values of the parameter λ from (30). To this end, substituting (30) into the second condition (19), we obtain

$$J_{v_l}\left(\sqrt{\lambda}\right)=0,\ \ l\in N. \tag{36}$$

It is known that for $p>-1$ the Bessel function $J_p(z)$ has a countable number of zeros, all of which are real and with pairwise opposite signs [5]. Since $v_l\geq 1/2+\beta+\gamma$, the equation (36) has a countable number of real roots. Denoting by is $\sigma_{ml}-$ m the positive root of equation (36), we obtain the values of the parameter λ, at which nontrivial solutions of the problem exist (i.e. eigenvalues of the problem $\mathrm{DN}_\lambda^{\beta\gamma}$) {(18),(19)} $\lambda_{ml}=\sigma_{ml}^2,\ m,l\in N$.

Assuming in (30) $\lambda=\sigma_{ml}^2$ and $b_3=b_{ml}$, where $b_{ml}\neq 0-$ is a spontaneous constant, $m,l\in N$ we obtain nontrivial solutions (eigenfunction) of the problem {(18),(19)}:

$$R_{ml}(r)=b_{ml}r^{-(1/2+\beta+\gamma)}J_{v_l}(\sigma_{ml}r),\ m,l\in N. \tag{37}$$

Hence, the $DN_\lambda^{\beta\gamma}$ problem has a countable number of eigenvalues and eigenfunctions. Its eigenvalues are numbers $\lambda = \sigma_{ml}^2$, $m, l \in N$, and the eigenfunctions, by virtue of formulae (27), (35), and (37), are defined by the equations

$$u_{nlm}(x,y,z) = b_{nml} r^{-(1/2+\beta+\gamma)} J_{v_l}(\sigma_{ml} r)$$

$$\times F\left(-n, n+2\beta; \beta + \frac{1}{2}; \sin^2\frac{\varphi}{2}\right)$$

$$\times F_3\left(1/2+\beta+\gamma+l, \beta+n/2, -l, -n/2, 1+\beta; \sin^2\theta, 1\right),$$

where $b_{nml} \neq 0$ are arbitrary constants.

This completes the study of the $DN_\lambda^{\beta\gamma}$.

The following problem is investigated in a similar way:

Problem $DN_\lambda^{\alpha\beta\gamma}$. Find the values of the parameter λ and their corresponding nontrivial functions $u(x,y,z) \in C\left(\overline{\tilde{\Omega}}\right) \cap C^2\left(\tilde{\Omega}\right)$, satisfying the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y + \frac{2\gamma}{z}u_z + \lambda u = 0, \tag{38}$$

in the region's $\tilde{\Omega}$ and regional conditions

$$u(x,y,z) = 0, (x,y,z) \in \bar{S}_3, \ \lim_{x\to 0} x^{2\alpha} u_x(x,y,z) = 0, \ (x,y,z) \in S_4,$$

$$\lim_{y\to 0} y^{2\beta} u_y(x,y,z) = 0, \ (x,y,z) \in S_5,$$

$$\lim_{z\to 0} z^{2\gamma} u_z(x,y,z) = 0, \ (x,y,z) \in S_6,$$

where $\tilde{\Omega} = \Omega \cap \{x > 0\}$, $S_3 = S_0 \cap \{x > 0\}$, $S_5 = S_1 \cap \{x > 0\}$, $S_6 = S_2 \cap \{x > 0\}$, $S_4 = \left\{(x,y,z) : y^2 + z^2 < 1, x = 0, y > 0, z = 0\right\}$, $\alpha = const \in (0, 1/2)$.

By performing the same arguments as in the solution of the $DN_\lambda^{\alpha\beta\gamma}$ problem, we can see that under $\lambda \leq 0$ the problem $DN_\lambda^{\alpha\beta\gamma}$ has only trivial solution, i.e. in this interval the eigenvalues do not exist, and at $\lambda > 0$ eigenvalues $\tilde{\lambda}_{ml} = \tilde{\sigma}_{ml}^2$ $(m, l \in N)$ of the $DN_\lambda^{\alpha\beta\gamma}$ problem are defined as the roots of equations $J_{\tilde{v}_l}\left(\sqrt{\lambda}\right) = 0$, $l \in N$, where $\tilde{v}_l = 2l + 1/2 + \alpha + \beta + \gamma$, $l \in N$, and the corresponding eigenfunctions in the domain $\tilde{\Omega}$ are given by the formulas

$$\tilde{u}_{nlm}(x,y,z) = \tilde{b}_{nml} r^{-(1/2+\alpha+\beta+\gamma)} J_{\tilde{v}_l}(\tilde{\sigma}_{ml} r)$$

$$\times F\left(n+\alpha+\beta,-n;1/2+\beta;\sin^2\varphi\right)$$
$$\times F_3\left(1/2+\alpha+\beta+\gamma+l,\beta+n/2,-l,-n/2;1+\alpha+\beta;\sin^2\theta,1\right),$$

where $n,m,l\in N$, $\varphi,\theta\in[0,\pi/2]$, $r\in[0,1]$, $\tilde{b}_{nml}\neq 0$ are arbitrary constants.

3. Conclusion

In this paper, Dirichlet-Neumann spectral problems have been formulated for elliptic type equations with two and three singular coefficients in domains consisting of parts of a sphere. The region of values of the parameter λ where there are no eigenvalues of the problem, and a countable number of eigenvalues of the problem are found and eigenfunctions corresponding to the found eigenvalues are constructed.

The set problems are mapped in spherical coordinates, and then three one-dimensional problems on eigenvalues are obtained by the method of separation of variables. Using substitution of variables, the one-dimensional equations are reduced to Bessel equations, hypergeometric equations and degenerate Goyne equations. Generated Goyne equations are a special case of the Goyne equation, which appears by means of a limit transition of one of the parameters of the equation. This type of equation has not yet been investigated. Therefore, the introduction gives information about this equation and finds its general solution.

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