

**STABILITY OF THE TIME-DEPENDENT IDENTIFICATION
PROBLEM FOR THE TELEGRAPH EQUATION
WITH INVOLUTION**

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Abstract: In the present paper, a time-dependent source identification problem for a one dimensional telegraph equation with involution is studied. Theorems on the stability estimates for the solution of this problem are established.

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Key Words: source identification problem; telegraph equation; involution; stability

1. Introduction

There has always been a major interest for the theory of source identification problems for partial differential equations since they have widespread applications in modern physics and technology. The stability of various source identification problems for partial differential and difference equations has also been

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studied extensively by many researchers (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [27], [28], [29], [30], [31], [32], [33], and the references given therein).

The telegraph hyperbolic partial differential equation is important for modeling several relevant problems such as in signal analysis, wave propagation, random walk theory [19], [20], [21], [22], [23], [24], [25], [26]. To deal with this equation, various mathematical methods have been proposed for obtaining exact and approximate analytic solutions.

In the present paper, the time-dependent source identification problem is considered

$$\left\{ \begin{array}{l} u_{tt}(t, x) + \alpha u_t(t, x) \\ - (a(x)u_x(t, x))_x - \beta (a(-x)u_x(t, -x))_x + \sigma u(t, x) \\ = p(t)q(x) + f(t, x), \quad 0 < t < T, \quad x \in (-l, l) \\ u(0, x) = \varphi(x), u_t(0, x) = \omega(x), \quad x \in [-l, l], \\ u_x(t, -l) = u_x(t, l) = 0, \quad t \in [0, T], \\ \int_{-l}^l u(t, x) dx = \psi(t), \quad t \in [0, T] \end{array} \right. \quad (1)$$

for the telegraph equation with convolution and Neumann conditions. Under compatibility conditions problem (1) has a unique solution $(u(t, x), p(t))$ for the smooth functions $f(t, x)$ $((t, x) \in (0, T) \times (-l, l))$, $a(x), a \geq a(-x) \geq \delta > 0$, $\delta - a|\beta| \geq 0$, $\sigma > \frac{\alpha^2}{4}$, $\varphi(x), \omega(x), x \in [0, l], \psi(t), t \in [0, T], q'(-l) = q'(l) = 0$, and $\int_{-l}^l q(x) dx \neq 0$.

Problem (1) can be written as the time dependent identification problem

$$\left\{ \begin{array}{l} \frac{d^2 u(t)}{dt^2} + \alpha \frac{du(t)}{dt} + Au(t) = p(t)q + f(t), \quad t \in (0, T), \\ u(0) = \varphi, u'(0) = \omega, P[u(t)] = \psi(t), \quad t \in [0, T] \end{array} \right. \quad (2)$$

in a Hilbert space $H = L_2[-l, l]$ with the with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$Au(x) = - (a(x)u_x(x))_x - \beta (a(-x)u_x(-x))_x + \sigma u(x) \quad (3)$$

with the domain $D(A) = \{u \in W_2^2[-l, l] : u'(-l) = u'(l) = 0\}$. Here $P : H \rightarrow \mathbb{R}$ is a given linear bounded functional and $\psi(t) : [0, T] \rightarrow \mathbb{R}$ is a given smooth function and $q \in D(A), Pq \neq 0$.

The study in this paper is organized as follows. Section 1 is introduction. In Section 2, auxiliary statements are given. In Section 3, theorems on stability of differential problem (1) are established. Finally, Section 4 contains conclusions.

2. Auxiliary Statements

Let $c(t)$ be operator-function generated by the operator B and defined as the solution of the initial value problem for a second order differential equation

$$u_{tt}(t) + Bu(t) = 0, \quad 0 < t < \infty, \quad u(0) = \varphi, \quad u_t(0) = 0 \quad (4)$$

in a Hilbert space H , that is, $u(t) = c(t)\varphi$. Similarly, $s(t)$ is operator-function generated by the operator B and defined as the solution of the initial value problem for a second order differential equation

$$v_{tt}(t) + Bv(t) = 0, \quad 0 < t < \infty, \quad v(0) = 0, \quad v_t(0) = \psi \quad (5)$$

in a Hilbert space H , namely, $v(t) = s(t)\psi$. By the definitions of $c(t)$ and $s(t)$, we have that

$$s'(t) = c(t), \quad c'(t) = -Bs(t). \quad (6)$$

For the positive-definite and self-adjoint operator $B \geq \left(\sigma - \frac{\alpha^2}{4}\right) I$ in H we the following estimates

$$\begin{cases} \|B^{-\frac{1}{2}}\|_{H \rightarrow H} \leq \left(\sigma - \frac{\alpha^2}{4}\right)^{-\frac{1}{2}}, \quad \|s(t)\|_{H \rightarrow H} \leq t, \\ \|c(t)\|_{H \rightarrow H} \leq 1, \quad \|B^{\frac{1}{2}}s(t)\|_{H \rightarrow H} \leq 1. \end{cases} \quad (7)$$

3. Stability of Differential Problem (1)

Theorem 1. Assume that $\int_{-l}^l q(x)dx \neq 0, \sigma > \frac{\alpha^2}{4}$ and $\varphi \in W_2^2[-l, l], \omega \in W_2^1[-l, l]$ and $f(t, x)$ is a continuously differentiable function in t and square integrable in x , $\psi(t)$ is a twice continuously differentiable function. Then the time source time-dependent source identification problem (1) has a unique solution $u \in C(L_2[-l, l]) = C([0, T], L_2[-l, l]), p \in C[0, T]$ and for the solution

of the time-dependent source identification problem (1) the following stability estimates hold

$$\begin{aligned} & \|u_{tt}\|_{C(L_2[-l,l])} + \|u\|_{C(W_2^2[-l,l])} + \|p\|_{C[0,T]} \leq M_1(q, \sigma, \alpha) \left[\|\varphi\|_{W_2^2[-l,l]} \right. \\ & \left. + \|\omega\|_{W_2^1[-l,l]} + \|f(0)\|_{L_2[-l,l]} + |\psi(0)| + \|f_t\|_{C(L_2[-l,l])} + \|\psi''\|_{C[0,T]} \right]. \end{aligned}$$

Here $\mathbb{L}_2[-l, l]$ is the space of all square integrable functions $w(x)$ defined on $[-l, l]$ and $\mathbb{W}_2^k[-l, l]$, $k = 1, 2$ are Sobolev spaces equipped with norms

$$\|w\|_{\mathbb{W}_2^1[-l,l]} = \left(\int_{-l}^l [w^2(z) + w_z^2(z)] dz \right)^{\frac{1}{2}},$$

$$\|w\|_{\mathbb{W}_2^2[-l,l]} = \left(\int_{-l}^l [w^2(z) + w_z^2(z) + w_{zz}^2(z)] dz \right)^{\frac{1}{2}},$$

respectively.

Proof. We will seek $u(t, x)$, using the substitution

$$u(t, x) = w(t, x) + \eta(t)q(x), \quad (8)$$

where $\eta(t)$ is the function defined by the formula

$$\eta(t) = \int_0^t (t-s)p(s)ds, \quad \eta(0) = \eta'(0) = 0. \quad (9)$$

It is easy to see that $w(t, x)$ is the solution of the problem

$$\left\{ \begin{aligned} & \frac{\partial^2 w(t,x)}{\partial t^2} + \alpha \frac{\partial w(t,x)}{\partial t} - (a(x)w_x(t,x))_x \\ & - \beta (a(-x)w_x(t,-x))_x + \delta w(t,x) \\ & = - (a(x)q_x(x))_x - \beta (a(-x)q_x(-x))_x + \sigma q(x) \\ & + \alpha \eta'(t)q(x) + f(t,x), \quad 0 < t < T, \quad x \in (-l, l), \\ & w(0,x) = \varphi(x), \quad w_t(0,x) = \omega(x), \quad x \in [-l, l], \\ & w_x(t, -l) = w_x(t, l) = 0, \quad t \in [0, T]. \end{aligned} \right. \quad (10)$$

Now we will take an estimate for $|p(t)|$. Applying the integral overdetermined condition $\int_{-l}^l u(t, x) dx = \psi(t)$ and substitution (8), we get

$$\eta(t) = \left(\int_{-l}^l q(x) dx \right)^{-1} \left(\psi(t) - \int_{-l}^l w(t, x) dx \right).$$

From that and $p(t) = \eta''(t)$, it follows that

$$p(t) = \left(\int_{-l}^l q(x) dx \right)^{-1} \left(\psi''(t) - \int_{-l}^l \frac{\partial^2}{\partial t^2} w(t, x) dx \right).$$

Then, using the triangle inequality, we obtain

$$|p(t)| \leq \left(\left| \int_{-l}^l q(x) dx \right| \right)^{-1} \left(|\psi''(t)| + \int_{-l}^l \left| \frac{\partial^2}{\partial t^2} w(t, x) \right| dx \right) \quad (11)$$

$$\leq k(q, l) \left[|\psi''(t)| + \left\| \frac{\partial^2}{\partial t^2} w(t, \cdot) \right\|_{\mathbb{L}_2[-l, l]} \right]$$

for all $t \in (0, T)$. Now, using substitution (8), we get

$$\frac{\partial^2 u(t, x)}{\partial t^2} = \frac{\partial^2 w(t, x)}{\partial t^2} + p(t)q(x).$$

Applying the triangle inequality, we obtain

$$\left\| \frac{\partial^2 u(t, \cdot)}{\partial t^2} \right\|_{\mathbb{L}_2[-l, l]} \leq \left\| \frac{\partial^2 w(t, \cdot)}{\partial t^2} \right\|_{\mathbb{L}_2[-l, l]} + |p(t)| \|q\|_{\mathbb{L}_2[-l, l]} \quad (12)$$

for all $t \in (0, T)$. Therefore, the proof of Theorem 1 is based on the following theorem. \square

Theorem 2. *Under the assumptions of Theorem 1, for the solution of problem (10) the following stability estimate holds:*

$$\max_{0 \leq t \leq T} \|w_{tt}\|_{\mathbb{L}_2[-l, l]}, \max_{0 \leq t \leq T} \|w_t\|_{\mathbb{W}_2^1[-l, l]}, \max_{0 \leq t \leq T} \|w\|_{\mathbb{W}_2^2[-l, l]} \quad (13)$$

$$\begin{aligned} &\leq M_1(q, \alpha, \sigma) \left[\|\varphi\|_{W_2^2[-l, l]} + \|\omega\|_{W_2^1[-l, l]} \right. \\ &\quad \left. + \|f(0)\|_{L_2[-l, l]} + |\psi(0)| + \|f_t\|_{C(L_2[-l, l])} + \|\psi''\|_{C[0, T]} \right]. \end{aligned}$$

Proof. It is clear that the mixed problem (10) can be written as the initial value problem

$$\begin{cases} w_{tt}(t) + \alpha w_t(t) + Aw(t) = F(t), \\ F(t) = -\alpha\eta'(t)q - \eta(t)Aq + f(t), \quad t \in (0, T), \\ w(0) = \varphi, \quad w'(0) = \omega \end{cases} \quad (14)$$

in a Hilbert space $\mathbb{H} = \mathbb{L}_2[-l, l]$ with A determining by (3). From (11) it follows that

$$|p(t)|, |\eta(t)|, |\eta'(t)| \leq k(q, \sigma, \alpha, l) \left[|\psi''(t)| + \|w_{tt}(t)\|_H \right] \quad (15)$$

for all $t \in (0, T)$. Therefore, the proof of Theorem 2 is based on the boundedness in $L_2[-l, l]$ of a linear functional P defined by the formula

$$Pu(t, x) = \int_{-l}^l u(t, x) dx, \quad t \in [0, T].$$

and on the following abstract theorem. □

Theorem 3. *Under the assumptions of Theorem 1, for the solution of problem (14) the following stability estimate holds:*

$$\begin{aligned} &\max_{0 \leq t \leq T} \|w_{tt}\|_H, \max_{0 \leq t \leq T} \left\| A^{\frac{1}{2}} w_t \right\|_H, \max_{0 \leq t \leq T} \|Aw\|_H \\ &\leq M(\delta, q, \sigma, \alpha) \left[\|A\varphi\|_H + \left\| A^{\frac{1}{2}} \omega \right\|_H + \|f(0)\|_H + \|f_t\|_{C(H)} \right. \\ &\quad \left. + |\psi(0)| + \|\psi_{tt}\|_{C[0, T]} \right]. \end{aligned} \quad (16)$$

Proof. The initial value problem (14) is equivalent to the integral equation

$$w(t) = e^{-\frac{\alpha}{2}t} c(t)w(0) + \frac{\alpha}{2} e^{-\frac{\alpha}{2}t} s(t)w(0) + e^{-\frac{\alpha}{2}t} s(t)w'(0) \quad (17)$$

$$\begin{aligned}
& + \int_0^t e^{-\frac{\alpha}{2}(t-z)} s(t-z) \left\{ -\frac{1}{Pq} (\psi(z) - P[w(z)]) Aq + f(z) \right\} dz \\
& + \int_0^t e^{-\frac{\alpha}{2}(t-z)} s(t-z) \left\{ -\frac{\alpha}{Pq} (\psi'(z) - P[w'(z)]) q \right\} dz.
\end{aligned}$$

Here, $c(t)$ and $s(t)$ are strongly continuous cosine and sine operator-functions generated by the operator $B = A - \frac{\alpha^2}{4}I$. Taking the first and second order derivatives with respect to t , we can write

$$\begin{aligned}
w'(t) = & -e^{-\frac{\alpha}{2}t} \left(B + \frac{\alpha^2}{4}I \right) s(t) \varphi + e^{-\frac{\alpha}{2}t} \left(-\frac{\alpha}{2}s(t) + c(t) \right) \omega \\
& + \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(-\frac{\alpha}{2}s(t-z) + c(t-z) \right) \\
& \times \left\{ -\frac{1}{Pq} (\psi(z) - P[w(z)]) Aq + f(z) \right\} dz \\
& + \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(-\frac{\alpha}{2}s(t-z) + c(t-z) \right) \\
& \times \left\{ -\frac{\alpha}{Pq} (\psi'(z) - P[w'(z)]) q \right\} dz
\end{aligned}$$

and

$$\begin{aligned}
w''(t) = & -e^{-\frac{\alpha}{2}t} \left(B + \frac{\alpha^2}{4}I \right) \left(-\frac{\alpha}{2}s(t) + c(t) \right) \varphi \\
& + e^{-\frac{\alpha}{2}t} \left(-\left(B - \frac{\alpha^2}{4}I \right) s(t) - \alpha c(t) \right) \omega \\
& - \frac{1}{Pq} (\psi(t) - P[w(t)]) Aq - \frac{\alpha}{Pq} (\psi'(t) - P[w'(t)]) q + f(t) \\
& + \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(-\left(B - \frac{\alpha^2}{4}I \right) s(t-z) - \alpha c(t-z) \right) \\
& \times \left\{ -\frac{1}{Pq} (\psi(z) - P[w(z)]) Aq - \frac{\alpha}{Pq} (\psi'(z) - P[w'(z)]) q + f(z) \right\} dz.
\end{aligned} \tag{18}$$

Applying the formulas

$$\int_0^t e^{-\frac{\alpha}{2}(t-z)} B s(t-z) g(z) dz = g(t) - e^{-\frac{\alpha}{2}t} c(t) g(0)$$

$$\begin{aligned}
& - \int_0^t e^{-\frac{\alpha}{2}(t-z)} c(t-z) \left(\frac{\alpha}{2} g(z) + g'(z) \right) dz, \\
w(t) &= w(0) + w'(0)t + \int_0^t (t-y)w''(y) dy, \\
w'(t) &= w'(0) + \int_0^t w''(y) dy
\end{aligned}$$

and formula (18), we get

$$\begin{aligned}
w''(t) &= -e^{-\frac{\alpha}{2}t} \left(B + \frac{\alpha^2}{4}I \right) \left(-\frac{\alpha}{2}s(t) + c(t) \right) \varphi \\
&+ e^{-\frac{\alpha}{2}t} \left(- \left(B - \frac{\alpha^2}{4}I \right) s(t) - \alpha c(t) \right) \omega \\
&- \frac{1}{Pq} (\psi(t) - P[w(t)]) Aq - \frac{\alpha}{Pq} (\psi'(t) - P[w'(t)]) q + f(t) \\
&+ \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(- \left(B - \frac{\alpha^2}{4}I \right) s(t-z) - \alpha c(t-z) \right) \\
&\times \left\{ -\frac{1}{Pq} (\psi(z) - P[w(z)]) Aq - \frac{\alpha}{Pq} (\psi'(z) - P[w'(z)]) q + f(z) \right\} dz \\
&= -e^{-\frac{\alpha}{2}t} \left[\left(B + \frac{\alpha^2}{4}I \right) \left(-\frac{\alpha}{2}s(t) + c(t) \right) \varphi \right. \\
&\quad \left. + \left(- \left(B - \frac{\alpha^2}{4}I \right) s(t) - \alpha c(t) \right) \omega \right] \\
&+ \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(\frac{\alpha^2}{4}s(t-z) - \alpha c(t-z) \right) \\
&\times \left\{ -\frac{1}{Pq} (\psi(z) - P[w(z)]) Aq - \frac{\alpha}{Pq} (\psi'(z) - P[w'(z)]) q + f(z) \right\} dz \\
&- \left\{ -\frac{1}{Pq} (\psi(t) - P[w(t)]) Aq - \frac{\alpha}{Pq} (\psi'(t) - P[w'(t)]) q + f(t) \right\} \\
&+ e^{-\frac{\alpha}{2}t} c(t) \left\{ -\frac{1}{Pq} (\psi(0) - P[\varphi]) Aq - \frac{\alpha}{Pq} (\psi'(0) - P[\omega]) q + f(0) \right\} \\
&+ \int_0^t e^{-\frac{\alpha}{2}(t-z)} c(t-z)
\end{aligned}$$

$$\times \left\{ -\frac{1}{Pq} (\psi'(z) - P[w'(z)]) Aq - \frac{\alpha}{Pq} (\psi''(z) - P[w''(z)]) q + f'(z) \right\} dz.$$

Then, applying the triangle inequality and estimate (7) and $Pq \neq 0$, we get the estimate

$$\begin{aligned} \|w''(t)\|_H &\leq e^{-\frac{\alpha}{2}t} \left(\frac{\alpha}{2} \|s(t)\|_{H \rightarrow H} + \|c(t)\|_{H \rightarrow H} \right) \|A\varphi\|_H + e^{-\frac{\alpha}{2}t} \\ &\quad \times \left(\left\| \left(B - \frac{\alpha^2}{4} I \right)^{\frac{1}{2}} A^{-\frac{1}{2}} \right\|_{H \rightarrow H} \|B^{\frac{1}{2}} s(t)\|_{H \rightarrow H} \right. \\ &\quad \left. + \alpha \|A^{-\frac{1}{2}}\|_{H \rightarrow H} \|c(t)\|_{H \rightarrow H} \right) \\ &\times \left\| A^{\frac{1}{2}} \omega \right\|_H + \int_0^t e^{-\frac{\alpha}{2}(t-z)} \left(\frac{\alpha^2}{4} \|s(t-z)\|_{H \rightarrow H} + \alpha \|c(t-z)\|_{H \rightarrow H} \right) \\ &\quad \times \left\{ \frac{1}{|Pq|} (|\psi(z)| + |P[\varphi + \omega z] \right. \\ &\quad \left. + \int_0^z (z-y)w''(y)dy) \right\} \|Aq\|_H + \|f(z)\|_H \Big\} dz \\ &+ \frac{1}{|Pq|} \left(|\psi(t)| + \left| P[\varphi + \omega t + \int_0^t (z-y)w''(y)dy] \right| \right) \|Aq\|_H + \|f(t)\|_H \\ &\quad + e^{-\frac{\alpha}{2}t} \|c(t)\|_{H \rightarrow H} \frac{1}{|Pq|} (|\psi(0)| + |P[\varphi]|) \|Aq\|_H + \|f(0)\|_H \\ &\quad + \frac{1}{|Pq|} \left(\alpha \left| P[\omega + \int_0^t w'(y)dy] \right| \right) \|q\|_H + \frac{1}{|Pq|} (\alpha |P[\omega]|) \|q\|_H \\ &\quad + \int_0^t e^{-\frac{\alpha}{2}(t-z)} \|c(t-z)\|_{H \rightarrow H} \\ &\quad \times \frac{1}{|Pq|} \left\{ \left(|\psi'(z)| + \left| P \left[\omega + \int_0^z w''(y)dy \right] \right| \right) \|Aq\|_H \right. \\ &\quad \left. + \alpha (|\psi''(z)| + |P[w''(z)]|) \|q\|_H + \|f'(z)\|_H \right\} dz \\ &\leq M_3(\delta, q, \sigma, \alpha) [|\psi(0)| + \|f(0)\|_H + \|A\varphi\|_{\mathbb{H}}] \\ &\quad + \|\psi_{tt}\|_{C[0,T]} + \|f_t\|_{C(H)} \Big] + M_4(\delta, q, \alpha) \int_0^t \|w_{ss}(s)\|_H ds \end{aligned} \tag{19}$$

for $0 \leq t \leq T$. Then, applying the integral inequality, we conclude that the following stability estimate

$$\begin{aligned} \|w_{tt}(t)\|_H &\leq M(\sigma, \delta, q, \alpha) [|\psi(0)| + \|f(0)\|_H + \|A\varphi\|_H \\ &\quad + \|A^{\frac{1}{2}}\omega\|_H + \|\psi_{tt}\|_{C[0,T]} + \|f_t\|_{C(H)}] e^{M_4(\delta, \sigma, q)t} \end{aligned} \quad (20)$$

is satisfied for the solution of initial value problem (14) for every $t \in [0, T]$. From estimate (20) it follows estimate (16). Theorem 3 is established. \square

4. Conclusions

In the present paper, a time-dependent source identification problem for a one dimensional telegraph equation is studied. Operator-functions generated by the positive operator are considered. The main theorems on stability estimates for the solution of the problem (1) are established.

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