

ON ZEROS OF AN ENTIRE FUNCTION HAVING
AN INTEGRAL REPRESENTATION AND COINCIDING
WITH EXPONENTIAL-TYPE QUASIPOLYNOMS

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Abstract: In this paper, we study zeros of an entire function of the following special form:

$$\Delta(\lambda) = \sum_{k=1}^N P_k \cdot \lambda^{m_k} \cdot e^{\alpha_k \cdot \lambda} + \int_{-1}^1 e^{\lambda t} \cdot \Phi(t) dt,$$

which is a linear combination of functions previously studied in [18], [19], [20], [21] associated with regular differential operators of the third and first orders on an interval.

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1. Introduction

In their work, V.B. Lidsky and V.A. Sadovnichy [1] introduced a class K of entire functions admitting a representation of the form:

$$\Delta(\lambda) = \sum_{k=1}^N e^{\alpha_k \cdot \lambda} \cdot \lambda^{m_k} \cdot \left[\sum_{\nu}^h \beta_{\nu}^k \cdot \lambda^{-\nu} + O(\lambda^{-\nu}) \right],$$

as $\lambda \rightarrow \infty$, where α_k are complex constants, m_k is some integer number, and $\beta_0^k \neq 0$. It is assumed that this asymptotic expansion admits term-by-term differentiation. The class K arose in solving differential equations containing a parameter. It turned out that functions of the class K correspond to boundary problems for ordinary differential equations. If the “boundary” conditions are non-local, then the corresponding entire functions may not belong to the class K .

Necessity to study the asymptotics and distributions of zeros of entire functions arises in study of spectrum of a differential operator. However, in solving spectral problems on a segment with boundary conditions at internal points (multipoint problems), there often appear functions of the class K , for which the assumptions about location of exponents on the complex plane are not satisfied.

The study of the distribution and asymptotics of the zeros of entire functions that do not belong to the class K is far from complete. In particular, the classical works by E. Titchmarsh [2], M. Cartwright [3] are well known. A wide class of analogous entire functions was studied in the monographs [4], [5], [6], [7]. Among the latest works, we note [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

2. Problem statement and discussion

In [18], [19], [20], distribution of eigenvalues of a third-order differential operator u of a composite-type differential operator are studied in the functional space $W_2^3(0, 1)$ with periodic boundary value conditions, i.e.

$$Lu \equiv u_{xxx} + u_{yyy}\lambda u = 0, \quad (1)$$

$$u|_{\partial D} = 0, \quad u_x(0, y) = u_x(1, y), \quad u_y(x, 0) = u_y(x, 1), \quad (2)$$

where $D = \{x, y : 0 < x < 1, 0 < y < 1\}$, which, after applying the Fourier variable separation method, the problem (1) - (2) decomposes to the following

spectral problems for third-order ordinary differential operators with periodic boundary value conditions in the space $W_2^3(0, 1)$:

$$L_0 X \equiv X''' + \mu X = 0; X(0) = X(1) = 0; X'(0) = X'(1), \quad (3)$$

$$L_1 Y \equiv Y''' + \nu Y = 0; Y(0) = Y(1) = 0; Y'(0) = Y'(1), \quad (4)$$

where $\lambda = \mu + \nu$ is a spectral parameter, complex number. The problems (3) and (4) are reduced to study zeros of an entire function coinciding with exponential-type quasi-polynomials from the class K , which is the characteristic determinant of the problem (3)-(4):

$$\begin{aligned} \Delta(a) = & (\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})a} + (\sqrt{3} - 3i) \cdot e^{(1-i\sqrt{3})a} + (\sqrt{3} + 3i) \cdot e^{-(1+i\sqrt{3})a} \\ & + (\sqrt{3} - 3i) \cdot e^{-(1-i\sqrt{3})a} - 2\sqrt{3}e^{2a} - 2\sqrt{3}e^{-2a}, \end{aligned}$$

where $a = \frac{\sqrt[3]{-\mu}}{2} = \frac{\sqrt[3]{-\nu}}{2} \neq 0$ and all zeros, i.e. all eigenvalues and corresponding eigenfunctions of the operator L are found.

In [21], [22], [23], [24], characteristic determinant of spectral problems of a first-order differential equation on a segment with a spectral parameter in a boundary value condition with an integral perturbation and a second-order differential equation with periodic boundary value conditions, one of the boundary value conditions with integral perturbation, which is an entire analytic function of the spectral parameter, is constructed. Due to the formula for the characteristic determinant, conclusions about the asymptotic behavior of the spectrum of perturbed spectral problems are proved. In particular, in [21], in the space $W_2^1(-1, 1)$ the spectral problem is considered for the operator

$$L_0 y \equiv y'(t) = \lambda y(t), \quad -1 < t < 1 \quad (5)$$

with perturbed boundary value conditions

$$y(-1) - y(1) = \lambda \int_{-1}^1 y(t) \Phi(t) dt. \quad (6)$$

In the case when $\Phi(t)$ is a function of bounded variation, and $\Phi(-1) = \Phi(1) = 1$, the problem (5) - (6) is reduced to the characteristic determinant $\Delta_0(\lambda) =$

$$\frac{e^{-\lambda}}{\lambda} - \frac{e^{\lambda}}{\lambda} - \lambda \int_{-1}^1 e^{\lambda t} \Phi(t) dt, \text{ which is an entire analytical function of } \lambda.$$

Theorem 1. Let $\Phi(t)$ be a function of bounded variation and $\Phi(-1) = \Phi(1) = 1$, then all zeros of the function $\Delta_0(\lambda)$, that is all eigenvalues of the operator L_0 belong to the strip $\operatorname{Re}\lambda = |\lambda| < k$ for some k and form a countable set, having asymptotics $\lambda_n^0 = i\pi n + \underline{Q}(1)$ as $n \rightarrow \infty$.

Theorem 2. Let $\Phi(t)$ be a continuous function and $\Phi(-1) = \Phi(1) = 1$, in the boundary value condition (6) $\lambda = 1$, then the characteristic determinant of the problem (5) - (6) will be $\Delta_0^*(\lambda) = e^{-\lambda} - e^\lambda - \int_{-1}^1 e^{\lambda t} \Phi(t) dt$ - entire analytical function of λ and zeros belong to the strip $\operatorname{Re}\lambda = |\lambda| \leq k \cdot r \cdot \omega(\frac{1}{r})$ for some k , form a countable set, and for eigenvalues of the operator L_0 the asymptotic formula $\lambda_n^* = i\pi n + \underline{Q}(n\omega(\frac{1}{n}))$ holds as $n \rightarrow \infty$, where $\omega(h)$ is a module of continuity of $\Phi(t)$.

Remark 3. If $\Phi(t)$ is continuous, then the strip expands depending on properties of the continuity module of $\Phi(t)$.

Let $\{P_k\}$, $\{\alpha_k\}$ be complex numbers, moreover $P_n \neq 0$ for all $k = \overline{1, N}$.

In this paper, we consider distribution of zeros of an entire function of the form

$$\Delta(\lambda) = \sum_{k=1}^N P_k \cdot \lambda^{m_k} \cdot e^{\alpha_k \cdot \lambda} + \int_{-1}^1 e^{\lambda t} \cdot \Phi(t) dt, \quad (7)$$

which is a linear combination of entire functions previously studied in papers [1], [18], [19], [20], [21].

3. Distribution of zeros of the entire function $\Delta(\lambda)$ in (7)

Depending on properties of $\Phi(t)$, the function $\Delta(\lambda)$ in (7) may not belong to the class K . The growth of individual members of the function $\Delta(\lambda)$ is determined primarily by the exponents included in (7). Since module of the exponent is found by the value of the real part of its exponent, that is

$$\left| e^{\alpha_k \lambda} \right| = e^{\operatorname{Re}(\alpha_k \lambda)},$$

then the terms with maximum real parts of the exponentials make the greatest contribution. Therefore, it is necessary to calculate

$$\max(\operatorname{Re}(\alpha_k \lambda), \operatorname{Re}(\lambda t)), \quad (8)$$

when α_k runs through values $\alpha_1, \alpha_2, \dots, \alpha_N$, the value t changes from -1 to +1. To find the maximum in (8), it is enough to use the following geometric technique.

On the complex plane, we plot the numbers $\overline{\alpha_1}, \overline{\alpha_2}, \dots, \overline{\alpha_N}$, and also the segment $[-1, 1]$. Let us find the smallest convex hull of the marked points so that exactly these points are vertices of the convex polygon. Let us denote the sides of the resulting polygon as L_1, L_2, \dots, L_s . If λ changes along a ray that is not perpendicular to any of these sides of the polygon L_1, L_2, \dots, L_s , then growth is determined by one of the exponents $e^{\alpha_1\lambda}, e^{\alpha_2\lambda}, \dots, e^{\alpha_N\lambda}, e^{-\lambda a}, e^{\lambda b}$, where $a = \min \text{Supp}\Phi$, $b = \max \text{Supp}\Phi$, and, therefore, as $|\lambda| \rightarrow \infty$ along the specified ray, the function $\Delta(\lambda)$ in (7) grows commensurately with the corresponding exponent, that is, the following estimate holds

$$|\Delta(\lambda)| > C \cdot e^{\text{Re}(\beta\lambda)},$$

where

$$\text{Re}(\beta\lambda) = \max(\text{Re}(\alpha_1\lambda), \text{Re}(\alpha_2\lambda), \dots, \text{Re}(\alpha_N\lambda), \text{Re}(\lambda b), -\text{Re}(\lambda a)).$$

When λ changes along a ray perpendicular to some side L_j , then the main part of $\Delta(\lambda)$ is determined by the exponential terms from (7), which correspond to the side L_j . Note that, in this case, the comparison function consists of exponential terms, the exponents of which fall on the side L_j . From Rouché's theorem [25], it follows that, zeros of $\Delta(\lambda)$ in the principal are determined by zeros of the comparison functions. Therefore, the previous study of the zeros of entire functions, which has an integral representation [21] and coincides with exponential-type quasi-polynomials [18], [19], [20], [26], can be considered as finding the zeros of the comparison function. Consequently, zeros of the function $\Delta(\lambda)$ in (7) asymptotically coincide with zeros of the comparison function. The number of comparison functions is determined by the number of sides of the constructed convex polygon.

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