International Journal of Applied Mathematics

Volume 35 No. 3 2022, 423-438
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v35i3.5

ON A TIME-NONLOCAL BOUNDARY VALUE PROBLEM FOR TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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Abstract: A boundary value problem with a nonlocal m-point condition in time for a space-degenerate partial differential equation involving the bi-ordinal Hilfer fractional derivative is the main subject of the present investigation. We aim to prove a unique solvability of this problem based on certain properties of the Legendre polynomials and the two-parameter Mittag-Leffler function. An explicit solution to the considered problem is found in the form of the Fourier-Legendre series.

AMS Subject Classification: 35R11, 35C11, 35K65
Key Words: time-fractional PDE; bi-ordinal Hilfer derivative; nonlocal condition; Fourier-Legendre series

Received: March 29, 2022

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1. Introduction

In fractional calculus [2], [1], proposing several definitions of fractional order of derivatives have been interesting target because of their various applications in science and engineering. While using the Riemann-Liouville and Caputo-Gerasimov fractional differential operators as convenient tool in analysing the problems in the theory of differential equations [2], the Hilfer differential operator is also used as a generalization of these operators. However, we note that the generalization of the Riemann-Liouville and Caputo-Gerasimov derivatives, even for the Hilfer fractional derivative was already considered in [3] by Soviet mathematicians M.M. Dzhrbashyan and A.B. Nersesyan in the following form:

$$D_{0x}^{\sigma_n}g(x)=I_{0x}^{1-\gamma_n}D_{0x}^{\gamma_{n-1}}...D_{0x}^{\gamma_1}D_{0x}^{\gamma_0}g(x),\quad n\in\mathbf{N},\ x>0, \tag{1}$$

where I_{0x}^{α} and D_{0x}^{α} are the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative of order α respectively, $\sigma_n\in(0,n]$ which is defined by

$$\sigma_n=\sum_{j=0}^{n}\gamma_j-1>0,\ \gamma_j\in(0,1].$$

Admittedly, the fame of Dzhrbashyan-Nersesyan fractional derivative is arising into a trend again after releasing the translation of the original work in the FCAA journal, [4].

Finding effective and convenient methods for solving fractional partial differential equations (PDEs) is also an interesting part of the research among its applications. Certain aspects of the equations and the properties of the fractional order derivatives allow us to choose the methods to solve the problems. For example, the series method is often used to solve PDEs with any arbitrary order of derivatives and in this problem, it can be divided into two problems of solving ordinary differential equations.

Modeling the phenomena in physics or engineering often requires to study fractional order partial differential equation with variable coefficients. For example, in [5] the authors considered fractional PDEs with the space-dependent coefficient and analyzed the uniqueness and existence of the solution with the help of the properties of the Legendre polynomials.

In physics, fractional order variant of the of Langevin equation plays an important role as a more detailed description of Brownian motion (see [6], Sect. 15.5), when we consider the concept of diffusion process which is associated with random motion of particle in the space. Despite several other applications, it can be said that the Langevin equation itself is attractive and many various differential equations have been considered during recent years. Langevin equation and the idea of further development and generalization of [5] were a key motivation for the studies in the present work.

We would like to note related works [7] - [10] considering nonlocal conditions in time for parabolic and sub-diffusion equations. Precisely, in [10] for sub-diffusion equations with the Caputo and Riemann-Liouville fractional derivatives forward problems with the following time-nonlocal conditions

$$u(\xi) = \alpha u(0) + \varphi, \; I_{0t}^{1-\rho} u(t)|_{t=\xi} = \alpha \lim_{t\to 0} I_{0t}^{1-\rho} u(t), \; \xi \in (0, T],$$

($0 < \rho < 1, \alpha \neq 0$, ξ is a fixed number and φ is a given function) have been studied.

In [11] a boundary value problem with nonlocal condition on time was investigated for a time-fractional and space-singular wave equation. In another work [12], more general PDEs were under investigation with the same time-nonlocal conditions.

In the present paper, we are interested in investigating the following space-degenerate partial fractional differential equation

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1} \left(D_{0+}^{(\alpha_2,\beta_2)\mu_2} u(x,t) - \frac{\partial}{\partial x} \left[(1-x^2) u_x(x,t) \right] \right) = f(x,t) \tag{2}$$

in the domain $\Omega = \{(x,t) : -1 < x < 1, \, 0 < t \leq T\}$. Here $D_{0t}^{(\alpha_s,\beta_s)\mu_s}$ is a bi-ordinal Hilfer fractional derivative defined by

$$D_{0t}^{(\alpha_s,\beta_s)\mu_s} y(x) := I_{0+}^{\mu_s(s-\alpha_s)} \left(\frac{d}{dx} \right)^s I_{0+}^{(1-\mu_s)(s-\beta_s)} y(x), \tag{3}$$

where $0 < \alpha_s, \beta_s < 1, \quad 0 \leq \mu_s \leq 1, \; s = \overline{1,2}$ and $I_{0+}^{\gamma} y(x)$ is the Riemann-Liouville integral operator of order γ of a function $y(x)$ [2].

The bi-ordinal Hilfer fractional differential operator is considered by V. M. Bulavatsky with he help of the well-known generalizing Hilfer fractional derivative of given order and type by expressing it with two orders and type [13]. Note that (3) is still preserving its interpolation concept between the Riemann-Liouville and the Caputo-Gerasimov fractional differential operators.

Remark 1. The bi-ordinal Hilfer derivative $D_{0+}^{(\alpha,\beta)\mu} g(x)$ can be written as

$$D_{0+}^{(\alpha,\beta)\mu} g(x) = I_{0+}^{\mu(n-\alpha)} \left(\frac{d}{dt} \right)^n I_{0+}^{(1-\mu)(n-\beta)} g(x)$$

$$= I_{0+}^{\mu(n-\alpha)} \left(\frac{d}{dt} \right)^n I_{0+}^{n-\gamma} g(x) = I_{0+}^{\mu(n-\alpha)} D_{0+}^{\gamma} g(x) = I_{0+}^{\gamma-\delta} D_{0+}^{\gamma} g(x) \tag{4}$$

for $x \in [0,T]$, where $\gamma = \beta + \mu(n-\beta)$ and $\delta = \beta + \mu(\alpha - \beta)$.

From Remark 1 and (1) it is not difficult to show that the bi-ordinal Hilfer fractional differential operator can be represented as particular case of the Dzhrbashyan-Nersesyan fractional differential operator for $n=1$, i.e.

$$D_{0+}^{\sigma_1} g(x) = I_{0+}^{1-\gamma_1} D_{0+}^{\gamma_0} g(x).$$

We also emphasize that in [14]-[15] direct and inverse problems for the generalized fractional diffusion equation with the bi-ordinal Hilfer derivative were considered.

2. Statement of the problem and main results

Problem A. Find a solution $u(x,t)$ of the equation (2) satisfying regularity conditions

$$t^{1-\gamma_2} u, \; t^{1-\gamma_2} D_{0+}^{(\alpha_2,\beta_2)\mu_2} u, \quad t^{1-\gamma_2} u_x \in C(\overline{\Omega}),$$

$$t^{1+\delta_2-\gamma_1} D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2} u \in C(\overline{\Omega}), \quad u_{xx} \in C(\Omega)$$

and initial condition

$$\lim_{t\to 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)} u(x,t) = \psi(x), \; -1 \le x \le 1, \tag{5}$$

and subject to the nonlocal condition

$$u(x,T) = \sum_{i=1}^{m} p_i I_{0+}^{q_i} D_{0+}^{\delta_2+\gamma_1} u(x,\tau_i), \; -1 < x < 1, \tag{6}$$

where $\psi(x)$, $f(x,t)$ are given functions and $q_i > 0$, $\delta_j = \beta_j + \mu_j(\alpha_j - \beta_j)$, $\gamma_j = \beta_j + \mu_j(1-\beta_j)$, $j = \overline{1,2}$, $p_i \in \mathbf{R}$, $0 < \tau_1 < \tau_2 < ... < \tau_m \le T$ and also we assume $0 < \gamma_2 - \gamma_1 < \delta_2$.

We investigate a unique solvability of this problem and present the solution in the form of Fourier-Legendre series as stated in the following theorem.

Theorem 1. *If* $\sum_{i=1}^{m} \frac{p_i \tau_i^{q_i-1}}{\Gamma(q_i)} > 0$, $\psi(x) \in C^1[-1,1]$, $\psi''(x) \in L^2(-1,1)$, $f(x,\cdot) \in C_{-1}^1[0,T]$ *and* $f(\cdot,t) \in C[-1,1]$, $f_{xx}(\cdot,t) \in L^2(-1,1)$, *then Problem A has a unique solution which can be represented as*

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) P_k(x).$$

Here $\lambda_k = k(k+1)$, $k = 0,1,2,...$, ψ_k and $f_k(t)$ are the Fourier-Legendre coefficients of functions $\psi(x)$ and $f(x,t)$, respectively,

$$u_k(t) = \psi_k t^{\gamma_2 - 1} E_{\delta_2,\gamma_2}\left(-\lambda_k t^{\delta_2}\right) + C_0 t^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}(-\lambda_k t^{\delta_2}$$

$$+ \int\limits_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2,\delta_2+\delta_1}\left[-\lambda_k (t-s)^{\delta_2}\right] f_k(s) ds,$$

C_0 is defined by the formula (16).

We note that the vector space C_{-1} is defined to be the set of all functions $f(x)$, $x > 0$, expressible as $f(x) = x^p f_1(x)$ for some real number $p > -1$ and function $f_1 \in C[0,\infty)$ and the vector space C^1_{-1} is defined to consist of all functions $f(x)$, $x > 0$, such that f is one times differentiable and $f' \in C_{-1}$ (see [16]).

Proof. We intend to investigate this problem by applying the method of separation variables. From the equation (2) in the homogeneous case and considering $u(x,t)$, $u_x(x,t)$ are bounded at $x = \pm 1$ which are come from regularity conditions, yield the following Legendre equation:

$$(1-x^2)X''(x) - 2xX'(x) + \lambda X(x) = 0 \tag{7}$$

and it has a bounded solution in $[-1,1]$ only if $\lambda_k = k(k+1)$, $k = 0,1,2,...$ and it is given by

$$X(x) = P_k(x) = \frac{1}{2^k \cdot k!} \frac{d^k (x^2-1)^k}{dx^k},$$

where $P_k(x)$ are the Legendre polynomials.

It is known that (W. Kaplan [17], p. 511) that the Legendre polynomials form a complete orthogonal system in $[-1,1]$ and any piece-wise continuous function g can be expressed in the form of Fourier-Legendre series with respect to the system $\{P_k(x)\}$:

$$g(x) = \sum_{k=0}^{\infty} c_k P_k(x), \;\; c_k = \frac{(g, P_k)}{\|P_k\|^2} = \frac{2k+1}{2} \int\limits_{-1}^{1} g(x) P_k(x) dx.$$

Hence, we represent a sought function $u(x,t)$ and the given function $f(x,t)$ in the following forms:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) P_k(x), \tag{8}$$

$$f(x,t)=\sum_{k=0}^{\infty} f_k(t)P_k(x), \tag{9}$$

where $u_k(t)$ is unknown and $f_k(t)$ is the Fourier-Legendre coefficient of $f(x,t)$, i.e.

$$f_k(t)=\frac{2k+1}{2}\int_{-1}^{1} f(x,t)P_k(x)dx.$$

By substituting (8) and (9) into the equation (2) and initial conditions (5) one can obtain the following fractional differential equation

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\left(D_{0+}^{(\alpha_2,\beta_2)\mu_2}+\lambda_k\right)u_k(t)=f_k(t), \tag{10}$$

with initial condition

$$\lim_{t\to 0+} I_{0+}^{(1-\mu_2)(1-\beta_2)}u_k(t)=\psi_k, \tag{11}$$

and nonlocal condition

$$u_k(T)=\sum_{i=1}^{m} p_i I_{0+}^{q_i} D_{0+}^{\delta_2+\gamma_1}u_k(\tau_i), \tag{12}$$

where

$$\psi_k=\frac{2k+1}{2}\int_{-1}^{1}\psi(x)P_k(x)dx.$$

Lemma 1. *If $g\in L^1(a,b)$ and $I_{0+}^{1-\gamma}g\in AC^n(a,b)$, $n-1<\alpha,\beta\le n$ and $0\le\mu\le 1$, then*

$$I_{0+}^{\delta}D_{0+}^{(\alpha,\beta)\mu}g(t)=g(t)-\sum_{k=1}^{n}\frac{t^{\gamma-k}}{\Gamma(\gamma-k+1)}\left[\lim_{t\to 0+}(\frac{d}{dt})^{n-k}I_{0+}^{n-\gamma}g(t)\right]$$

$$=g(t)-\sum_{k=1}^{n}\frac{C_{n-k}t^{\gamma-k}}{\Gamma(\gamma-k+1)},$$

where $\delta=\beta+\mu(\alpha-\beta)$, $\gamma=\beta+\mu(n-\beta)$.

Proof. The proof of Lemma 1 can be derived from Remark 1 and the composition $I_{a+}^{\alpha}D_{a+}^{\alpha}$ of the Riemann-Liouville fractional integration I_{a+}^{α} and differentiation operator D_{a+}^{α}. □

Applying the operator $I_{0+}^{\delta_1}$ to both sides of (10) and using Lemma 1, we obtain the following fractional differential equation

$$D_{0+}^{(\alpha_2,\beta_2)\mu_2} u_k(t) + \lambda_k u_k(t) = h(t), \tag{13}$$

where

$$h(t) = I_{0+}^{\delta_1} f_k(t) + \frac{C_0 t^{\gamma_1 - 1}}{\Gamma(\gamma_1)}.$$

The solution of the problem (13), (11) can be represented as follows ([13]):

$$\begin{aligned} u_k(t) &= \psi_k t^{\gamma_2-1} E_{\delta_2,\gamma_2}\left(-\lambda_k t_2^{\delta}\right) \\ &+ \int_0^t (t-\tau)^{\delta_2-1} E_{\delta_2,\delta_2}\left[-\lambda_k (t-\tau)^{\delta_2}\right] h(\tau) d\tau. \end{aligned} \tag{14}$$

By substituting $h(t)$ into the solution (14) and after some evaluations, we can rewrite the solution of (10) satisfying (11) as follows:

$$u_k(t) = \psi_k t^{\gamma_2-1} E_{\delta_2,\gamma_2}\left(-\lambda_k t_2^{\delta}\right) + C_0 t^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}(-\lambda_k t^{\delta_2}$$

$$+ \int_0^t (t-s)^{\delta_2+\delta_1-1} E_{\delta_2,\delta_2+\delta_1}\left[-\lambda_k (t-s)^{\delta_2}\right] f_k(s) ds. \tag{15}$$

In order to find C_0 we use the nonlocal condition (12), and obtain

$$\psi_k T^{\gamma_2-1} E_{\delta_2,\gamma_2}(-\lambda_k T^{\delta_2}) + C_0 T^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}(-\lambda_k T^{\delta_2})$$

$$+ \int_0^T (T-s)^{\delta_2+\delta_1-1} E_{\delta_2,\delta_2+\delta_1}\left[-\lambda_k (T-s)^{\delta_2}\right] f_k(s) ds$$

$$= \psi_k \sum_{i=1}^m p_i \tau_i^{\gamma_2-\delta_2-\gamma_1+q_i-1} E_{\delta_2,\gamma_2-\gamma_1-\delta_2+q_i}(-\lambda_k \tau_i^{\delta_2})$$

$$-C_0 \lambda_k \sum_{i=1}^m p_i \tau_i^{\delta_2+q_i-1} E_{\delta_2,\delta_2+q_i}(-\lambda_k \tau_i^{\delta_2})$$

$$+f_k(0) \sum_{i=1}^m p_i \tau_i^{\delta_1-\gamma_1+q_i} E_{\delta_2,\delta_1-\gamma_1+q_i+1}(-\lambda_k \tau_i^{\delta_2})$$

$$+\sum_{i=1}^m p_i \int_0^{\tau_i} (\tau_i - s)^{\delta_1-\gamma_1+q_i} E_{\delta_2,\delta_1-\gamma_1+q_i+1}\left[-\lambda_k (\tau_i - s)^{\delta_2}\right] f_k'(s) ds.$$

From the last equality we find C_0 as

$$C_0 = \frac{1}{\Delta_k}(B_k + F_k), \tag{16}$$

where

$$\Delta_k = T^{\delta_2+\gamma_1-1} E_{\delta_2,\delta_2+\gamma_1}\left(-\lambda_k T^{\delta_2}\right) + \lambda_k \sum_{i=1}^{m} p_i \tau_i^{\delta_2+q_i-1} E_{\delta_2,\delta_2+q_i}\left(-\lambda_k \tau_i^{\delta_2}\right),$$

$$B_k = \psi_k \left[\sum_{i=1}^{m} p_i \tau_i^{\gamma_2-\delta_2-\gamma_1+q_i-1} E_{\delta_2,\gamma_2-\gamma_1-\delta_2+q_i}\left(-\lambda_k \tau_i^{\delta_2}\right) - T^{\gamma_2-1} E_{\delta_2,\gamma_2}\left(-\lambda_k T^{\delta_2}\right) \right],$$

$$F_k = f_k(0) \sum_{i=1}^{m} p_i \tau_i^{\delta_1-\gamma_1+q_i} E_{\delta_2,\delta_1-\gamma_1+q_i+1}\left(-\lambda_k \tau_i^{\delta_2}\right)$$
$$+ \sum_{i=1}^{m} p_i \int_0^{\tau_i} (\tau_i - s)^{\delta_1-\gamma_1+q_i} E_{\delta_2,\delta_1-\gamma_1+q_i+1}\left[-\lambda_k (\tau_i - s)^{\delta_2}\right] f_k'(s) ds$$
$$- \int_0^{T} (T - s)^{\delta_2+\delta_1-1} E_{\delta_2,\delta_2+\delta_1}\left[-\lambda_k (T - s)^{\delta_2}\right] f_k(s) ds.$$

We assume that $\Delta_k \neq 0$ for any k, then (15) will be the solution of the problem (10) - (12). For that reason we recall some properties of the Mittag-Leffler which are reduced from the properties Wright-type function investigated by A. Pskhu [18].

Lemma 2. *If $\pi \geq |argz| > \frac{\pi\alpha}{2} + \varepsilon$, $\varepsilon > 0$, then the following relations are valid for $|z| \to +\infty$:*

$$\lim_{|z|\to+\infty} E_{\alpha,\beta}(z) = 0, \qquad \lim_{|z|\to+\infty} z E_{\alpha,\beta}(z) = -\frac{1}{\Gamma(\beta-\alpha)}.$$

Lemma 3. ([19]) *For every $\alpha \in (0,1]$, $\beta > \alpha$ and $x \geq 0$ one has*

$$\frac{1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)} x} \leq \Gamma(\beta) E_{\alpha,\beta}(-x) \leq \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x}.$$

By considering Lemma 2 it can be shown that

$$\lim_{k\to+\infty}\Delta_k=\lim_{\lambda_k\to+\infty}\Delta_k=\lim_{|z_1|\to+\infty}T^{\delta_2+\gamma_1-1}E_{\delta_2,\delta_2+\gamma_1}(z_1)$$

$$-\lim_{|z_2|\to+\infty}\sum_{i=1}^{m}p_i\tau_i^{q_i-1}z_2E_{\delta_2,\delta_2+q_i}(z_2)=\sum_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}.$$

Assuming $\sum\limits_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}\neq 0$, then it confirms that $\Delta_k\neq 0$ for any sufficiently large k. According to Lemma 3 we can find lower bound of Δ_k for any k as

$$\Delta_k\geq\frac{T^{\delta_2+\gamma_1-1}}{\Gamma(\delta_2+\gamma_1)+\Gamma(\gamma_1)\lambda_kT^{\delta_2}}+\sum_{i=1}^{m}\frac{p_i\lambda_k\tau_i^{\delta_2+q_i-1}}{\Gamma(\delta_2+q_i)+\Gamma(q_i)\lambda_k\tau_i^{\delta_2}}$$

$$\geq\sum_{i=1}^{m}\frac{p_i\lambda_k\tau_i^{\delta_2+q_i-1}}{\Gamma(\delta_2+q_i)+\Gamma(q_i)\lambda_k\tau_i^{\delta_2}}=\sum_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}.$$

If $\sum\limits_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}>0$ for any k. Moreover, we may write upper bound of $\frac{1}{\Delta_k}$ as

$$\frac{1}{\Delta_k}\leq M_1=\frac{1}{\sum\limits_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}}.$$

Lemma 4. ([20]) *As* $\alpha<2$, $\beta\in\mathbf{R}$ *and* $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$, *then*

$$|E_{\alpha,\beta}(z)|\leq\frac{M}{1+|z|},\ \ \mu\leq|argz|\leq\pi,\ \ |z|\geq 0, M>0.$$

Now we find upper bounds of B_k and F_k by using the well-known estimation of the Mittag-Leffler function from Lemma 4:

$$|B_k|\leq|\psi_k|\left[\sum_{i=1}^{m}|p_i|\frac{\tau_i^{\gamma_2-\delta_2-\gamma_1+q_i-1}M}{1+\lambda_k\tau_i^{\delta_2}}+T^{\gamma_2-1}\frac{M}{1+\lambda_kT^{\delta_2}}\right]\leq\frac{|\psi_k|M_2}{\lambda_k},$$

where $M_2=\sum\limits_{i=1}^{m}|p_i|\tau_i^{\gamma_2-2\delta_2-\gamma_1-1}+T^{\gamma_2-\delta_2-1}$,

$$|F_k|\leq|f_k(0)|\sum_{i=1}^{m}|p_i|\frac{\tau_i^{\delta_1-\gamma_1+q_i}M}{1+\lambda_k\tau_i^{\delta_2}}$$

$$+\sum_{i=1}^{m}|p_i|\int\limits_0^{\tau_i}\frac{|\tau_i-s|^{\delta_1-\gamma_1+q_i}M}{1+\lambda_k|\tau_i-s|^{\delta_2}}|f_k'(s)|ds$$

$$+\int\limits_0^{T}|T-s|^{\delta_2+\delta_1-1}\frac{M}{1+\lambda_k|T-s|^{\delta_2}}|f_k(s)|ds.$$

If $f_k'(s)\in C_{-1}[0,T]$, then we may consider for $k\to\infty$ that

$$|F_k|\leq M_3<+\infty,\;\; M_3=const>0.$$

Considering upper bounds of B_k and F_k, we have that

$$|C_0|\leq\frac{B_k+F_k}{\Delta_k}\leq\frac{|\psi_k|M_2+M3}{\lambda_k\sum\limits_{i=1}^{m}\frac{p_i\tau_i^{q_i-1}}{\Gamma(q_i)}}=\frac{M_4}{\lambda_k},$$

where we have assumed that $\psi(x)\in C[-1,1]$ and $f(\cdot,t)\in C[-1,1]$ and $f(x,\cdot)\in C_{-1}^1[0,T]$.

2.1. Uniqueness of the solution

Let there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of the main problem and consider the function $u(x,t)=u_1(x,t)-u_2(x,t)$ which is a solution of the equation (2) in the homogeneous case with homogeneous initial conditions

$$\lim_{t\to0+}I_{0+}^{(1-\mu_2)(1-\beta_2)}u(x,t)=0,\;\;-1\leq x\leq1. \tag{17}$$

Let us consider the following function

$$u_k(t)=\int\limits_{-1}^{1}u(x,t)P_k(x)dx,\;\; k=0,1,2,...,. \tag{18}$$

Based on (18), we consider the function below

$$v_k(t)=\int\limits_{\varepsilon-1}^{1+\varepsilon}u(x,t)P_k(x)dx,\;\; k=0,1,2,..., \tag{19}$$

where ε is very small positive number.

Applying the operator $D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2}$ with respect to t to both sides of equality (19) and using the homogeneous equation corresponding with (2) yields that

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu} v_k(t) = \int\limits_{\varepsilon-1}^{1+\varepsilon} D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2} u(x,t) P_k(x) dx$$

$$= \int\limits_{\varepsilon-1}^{1+\varepsilon} P_k(x) D_{0+}^{(\alpha_1,\beta_1)\mu_1} \frac{\partial}{\partial x}\left[(1-x^2)u_x(x,t)\right] dx,$$

then integrating by parts twice the right side of the last equality and calculating the limit as $\varepsilon \to 0$, gives that

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\left[D_{0+}^{(\alpha_2,\beta_2)\mu_2} + \lambda_k\right] u(t) = 0.$$

Obviously, it can be shown that this equation with homogeneous conditions (17) has only trivial solution $u_k(t) \equiv 0,\ t \in [0,T]$ and hence, from (18) we get

$$\int\limits_{-1}^{1} u(x,t) P_k(x) dx = 0,\ k = 0,1,2,\dots .$$

Therefore, using the fact of completeness property of system $\{P_k(x)\}$, it is deduced that $u(x,t) \equiv 0$ in Ω, which proves the uniqueness of the solution of Problem A.

2.2. Existence of the solution

To show the existence of the solution in the form of (8), we need to proof the uniform convergence of the series

$$u(x,t),\ D_{0+}^{(\alpha_1,\beta_1)\mu_1} \frac{\partial}{\partial x}\left[(1-x^2)u_x(x,t)\right],\ \text{and}\ D_{0+}^{(\alpha_1,\beta_1)\mu_1} D_{0+}^{(\alpha_2,\beta_2)\mu_2} u(x,t).$$

For $k = 1,2,3,\dots$ the Legendre polynomials satisfy the following identities and relations ([17]):

1) $P'_{k+1}(x) - P'_{k-1}(x) = (2k+1)P_k(x)$, 2) $\|P_k(x)\|^2 = \frac{2}{2k+1}$,

3) $P_k(1) = 1,\ \ P_k(-1) = (-1)^k$, 4) $|P_k(x)| \le 1,\ |x| \le 1$.

Let (f, g) be scalar product of the functions f and g in $L^2(-1,1)$. Using the above properties of the Legendre polynomials, we can write as

$$\psi_k = \frac{2k+1}{2}\int\limits_{-1}^{1}\psi(x)P_k(x)dx$$

$$= \frac{2k+1}{2}\int\limits_{-1}^{1}\psi(x)\frac{1}{2k+1}\left[P'_{k+1}(x) - P'_{k-1}(x)\right]dx$$

and integrating by parts,

$$\psi_k = -\frac{1}{2}\int\limits_{-1}^{1}\psi(x)\left[P_{k+1}(x) - P_{k-1}(x)\right]dx = -\frac{1}{2}\left[(\psi', P_{k+1}) - (\psi', P_{k-1})\right].$$

Applying the Schwartz inequality $|(f,g)| \leq \|f\|\|g\|$, we can write the estimation of ψ_k:

$$|\psi_k| \leq \frac{1}{2}\left|(\psi', P_{k+1})\right| + \frac{1}{2}\left|(\psi', P_{k-1})\right|$$

$$\leq \frac{1}{2}\left[\|\psi'\|\cdot\|P_{k+1}\| + \|\psi'\|\cdot\|P_{k+1}\|\right]$$

$$\leq \frac{1}{2}\|\psi'\|\left(\frac{\sqrt{2}}{(2k+3)^{\frac{1}{2}}} + \frac{\sqrt{2}}{(2k-1)^{\frac{1}{2}}}\right) \leq \frac{\|\psi'\|\sqrt{2}}{(2k-1)^{\frac{1}{2}}},$$

where $\|\cdot\|$ is a norm of $L^2(-1,1)$.

Repeating this process one more time, one can obtain

$$|\psi_k| \leq \frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}}\|\psi''(x)\|. \tag{20}$$

As a similar way, we write the estimation of $f_k(t)$:

$$|f_k(t)| \leq \frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}}\|f''_{xx}(\cdot,t)\|. \tag{21}$$

Considering the above estimation for the Mittag-Leffler function we write the bound of $u(x,t)$ by virtue of the properties of the Legendre polynomials:

$$|u(x,t)| \leq \sum_{k=0}^{\infty}\left[|\psi_k|\frac{t^{\gamma_2-1}M}{1+|\lambda_k||t_2^{\delta}|} + |C_0|\frac{t^{\delta_2+\gamma_1-1}M}{1+|\lambda_k||t_2^{\delta}|}\right.$$

$$+\int_0^t |t-s|^{\delta_2+\delta-1}\frac{M}{1+|\lambda_k||t-s|^{\delta_2}}|f_k(s)ds|\Bigg].$$

If $\psi(x) \in C[-1,1]$ and $f(\cdot,t) \in C_{-1}[-1,1]$, we can show that the series of $|u(x,t)|$ is bounded by convergent series in Ω domain, and therefore by Weierstrass M-test the series representation of $u(x,t)$ converges uniformly in Ω.

After that, by using the properties of the Legendre polynomials, we show the uniform convergence of the series of $D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}[(1-x^2)u_x]$ which is represented as follows:

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}[(1-x^2)u_x] = \sum_{k=0}^{\infty}\lambda_k D_{0+}^{(\alpha_1,\beta_1)\mu_1}u_k(t)P_k(x)$$

$$= \sum_{k=0}^{\infty}\Big[\psi_k t^{\gamma_2-\delta_1-1}E_{\delta_2,\gamma_2-\delta_1}(-\lambda_k t^{\delta_2}) + C_0 t^{\delta_2+\gamma_1-\delta_1-1}E_{\delta_2,\delta_2+\gamma_1-\delta_1}(-\lambda_k t^{\delta_2})$$

$$+\int_0^t (t-s)^{\delta_2-1}E_{\delta_2,\delta_2}[-\lambda_k(t-s)^{\delta_2}]f_k(s)ds\Big]P_k(x).$$

By means of the properties of the Legendre polynomials and the upper bounds of the Mittag-Leffler function presented by Lemma 4 and above estimations for given functions, we get the following estimation

$$\left|D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}\left[(1-x^2)u_x\right]\right| \le \sum_{k=0}^{\infty}|D_{0+}^{(\alpha_1,\beta_1)\mu_1}u_k(t)\lambda_k P_k(x)|$$

$$\le \sum_{k=0}^{\infty}\lambda_k\left[|\psi_k|\frac{t^{\gamma_2-\delta_1-1}M}{1+\lambda_k|t^{\delta_2}|} + |C_0|\frac{t^{\delta_2+\gamma_1-\delta_1-1}M}{1+\lambda_k|t^{\delta_2}|}\right]$$

$$+\sum_{k=0}^{\infty}\lambda_k\int_0^t |t-z|^{\delta_2-1}\frac{M}{1+\lambda_k||t-z|^{\delta_2}}|f_k(z)|dz$$

$$\le \sum_{k=0}^{\infty}\left[\frac{MT^{\gamma_2-\delta_1-\delta_2-1}4\sqrt{2}}{(2k-3)^{\frac{3}{2}}}\|\psi''(x)\| + \frac{MM_4T^{\gamma_1-\delta_1-1}}{\lambda_k}\right]$$

$$+\sum_{k=0}^{\infty}\int_0^t |t-z|^{\delta_2-1}\frac{M\lambda_k}{1+\lambda_k||t-z|^{\delta_2}}\frac{4\sqrt{2}}{(2k-3)^{\frac{3}{2}}}\|f''_{xx}(\cdot,z)\|dz],$$

where $\lambda_k = k(k+1)$.

Under necessary conditions for given functions are fulfilled, from the last inequalities one can show that the series of representation of $D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}[(1-x^2)u_x]$ is bounded by a convergent series which implies that it is convergent uniformly according to the Weierstrass M-test in Ω.

Finally, $D_{0+}^{(\alpha_1,\beta_1)\mu_1}D_{0+}^{(\alpha_2,\beta_2)\mu_2}u(x,t)$ which can be represented by the equation

$$D_{0+}^{(\alpha_1,\beta_1)\mu_1}D_{0+}^{(\alpha_2,\beta_2)\mu_2}u(x,t) = D_{0+}^{(\alpha_1,\beta_1)\mu_1}\frac{\partial}{\partial x}\left[(1-x^2)u_x(x,t)\right] + f(x,t)$$

and its uniform convergence can be shown as a similar way which we have done before to the $D_{0+}^{(\alpha_1,\beta_1)\mu_1}\dfrac{\partial}{\partial x}[(1-x^2)u_x]$.

Thus, we have proved Theorem 1. □

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