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WEAK SET-VALUED MARTINGALE DIFFERENCE AND ITS APPLICATIONS

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Abstract: The martingale difference is considered widely in finance and economic because of its application in efficient market, in which the conditional expectation $E[d_t|\mathcal{F}_{t-1}] = 0$ a.s., $\forall t \geq 2$ for a sequence of asset returns $\{d_t, t \geq 1\}$ and related historical information $\mathcal{F}_{t-1}$. However, the concept of martingale difference in set-valued random variables (i.e. random sets) has not been studied. This paper proves some properties of a set-valued random variable sequence called *a weak set-valued martingale difference.* By studying its characteristic properties, we propose a method of testing the weak set-valued martingale difference hypothesis and perform some simulations with real data.

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1. Introduction

The testing of martingale hypothesis has received a lot of attention from researchers in the economic field for a long time. Specifically, if we denote by y_t the price on day $t, t \geq 1$ of a stock A and $\mathcal{F}_t$ is the information related to day t of A, then $\{y_t, t \geq 1\}$ is a martingale if the conditional expectation $E[y_{t+1}|\mathcal{F}_t] = y_t, \forall t \geq 1$. It means that the best forecast for tomorrow's asset price is today's and then the asset price is unpredictable. This test is directly related to the efficient market, [16]. For computational convenience, instead of verifying whether $\{y_t\}$ is a martingale, we usually perform the testing of martingale difference hypothesis (called the MDH test) for its returns, meaning that $E[d_{t+1}|\mathcal{F}_t] = 0$, where $d_{t+1} = \ln y_{t+1} - \ln y_t \approx (y_{t+1} - y_t)/y_t$, [12, 29]. Various MDH tests depending on the relation of historical information $\mathcal{F}_{n-1}, \mathcal{F}_{n-2}, \cdots$ are found in [10, 13].

Although there have been many results showing that it is very difficult (even impossible) to predict the asset prices in short-term (e.g. [10] and Fama's studies (Nobel Prize in Economics 2013)), many recent short-term stock price prediction models are gaining great attention due to their high forecasting accuracy. In particular, forecasting models which are based on fuzzy time series (FTS) are a new trend [27, 11, 26]. What are the reasons for this contradiction?

To answer this question, we briefly learn about the prediction models based on FTS, basically as follows. Divide the universe of discourse U of a time series $\{d_t, t \geq 1\}$ by $\underbrace{u_1}_{A_1} \cup \underbrace{u_2}_{A_2} \cup ... \cup \underbrace{u_k}_{A_k}$, where $A_i, i = 1, \cdots, k$ are the fuzzy sets representing the states of $\{d_t\}$. Then for each t, we have $d_t \in D_t$ such that $D_t \in \{A_1, A_2, \cdots, A_k\}$. Hence, the sequence $D_1, D_2, ...D_t, ...$ is a sequence of set-valued random variables related to the sequence $\{d_t\}$. Based on the historical observations, the fuzzy rules with the form $A_i \xrightarrow{\text{fuzzy rules}} A_j, A_k, \cdots$ are estimated. By using some defuzzification methods, the asset price forecasts are obtained from these rules. We can see that these rules represent the tendency of $\{d_t\}$. Therefore, these forecasting models are also known as trend-based forecasting models [36]. Obviously these laws determine the quality of the forecasts. Since these laws are derived from statistical analyses of training data, the more stable these rules are (for an enough period of time) the higher accuracy of the forecasts. Thus, there can be an explanation for the appropriateness of trend-based forecasting models: testing the MDH for the predictability of asset prices is not in line with the predictability of the trend. Therefore, trend-based forecasting models have higher accuracy than classical models such as ARIMA, ANN, HMM (the comparison are found in [21, 20, 18, 31, 26] and the references therein). Although MDH has been used widely to test the predictability of stock prices, it is necessary to construct methods for testing the stock trends. In the FTS-based predictive models, since the trends of the time series form a sequence of set-valued random variables (its values are real-intervals), the testing for the predictability of these trends clearly relates to the theory of set-valued random variables or random sets.

For the theory of set-valued random variables (that is, random variables that taking their values in the space of all non-empty closed subsets of a Banach space $\mathfrak{X}$), most results in probability theory in the case of single-valued random variables are expanded [24, 22, 6]. Research directions on set-valued random variables are often related to limit theorems, the law of large numbers, and martingales [32, 25, 15].

Up to now, although most of the results on the limit theorems for sequences of single-valued random variables have been expanded and proven in the case of random sets [22, 24], the corresponding results for martingale difference in the case of set-valued random variables have not been adequately studied. In the other words, there is no method to test the predictability of a sequence of set-valued random variables in the same way as for the sequence of single-valued random variables (actually, the definition of a set-valued martingale difference does not exist). There are certain difficulties in expanding the concept of a martingale difference from single-valued to set-valued random variables. Specifically, if we define a set-valued martingale difference similar to the mean of a single-valued martingale difference, that is, a sequence of set-valued random variables $\{D_n, \mathcal{F}_n, n \geq 1\}$ is called a set-valued martingale difference if

$$\forall n \in \mathbb{N}, E[D_{n+1}|\mathcal{F}_n] = \{0\} \text{ a.s.}, \tag{1}$$

then according to Theorem 2.1.72 in [24], D_n degenerates to a singleton. That means $D_n = \{\xi_n\}$ for every $n \geq 1$ where $\{\xi_n\}$ is a single-valued martingale difference. Thus, there will not really exist a sequence of set-valued random variables (i.e. not a singleton) satisfying the definition of set-valued martingale difference as in (1). Therefore, defining a set-valued martingale difference as in this form has no meaning.

In another aspect, the problem posed by this paper is to test the predictability of the trend of a time series. When these trends are fuzzified into fuzzy sets (e.g. fuzzy time series prediction models), this fuzzy time series is a sequence of set-valued random variables, $\{D_n, \mathcal{F}_n, n \geq 1\}$ for instance, taking their values in the set of real-intervals. If these trends are always increase or decrease over time almost surely then $d(0, E[D_n|\mathcal{F}_{n-1}]) > 0$, a.s. (where $d(0, X) = \inf\{\|x\| : x \in X\}$). This means the trend of the time series is predictable. Thus, to test the unpredictability of $\{D_n\}$, we can check the condition $d(0, E[D_n|\mathcal{F}_{n-1}]) = 0,$ a.s. for all $n \geq 1$. This condition are equivalent to the following condition

$$0 \in E[D_n|\mathcal{F}_{n-1}], \text{ a.s. for all } n \geq 1, \tag{2}$$

where $E[D_n|\mathcal{F}_{n-1}]$ denotes the conditional expectation for set-valued random variables [2].

This paper studies the characteristics of the sequence of set-valued random variables satisfying the formula (2) and proposes a method of statistical test for it. This sequence was first studied by Ezzaki [14], and it is called *multi-valued martingale difference*. To avoid misunderstandings with the concept of set-valued martingale difference that really does not exist, we call it *weak set-valued martingale difference*. The structure of the paper is as follows. Section 2 presents some popular MDH tests and basic knowledge of set-valued random variables. Section 3 demonstrates some properties of weak set-valued martingale difference and prove its characteristic property under more general conditions than Ezzaki's with another way. Based on the results of Section 3, Section 4 proposes an approach to test predictability of asset trends. The correctness of the proposed method is tested on simulation data, then the method is applied for some actual data. Finally, there are some conclusions about the contribution of the paper as well as some future development directions.

2. Preliminaries

The martingale difference hypothesis (MDH) is in order to test whether the sequence of returns $d_n, n \geq 1$ of a financial or economic time series $y_n, n \geq 1$ supports a martingale difference in $\mathbb{R}$. Specifically, let $I_n = \{d_n, d_{n-1}, ...\}$ be the information set at time n and call $\mathcal{F}_n$ the σ-field generated by I_n. In generally case, the MDH holds for d_n when the conditional moment restriction $E[d_n|I_{n-1}] = \mu$ a.s. holds almost surely for some constant μ. The MDH says that the best predictor of the future values is just unconditional expectation. It means that past and current information is no use for forecasting future values for a martingale difference sequence (MDS for short). It also means that d_t is linearly unpredictable given any linear or nonlinear transformation $w\,(I_{n-1})$ of the past. From the following equivalence

$$E[d_n|I_{n-1}] = \mu\ (a.s.), \mu \in \mathbb{R} \Leftrightarrow E[(d_n - \mu)w(I_{n-1})] = 0, \tag{3}$$

for all $\mathcal{F}_{n-1}$-measurable weighting function $w(.)$ (provided that the moment exists), the MDH tests are usually based on how far from sample mean $E[(d_n -$

$\mu)w(I_{n-1})]$ to 0 to implement. However, it is impossible to fulfill this condition with every function $w(.)$, so existing MDH tests are only necessary conditions for a MDS through choosing some linear or nonlinear function $w(.)$. In all MDH tests, the null hypothesis is $H0 : E[(d_n - \mu)w(I_{n-1})] = 0$ and the alternate hypothesis $H1$ is otherwise.

This paper uses the MDH tests based on considering a linear function $w(.)$ in the simplest approach, i.e. $w(I_{n-1}) = d_{n-j}$ for all $j \geq 1$. Hence, a necessary (but not sufficient, in general) condition for the MDH to hold is that the time series is uncorrelated, i.e.

$$\gamma_j = Cov(d_n, d_{n-j}) = E\left[(d_n - \mu)d_{n-j}\right] = 0,\ \forall j \geq 1. \tag{4}$$

In the case of the lag (P) is finite, the most popular MDH test is Box-Pierce [4] Portmanteau Q_P test. This test is designed to test that the first P auto-correlations of a series are zero (i.e. $j = 1, 2, ..., P$). Given an observation sequence of returns $\{d_n, n \geq 1\}$ then the sample auto-covariance can be consistently estimated by

$$\widehat{\gamma}_j = (N-j)^{-1} \sum_{n=1+j}^{N} (d_n - \overline{Y})(d_{n-j} - \overline{Y}), \tag{5}$$

where $\overline{Y}$ is the sample mean. The j-th order auto-correlation now is denoted by $\widehat{\rho}_j = \widehat{\gamma}_j / \widehat{\gamma}_0$. The Q_P statistics is

$$Q_P = N \sum_{j=1}^{P} \widehat{\rho}_j^2. \tag{6}$$

Its adjusted version is modified by Ljung and Box [23] as below

$$LB_P = N(N+2) \sum_{j=1}^{P} (N-j)^{-1} \widehat{\rho}_j^2. \tag{7}$$

This statistics is compared to the Chi-square distribution. Specifically, the rejected area for the tests with the lag P is $LB_P > \chi_P^2$ where χ_k^2 is Chi-square distribution with k degree of freedom. This paper only uses LB_p statistics for all MDH tests.

We now summarize some knowledge about set-valued random variables that are used in this work. Denote by $(\Omega, \mathcal{F}, P)$ the complete probability space, let $(\mathfrak{X}, \|\cdot\|)$ be a separable Banach space with the norm $\|\cdot\|$. A random variable

f on $\mathfrak{X}$ is a measurable function $f : \Omega \longrightarrow \mathfrak{X}$. Denote by $E[f]$ and $E[f|\mathcal{A}]$ the expectation and the conditional expectation of f with respect to $\mathcal{A} \subseteq \mathcal{F}$, respectively . Let $L^1[\Omega; \mathfrak{X}]$ (if $\mathfrak{X} = \mathbb{R}$ then it is written by L^1 for short) be the set of all Bochner integrable random variables taking their values on $\mathfrak{X}$. For $1 \le p < \infty$ denote $L^p[\Omega, \mathcal{F}, P; \mathfrak{X}] = L^p[\Omega; \mathfrak{X}]$ be the Banach space of measurable functions $f : \Omega \longrightarrow \mathfrak{X}$ such that the norm

$$\|f\|_p = \left\{ \int_\Omega \|f(\omega)\|^p \, dP \right\}^{1/p}, \; 1 \le p < \infty,$$

is finite. Denote $L^p[\Omega, \mathcal{F}, P; \mathbb{R}] = L^p$ be the usual Banach space of real-valued functions with finite p-th moments.

Definition 1. Let $\mathcal{M}$ be a set of measurable functions $f : \Omega \to \mathfrak{X}$. $\mathcal{M}$ is called decomposable (with respect to $\mathcal{F}$) if for all $f_1, f_2 \in \mathcal{M}$ and $A \in \mathcal{F}$ then $I_A f_1 + I_{\Omega \setminus A} f_2 \in \mathcal{M}$.

Let $\{f_n, \mathcal{F}_n, n \ge 1\}$ be a sequence of $\mathfrak{X}$-valued $\mathcal{F}_n$-measurable random variables with $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$.

Definition 2. $\{f_n, \mathcal{F}_n, n \ge 1\}$ in $L^1[\Omega; \mathfrak{X}]$ is said to be a martingale (resp. martingale difference) if for all $n \ge 1$, $E[f_{n+1}|\mathcal{F}_n] = f_n$, a.s. (resp. $E[f_{n+1}|\mathcal{F}_n] = 0$, a.s.).

Suppose $\{f_n, \mathcal{F}_n, n \ge 1\}$ is an $\mathfrak{X}$-valued martingale, denote $d_n = f_n - f_{n-1}$, then $\{d_n, \mathcal{F}_n, n \ge 1\}$ is a $\mathfrak{X}$-valued martingale difference.

Now denote by $\mathcal{P}_0(\mathfrak{X})$ the family of all non-empty subsets of $\mathfrak{X}$ and $\mathbf{K}(\mathfrak{X})$ the family of all non-empty closed subsets of $\mathfrak{X}$. The suffixes b, k and c represent bounded, compact and convex respectively. For example, $\mathbf{K}_{kc}(\mathfrak{X})$ denotes the family of all non-empty compact convex subsets of $\mathfrak{X}$. These space are called hyperspace of the Banach space $\mathfrak{X}$. Denote $\|X\|_K = \sup\{\|x\| : x \in X\}$ for all $X \in \mathcal{P}_0(\mathfrak{X})$. Actually, $\|X\|_K$ is the Hausdorff distance between X and $\{0\}$.

Let $F : \Omega \to \mathcal{P}_0(\mathfrak{X})$ be a mapping from space Ω to the family of non-empty subsets of $\mathfrak{X}$ then F is called a set-valued mapping. The set

$$G(F) = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in F(\omega)\}$$

is called the graph of F, and denote

$$F^{-1}(C) = \{\omega \in \Omega : F(\omega) \cap C \ne \phi\},$$

with $C \in \mathcal{P}_0(\mathfrak{X})$.

Definition 3. ([22]) A set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is said to be strongly measurable if, for each closed subset C of $\mathfrak{X}$, $F^{-1}(C) \in \mathcal{F}$. A set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is called weakly measurable if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O) \in \mathcal{F}$. A weakly measurable set-valued mapping is called a set-valued random variable or a random set.

Castaing and Valadier [5] have proven that a strongly measurable set-valued mapping is weakly measurable and therefore a strongly measurable is also a random set.

Definition 4. ([22]) A function taking its values in $\mathfrak{X}$, $f : \Omega \to \mathfrak{X}$ is called a *selection* of a set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. A function f is called an *almost everywhere selection* of F if $f(\omega) \in F(\omega)$ for almost everywhere $\omega \in \Omega$.

Denote by $U[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})]$ the family of all set-valued random variables, simply by $U[\Omega; \mathbf{K}(\mathfrak{X})]$.

For $1 \leq p \leq \infty$, denote by

$$S_F^p = \{f \in L^p[\Omega; \mathfrak{X}] : f(\omega) \in F(\omega)\,, \text{a.s.}\}$$

the set of p-order integrable selections of the set-valued random variable F. For simplicity, the symbol S_F stands for S_F^1. Notice that S_F^p is a closed subset of $L^p[\Omega; \mathfrak{X}]$.

Theorem 5. ([22]) *Let $\mathcal{M}$ be a non-empty closed subset of $L^p[\Omega; \mathfrak{X}]$ and $1 \leq p < \infty$. Then there exists an $F \in U[\Omega; \mathbf{K}(\mathfrak{X})]$ such that $\mathcal{M} = S_F^p$, if and only if $\mathcal{M}$ is decomposable.*

The following definition presents a kind of integral for set-valued mapping, called Aumann integral, which is foundation for expectation and conditional expectation of a set-valued random variable.

Definition 6. ([2]) For a set-valued random variable F, the Aumann integral of F is defined by

$$\int_\Omega F dP = \left\{ \int_\Omega f dP :\ f \in S_F \right\},$$

where $\int_\Omega f dP$ is usual integral Bochner in $L^1[\Omega; \mathfrak{X}]$.

Define $\int_A FdP = \left\{ \int_A fdP : f \in S_F \right\}$ is Aumann integral of the restriction of F on A for $A \in \mathcal{F}$.

However, in order to define the expectation and conditional expectation of set-valued random variables according to the above integration, it is necessary to ensure that the set $\int_\Omega FdP$ is non-empty and closed (it is not closed in general). Hence, the expectation of set-valued random variables is defined as follows.

Definition 7. ([22]) Let $F \in U[\Omega; \mathbf{K}(\mathfrak{X})]$ with $S_F \neq \emptyset$. The expectation of F is given by

$$E[F] = \mathrm{cl} \int_\Omega FdP.$$

Then, the expectation of set-valued random variables possesses properties as the expectation of single-valued ones.

Definition 8 (Definition 1.3.8 in [22])**.** A set-valued random variable F is called integrable if S_F^1 is non-empty. F is called integrably bounded (or strongly integrable) if there exists $\rho \in L^1$ such that $\|x\| \leq \rho(\omega)$ for all $x \in F(\omega)$ and $\omega \in \Omega$.

It is easy to see that the Definition 8 is equivalent to $E[\|F\|_K] < \infty$ (see [24]).

Denote by $L^1[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})] = L^1[\Omega; \mathbf{K}(\mathfrak{X})]$ the family of all integrably bounded set-valued random variables.

Let $\mathcal{A}$ be a sub sigma-field of $\mathcal{F}$. For each $F \in U[\Omega, \mathcal{A}, P; \mathbf{K}(\mathfrak{X})]$ we denote

$$S_F(\mathcal{A}) = \left\{ f \in L^1[\Omega, \mathcal{A}, P; \mathfrak{X}] : f(\omega) \in F(\omega) \,\text{a.s.} \right\}.$$

Then we have the following theorem.

Theorem 9. ([22]) *Let $F \in U[\Omega; \mathbf{K}(\mathfrak{X})]$ with $S_F \neq \emptyset$. Then there exists a unique $\mathcal{A}$-measurable element of $U[\Omega, \mathcal{A}, P; \mathbf{K}(\mathfrak{X})]$, denoted by $E[F|\mathcal{A}]$, such that*

$$S_{E[F|\mathcal{A}]}(\mathcal{A}) = cl \{ E[f|\mathcal{A}] : f \in S_F \}, \tag{8}$$

where the closure are taken in $L^1[\Omega; \mathfrak{X}]$.

The set-valued random variable $E[F|\mathcal{A}]$ satisfying (8) is called the conditional expectation of F with respect to $\mathcal{A}$. Now, the conditional expectation of set-valued random variables have the same properties as the conditional expectation of single-valued random variables.

The following theorem gives a sufficient condition for the closeness of the set $\{E[f|\mathcal{A}] : f \in S_F\}$ in $L^1[\Omega; \mathfrak{X}]$ for a set-valued random variable F.

Theorem 10 (Theorem 2.3.9 in [22]). *If $\mathfrak{X}$ is a reflexive Banach space, $F \in L^1[\Omega, \mathcal{F}, P; \mathbf{K}_c(\mathfrak{X})]$ and $\mathcal{A}_0 = \sigma(\mathcal{U})$ where $\mathcal{U}$ is countable then the set $\{E[f|\mathcal{A}_0] : f \in S_F^1\}$ is closed in $L^1[\Omega, \mathfrak{X}]$.*

Moreover, we have from Theorem 2.1.72 in [24] the following.

Theorem 11. *Let X be a set-valued random variable and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$. If $E[X|\mathcal{A}]$ is a singleton almost surely, so is X.*

Finally, the definitions of set-valued martingale, sub-martingale and super-martingale are also expanded from the single ones.

Definition 12. ([22]) An adapted sequence $\{X_n, \mathcal{F}_n : n \in \mathbb{N}\}$ of integrable random convex closed sets is called a
(1) martingale if for all $n \in \mathbb{N}, X_n = E[X_{n+1}|\mathcal{F}_n]$ *a.s.*.
(2) sub-martingale if for all $n \in \mathbb{N}, X_n \subset E[X_{n+1}|\mathcal{A}_n]$ *a.s.*.
(3) super-martingale if for all $n \in \mathbb{N}, X_n \supset E[X_{n+1}|\mathcal{F}_n]$ *a.s.*.

3. Main results

As mentioned in the previous section, the extension of set-valued martingale difference given by $E[D_{n+1}|\mathcal{F}_n] = \{0\}$ has no sense because it degrades to a singleton. Tuyen [30] has proposed the concept of weak set-valued martingale difference (WSMD for short), where $E[D_{n+1}|\mathcal{F}_n] \ni 0$. We reintroduce this concept along with its martingale difference selections in more detail way.

Definition 13.
(i) An adapted set-valued random variable sequence $\{D_n, \mathcal{F}_n : n \in \mathbb{N}\}$ is said to

be a weak set-valued martingale difference if

$$0 \in S^1_{E[D_n|\mathcal{F}_{n-1}]}(\mathcal{F}_{n-1}) \ a.s., \forall n \geq 2 \text{ (or } 0 \in E[D_n|\mathcal{F}_{n-1}]).$$

(ii) An adapted $\mathfrak{X}$-valued random variable sequence $\{f_n, \mathcal{F}_n : n \in \mathbb{N}\}$ is said to be a martingale difference selection of $\{D_n\}$ if $\{f_n, \mathcal{F}_n : n \in \mathbb{N}\}$ is a martingale difference and $\forall n \geq 1, f_n \in S^1_{D_n}(\mathcal{F}_n)$.
(iii) An adapted $\mathfrak{X}$-valued random variable sequence $\{f_n, \mathcal{F}'_n : n \in \mathbb{N}\}$ is said to be martingale difference selection with natural filter of $\{D_n\}$ if $\{f_n, \mathcal{F}'_n : n \in \mathbb{N}\}$ is a martingale difference and $\forall n \geq 1, f_n \in S^1_{D_n}(\mathcal{F}_n)$. Where, $\mathcal{F}'_n = \sigma(D_n, D_{n-1}, \cdots, D_1)$ is a sub σ-field generated by $D_n, D_{n-1}, \cdots, D_1$.

Example 14. Given $\{Y_n, \mathcal{F}_n, n \geq 1\}$ is a sequence of integrable random variables such that $0 \in Y_n$ $a.s.$ for all $n \geq 1$. Hence, $X_n = Y_1 + Y_2 + \cdots + Y_n$ and $Z_n = \overline{\text{co}}(Y_1 \cup Y_2 \cup \cdots \cup Y_n)$ are weak set-valued martingale differences.

Proof. By $0 \in Y_n$ $a.s.$ for all $n \geq 1$, $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a increasing set-valued random variable sequence almost surely, i.e, $X_n \subset X_{n+1}\, a.s., \forall n \in \mathbb{N}$.
Moreover, $0 \in X_n$ a.s.
We have $X_n \subset X_{n+1}$ a.s. $\Rightarrow E[X_{n+1}|\mathcal{F}_n] \supset E[X_n|\mathcal{F}_n] = X_n$ a.s.
Since $0 \in X_n$ a.s., then $0 \in E[X_{n+1}|\mathcal{F}_n]$ a.s.
Hence $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference.
The proof of Z_n is completely similar. □

Theorem 15. *Let $\{D_n, \mathcal{F}_n : n \geq 1\}$ be a weak set-valued martingale difference in $U[\Omega, \mathbf{K_{kc}}(\mathfrak{X})]$ and $S_n = \sum_{i=1}^n D_i$, then $\{S_n, \mathcal{F}_n, n \geq 1\}$ is a set-valued sub-martingale.*

Proof. We have, for all $n \geq 2$,

$$E[S_n|\mathcal{F}_{n-1}] = E[D_n + S_{n-1}]|\mathcal{F}_{n-1} = E[D_n|\mathcal{F}_{n-1}] + S_{n-1}.$$

Since $0 \in E[D_n|\mathcal{F}_{n-1}]$, $S_{n-1} \subset E[D_n|\mathcal{F}_{n-1}]+S_{n-1}$. Hence, $S_{n-1} \subset E[S_n|\mathcal{F}_{n-1}]$.
□

Theorem 16 (The characteristic of WSMD)**.** *Assume $\mathfrak{X}$ is a reflexive separable Banach space, $\{D_n, \mathcal{F}_n, n \geq 1\}$ is an adapted sequence in $L^1[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})]$. We have the following statements:*
(i) If $\{D_n, \mathcal{F}_n, n \geq 1\}$ has a martingale difference selection then $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference.

(ii) If $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference then there exists a martingale difference selection with natural filter.

Proof.

(i) Suppose that $\{D_n, \mathcal{F}_n, n \geq 1\}$ has a martingale difference selection $\{d_n, \mathcal{F}_n, n \geq 1\}$, i.e. $E[d_{n+1}|\mathcal{F}_n] = 0$ *a.s.* for all $n \geq 1$ and $d_{n+1} \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})$.

From $S^1_{E[D_{n+1}|\mathcal{F}_n]}(\mathcal{F}_n) = \mathrm{cl}\left\{E[f|\mathcal{F}_n] : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})\right\}$ we have that $0 \in S^1_{E[D_{n+1}|\mathcal{F}_n]}(\mathcal{F}_n)$ *a.s.*. Hence $\{D_n, \mathcal{F}_n, n \geq 1\}$ is a weak set-valued martingale difference.

(ii) We have $D_{n+1} \in L^1[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})] \Rightarrow \exists \rho_{n+1} \in L^1$ such that $\forall f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})$, $\forall \omega \in \Omega$: $\|f(\omega)\| \leq \rho_{n+1}(\omega)$.

Assume that $0 \in \mathrm{cl}\left\{E[f|\mathcal{F}_n] : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})\right\}$,there exists a sequence $f^i_{n+1} \in S^1_{D_{n+1}}(\mathcal{F}_{n+1})$ such that $\lim_{i\to\infty} E[f^i_{n+1}|\mathcal{F}_n] = 0$ (by the norm of $L^1[\Omega; \mathfrak{X}]$).

For all $\omega \in \Omega$, set

$$r_{n+1}(\omega) = \sup_{i\geq 1} \left\|E[f^i_{n+1}|\mathcal{F}_n](\omega)\right\|, \tag{9}$$

$$T_{n+1}(\omega) = \left\{f(\omega) : f \in S^1_{D_{n+1}}(\mathcal{F}_{n+1}) \text{ và } \|E[f|\mathcal{F}_n](\omega)\| \leq r_{n+1}(\omega)\right\}; \tag{10}$$

$$R_{n+1} = \overline{\mathrm{co}}T_{n+1}(\omega).$$

From (9) and (10) we have $f^i_{n+1} \in S^1_{R_{n+1}}(\mathcal{F}_{n+1})$, hence $S^1_{R_{n+1}}(\mathcal{F}_{n+1}) \neq \emptyset$. Obviously that R_{n+1} is $\mathcal{F}_{n+1}$-measurable. We should prove $R_{n+1} \in L^1[\Omega, \mathcal{F}, P; \mathbf{K}_c(\mathfrak{X})]$. It is necessary to show that R_{n+1} is integrably bounded.

From the convexity of $\mathrm{co}T_{n+1}$ we have that for each $f \in S^1_{\mathrm{co}T_{n+1}}(\mathcal{F}_{n+1})$ there exists a sequence $\{f_k \in S^1_{D_{n+1}}(\mathcal{F}_{n+1}), k = 1, \cdots, m\}$ and $\{\alpha_k(\omega) \geq 0, \sum_{k=1}^m \alpha_k(\omega) = 1\}$ such that $f(\omega) = \sum_{k=1}^m \alpha_k(\omega) f_k(\omega)$.

Hence $\|f(\omega)\| \leq \sum_{k=1}^m \alpha_k(\omega)\|f_k(\omega)\| \leq \rho_{n+1}(\omega)$.

For each $d \in S^1_{R_{n+1}}(\mathcal{F}_{n+1})$ there exists $f \in S^1_{\mathrm{co}T_{n+1}}(\mathcal{F}_{n+1})$ such that $\|d(\omega) - f(\omega)\| \leq 1$.

The consequence is

$$\|d(\omega)\| \leq \|d(\omega) - f(\omega)\| + \|f(\omega)\| \leq 1 + \rho_{n+1}(\omega).$$

From the fact that

$$\|R_{n+1}(\omega)\|_K = \sup\{\|d(\omega)\| : d \in S^1_{R_{n+1}}(\mathcal{F}_{n+1})\} \leq 1 + \rho_{n+1}(\omega)$$

$$\Rightarrow \int_\Omega \|R_{n+1}(\omega)\|_K dP \le \int_\Omega (1+\rho_{n+1}(\omega))dP$$
$$= 1 + \int_\Omega \rho_{n+1}(\omega)dP = 1 + \|\rho_{n+1}\|_{L^1} < \infty.$$

Hence R_{n+1} is integrably bounded.

Let $\mathcal{F}'_n = \sigma(D_n, D_{n-1}, \cdots, D_1)$ denote the σ-algebra generated by D_n, D_{n-1}, $\cdots$, D_1 then $\mathcal{F}'_n \subset \mathcal{F}_n$.

Let $B_n = \left\{E[f|\mathcal{F}'_n] : f \in S^1_{R_{n+1}}(\mathcal{F}_{n+1})\right\}$. Since $\mathfrak{X}$ is separable, reflexive and $\mathcal{F}'_n$ is countable generated, from Theorem 10 we have B_n is closed in $L^1[\Omega; \mathfrak{X}]$.

From $\lim_{i\to\infty} E[f^i_{n+1}|\mathcal{F}_n] = 0$ in $L^1[\Omega; \mathfrak{X}]$ we have that for all $\epsilon > 0$, there exists a positive number $N \in \mathbb{N}$ such that $\forall i \ge N$, $E[\|E[f^i_{n+1}|\mathcal{F}_n]\|] \le \epsilon$.

Moreover, because of $\mathcal{F}'_n \subset \mathcal{F}_n$ we have

$$E[\|E[f^i_{n+1}|\mathcal{F}'_n]\|] = E[\|E[E[f^i_{n+1}|\mathcal{F}_n]|\mathcal{F}'_n]\|] \le E[\|E[f^i_{n+1}|\mathcal{F}_n]\|] < \epsilon.$$

Hence $\lim_{i\to\infty} E[f^i_{n+1}|\mathcal{F}'_n] = 0$. Furthermore, from $E[f^i_{n+1}|\mathcal{F}'_n] \in B_n$ and B_n is closed we have $0 \in B_n$. So there exists $d_{n+1} \in S^1_{R_{n+1}}(\mathcal{F}_{n+1})$ such that $E[d_{n+1}|\mathcal{F}'_n] = 0$. Let $d_1 \in S^1_{D_1}$ then $\{d_n, \mathcal{F}'_n, n \ge 1\}$ is a martingale difference selection with natural filter of $\{D_n, \mathcal{F}_n, n \ge 1\}$. □

Remark 17.
(1) For the case of $\{\mathcal{F}_n, n \ge 1\}$ is a natural filter of $\{D_n, n \ge 1\}$ then the conditions (i) and (ii) in Theorem 16 are equivalent.
(2) If $\mathfrak{X}$ is a separable metric space with countable base ($\mathbb{R}^d$ for instance) then its Borel σ-algebra is generated by countable open balls. It implies that (i) and (ii) are equivalent.

Theorem 18. *Let $\{D_n, \mathcal{F}_n : n \ge 1\}$ be a weak set-valued martingale difference in $L^1[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})]$ and $\{\mathcal{M}_n, n \ge 1\}$ be the family of all martingale difference selections of $\{D_n\}$. Then $\mathcal{M}_n$ is closed in $L^1[\Omega; \mathfrak{X}]$.*

Proof. Let $\{d^i_n, i \ge 1\}$ be a sequence in $\mathcal{M}_n$ such that $\lim_{i\to\infty} d^i_n = d^*_n$ by the norm of $L^1[\Omega; \mathfrak{X}]$, i.e. for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\int_\Omega \|d^i_n - d^*_n\|dP < \epsilon$ for all $i > N$. Hence

$$\int_{\mathcal{F}_{n-1}} \|d^i_n - d^*_n\|dP < \epsilon, i > N. \tag{11}$$

We have

$$\begin{aligned}\left\|\int_{\mathcal{F}_{n-1}} d_n^* dP\right\| &\le \left\|\int_{\mathcal{F}_{n-1}} (d_n^* - d_n^i) dP\right\| + \left\|\int_{\mathcal{F}_{n-1}} d_n^i dP\right\| \\ &\le \int\limits_{\mathcal{F}_{n-1}} \|d_n^i - d_n^*\| dP + \left\|\int_{\mathcal{F}_{n-1}} d_n^i dP\right\|.\end{aligned} \tag{12}$$

Since $d_n^i \in \mathcal{M}$, $E[d_n^i|\mathcal{F}_{n-1}] = 0$. By using (11) and (12) we have $\left\|\int_{\mathcal{F}_{n-1}} d_n^* dP\right\| < \epsilon$.

Hence $E[d_n^*|\mathcal{F}_{n-1}] = 0$, implies $d_n^* \in \mathcal{M}$. □

Remark 19.
Let $\{\mathcal{M}_n, n \ge 1\}$ be the family of all martingale difference selections of $\{D_n, n \ge 1\}$ then $\mathcal{M}_n$ is non-empty. For any $A \in \mathcal{F}_{n-1}$ and $d_n^1, d_n^2 \in \mathcal{M}_n$ we have

$$E[I_A d_n^1 + I_{\Omega\setminus A} d_n^2 | \mathcal{F}_{n-1}] = I_A E[d_n^1|\mathcal{F}_{n-1}] + I_{\Omega\setminus A} E[d_n^2|\mathcal{F}_{n-1}] = 0.$$

Hence $I_A d_n^1 + I_{\Omega\setminus A} d_n^2 \in \mathcal{M}_n$. This means that $\mathcal{M}_n$ is decomposable with respect to $\mathcal{F}_{n-1}$. By Theorem 18 and Theorem 5, there exists $\mathcal{D}_n \in L^1[\Omega, \mathcal{F}, P; \mathbf{K}(\mathfrak{X})]$ such that $S^1_{\mathcal{D}_n}(\mathcal{F}_n) = \mathcal{M}_n$. Hence, $E[\mathcal{D}_n|\mathcal{F}_{n-1}] = \{0\}$. By Theorem 11, we have $\{\mathcal{D}_n\} = \{\xi_n\}$ where ξ_n is a singleton. Therefore, the martingale difference selection in Theorem 16 can be degraded from a set of martingale difference selections. For example, let $\{X_n, n \ge 1\}$ be a i.i.d set-valued random variable sequence then it is easy to see that $\{D_n = X_n - E[X_n], n \ge 1\}$ is a weak set-valued martingale difference with natural filter where $A - B = \{a - b : a \in A, b \in B\}$ (the closeness of D_n is guaranteed by M. Thera [1]). Now, any selection of D_n with the form $f_n - E[f_n]$ for all $f_n \in S_{X_n}(\mathcal{F}_n)$ belongs to $\mathcal{M}_n$. This means that $\mathcal{M}_n$ does not have only one martingale difference selection but also numerous martingale difference selections. Although they theoretically degrade to a singleton, in practice we still have to work with the set of all martingale difference selections of $\{D_n; n \in \mathbb{N}\}$.

4. An application of weak set-valued martingale difference

4.1. Presenting the trends of stock prices by a sequence of set-valued random variables

Given $\{d_n, n \geq 1\}$ is a sequence of return observations of a financial asset. The following transformation allows converting $\{d_n, n \geq 1\}$ to a sequence of set-valued random variables $\{D_n, n \geq 1\}$:

- From the histogram of $\{d_n\}$, its universe of discourse, i.e. U, is determined.

- Divide U into k disjoint real intervals $U = E_1 \cup E_2 \cup \cdots \cup E_k$ in which $E_i = (u_{i-1}, u_i)$ for $i = 2, ..., k-1$ and $E_1 = (\min U - \delta; u_1)$ and $E_k = (u_{k-1}; \max U + \delta)$ for some constant $\delta > 0$.

- For each $n \in \mathbb{N}$, define a set-valued random variable D_n related to d_n as follows: if $d_n \in (u_{i-1}, u_i]$, $i \in \{1, 2, ..., k\}$ then $D_n(\omega) = \overline{E_i}$. So we have a sequence of set-valued random variables $\{D_n, n \geq 1\}$ on $\mathbf{K}(\mathbb{R})$.

We can see that D_n represents the trend of the returns because E_i represents its level of volatility.

For example, consider two daily exchange rate returns of Euro (EUR) and Pound (GBP) against the US dollar, and two daily stock index returns of S&P500 and VNI (Vietnam stock index), their histograms are given in Figure 1.

We can see that their values are concentrated from -0.01 to 0.01. Hence, the set of their trends $\{E_i, i = 1, 2, ..., 7\}$ can be defined as follows:

- E_1="very very low"=$(\min\limits_{n\geq 1} d_n - 0.5, -0.01)$
- E_5="high"=$(0.002, 0.006)$
- E_2="very low"=$(-0.01, -0.006)$
- E_6="very high"=$(0.006, 0.01)$
- E_3="low"=$(-0.006, -0.002)$
- E_7="very very high"=$(0.01, \max\limits_{n\geq 1} d_n + 0.5)$
- E_4="normal"=$(-0.002, 0.002)$

Now for each $d_n, n \geq 1$, we have D_n as a trend observation of the time series taking its value in $\{\overline{E_i}, i = 1, .., 7\}$ when $d_n \in E_i$. In this example, we set $\delta = 0.5$ which is large enough for containing future returns and $k = 7$ for the relatively

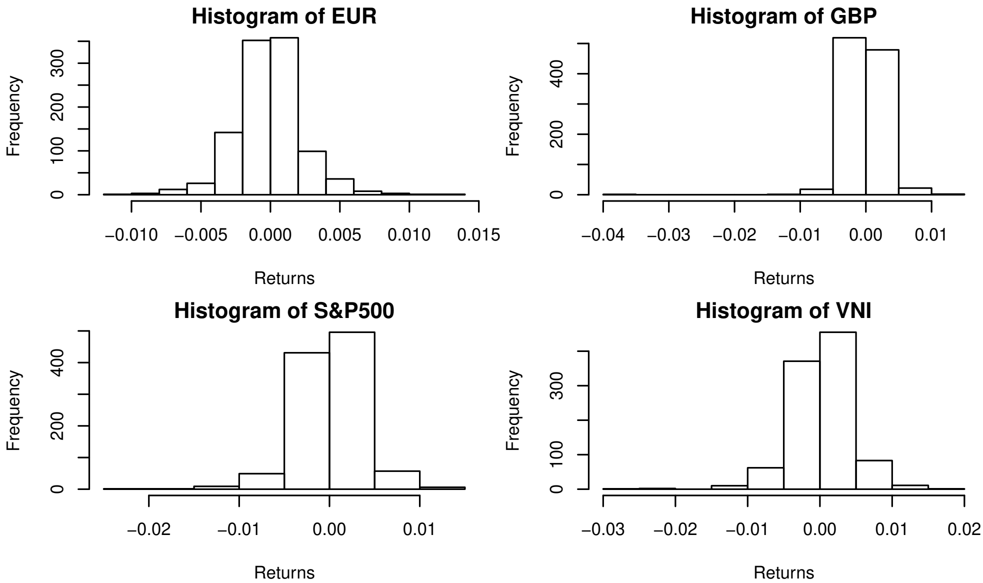

Figure 1: The histograms of daily return of some exchange rates and stock indices

complete possible states of d_n. These parameters can be changed according to the wishes of the users. If not misunderstanding the symbol between random variable and its observation, we have a sequence of set-valued random variables $\{D_n, n \geq 1\}$ which represents the trends $E_i, i = 1, ..., 7$ of $\{d_n, n \geq 1\}$.

4.2. Testing for weak set-valued martingale difference hypothesis

It is well-known that testing the martingale difference hypothesis (MDH) is to check whether the sequence of returns of a financial time series is a martingale difference. This test allows users to evaluate its predictability. For the predictability of its trend, one can think of testing whether the sequence of set-valued random variables $\{D_n, n \geq 1\}$ defined as in Section 4.1 is a weak set-valued martingale difference. We say that it is testing for the weak set-valued martingale difference hypothesis (WSMDH) for this task. Because of Theorem 16, one can think that if a sequence of random variables supports the MDH then it also supports WSMDH. Unfortunately, recall that the MDH tests mentioned above are all only the necessary condition for MDS. Hence, testing for WSMDH can achieve different results in comparison with testing for MDH.

Furthermore, from Remark 17 for $\mathfrak{X} = \mathbb{R}$ and by Theorem 16 we have that a sequence of random variables $\{D_n, n \geq 1\}$ is a WSMD if only if it has a MDS selection. But by Remark 19, the MDS selection of $\{D_n, n \geq 1\}$ can be merged from a set of MDS selections. Therefore, in statistical practice for testing the WSMDH we need to implement the MDH tests on a set of selections of $\{D_n, n \geq 1\}$. This approach also increases the confidence of being a martingale difference for each selection because the MDH tests are only the necessary condition for MDS as talking above.

The aim of this paper is to show that there are some financial or economic indicators which support the MDH (i.e. difficult to predict its values in short-term [16]) but not the WSMDH and vice versa. From this fact, one can say that the predictability of the trend of a time series and its price values is inconsistent. The consequence is that the market can be beaten by some novel strategies when a good enough trend-based forecasting model is applied.

Let $\{D_n, n \geq 1\}$ be a sequence of set-valued random variables that constructed as in Section 4.1, consider two following conflicting hypotheses:

$$(I): \text{The set of MDS selections of } \{D_n, n \geq 1\} \text{ is non empty.}$$
$$(II): \text{The set of MDS selections of } \{D_n, n \geq 1\} \text{ is empty.}$$

To make a piece of statistical evidence for these two hypotheses, it is reasonable to implement the MDH tests as described in Section 2 for a set of B random selections $\{d_n^i, i = 1, \cdots, B\}$ of $\{D_n, n \geq 1\}$ with B is large enough. This fact leads to the multiple testing problem in which the rejection rate (rejections) and the false discovery rate (FDR) are concerned, [3]. The hypothesis (I) is supported if the rejection rate for MDH is low and the FDR is high (then we can say that it support WSMDH). Conversely, (II) is supported if the rejection rate high and the FDR is low (then we can say that the WSMDH is rejected for some level of confident).

The multiple test problem has been developed and widely applied in many research fields, especially in gene analysis and in medicine [35, 33]. The challenge in the multiple testing problem is unable to rely on the p values $\{p_k, k = 1, \cdots, B\}$ to make the conclusion for the test. This is because of the following reason. For the significance of 0.05, when B is larger, the probability of existing a false rejection is bigger, even with an approximation of 1. For example, at 0.05 probability of false rejection for each selection then with the number of 100 rejections of $H0$, the probability of existing at lest one false rejection is $1 - 0.95^{100} \approx 0.994$. To solve this problem, these p-values must be adjusted depending on B (the simplest way is to compare p-valued with $0.05/B$ instead of 0.05). However, based on the possible distribution of these p-values, various sophisticated adjustments have been developed [34, 19] in order to optimize the rejection rate of $H0$ and the FDR in these rejections. These methods have been synthesized and packaged into many packages in R software from various authors [7, 17, 8, 28]. To give a decision for the above hypotheses (I) an (II), this paper applies the R-package **sgof** [7] for multiple MDH tests on the set of B random selections of $\{D_n, n \geq 1\}$ due to its up-to-date properties. Particularly, the multiple test based on Bayesian inference (i.e. the function **Bayesian.SGoF** in **sgof**-package) is used because it has been showed that there are many advantages in comparison with other multiple testing methods.

The set of p-values which is the input for this multiple test is obtained as follows:

- For each $n \in \mathbb{N}$, the observation of D_n is an real-interval E_k, $k = 1, \cdots, 7$.
- By taking a value $x_n \in E_k$ uniformly randomly, we have x_n is an observation of a selection of D_n.
- By applying a MDH test for $\{x_n, n \geq 1\}$, we have a p-value for this test.
- By implementing three above steps B times, we have a set of p-values using for the multiple test.

4.3. Simulation

This section implements the WSMDH test (by running the multiple MDH test for a set of random selections as described in Section 4.2) on simulation data for both martingale difference sequence and non-martingale difference sequence. The results of this test play a role as a measure of the hypothesis (I) and (II).

For a simulated martingale difference sequence, 500 observations representing the logarithm of assumed asset prices are generated by Brownian motion, which is given by the following formula

$$f_0 = 500;\ \ln f_n = \ln f_{n-1} + N(0, 0.01),\ \forall n \geq 1,$$

where $N(0, 0.01)$ denotes the normal distribution with a mean of zero and standard deviation of 0.01. Then, its returns $\{d_n = \ln f_n - \ln f_{n-1}, n \geq 1\}$ is a martingale difference. These sequences are plotted in Figure 2 where f_n (prices) are asset prices and d_n (returns) are its returns.

After fuzzifying the sequence $\{d_n, n \geq 1\}$ into the sequence of set-valued random variables $\{D_n, n \geq 1\}$ as exactly illustrated in Section 4.1 with $k = 7$, we apply the Bayesian multiple test function in R package **sgof** for the set of p-values generated from testing the MDH for the set of random selections

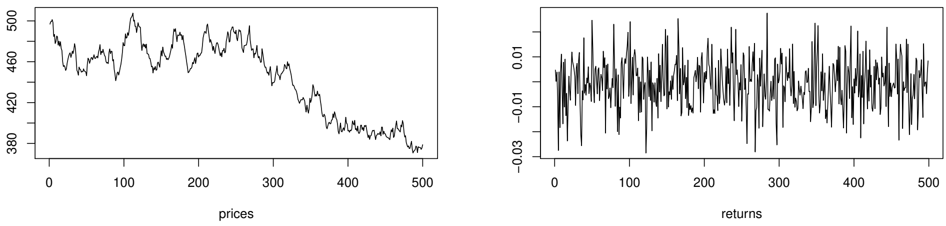

Figure 2: Simulation of a MDS

$\{d_n^i, n \geq 1, i = 1, 2, \cdots, B\}$ of $\{D_n, n \geq 1\}$. With the different sample sizes B of the selections, the results that include the rate of rejections (rejections) and the false discovery rate (FDR) are given in Table 1.

Table 1: MDH multiple testing for B random selections of a simulated WMDS

	Rejections(FDR)			
B	LB_5	LB_{15}	LB_{25}	LB_{50}
B=5	0(0)	0(0)	0(0)	0(0)
B=10	0(0)	0(0)	0(0)	0(0)
B=50	0(0)	0(0)	0(0)	0(0)
B=100	0(0)	0(0)	0(0)	0(0)
B=500	0.034 (0.18)	0(0)	0(0)	0(0)

These results have showed that none of the selections rejects the MDH at almost values of B, meaning that the hypothesis (I) is strongly supported. Hence, the proposed approach for testing the WSMDH is truly reflection of a weak set-valued martingale difference.

In order to check the proposed method of testing the WSMDH for the trend-predictability of an uptrend time series, we generate a time series with 500 observations from below formula

$$f_0 = 500, \ \ln f_n = \ln f_{n-1} + N(U(0, 0.005, 0.01), 0.01), \ \forall n \geq 1,$$

where $N(U(0, 0.005, 0.01), 0.01)$ is the normal distribution with uniformly random mean from $\{0, 0.005, 0.01\}$ and standard deviation of 0.01. This time series and its returns are represented in Figure 3.

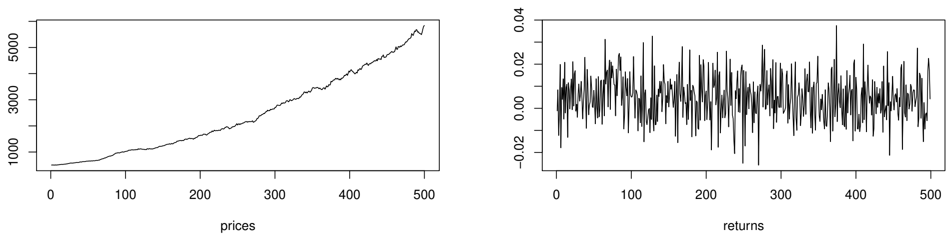

Figure 3: Simulation of a uptrend (predictable) time series

By applying the proposed method as in the MDS case, the results of the multiple test are given in Table 2.

Table 2: MDH multiple testing for B random selections of an uptrend simulated time series

	Rejections(FDR)			
B	LB_5	LB_{15}	LB_{25}	LB_{50}
B=5	0(0)	0.6(0)	0.6(0)	0.6(0)
B=10	0(0)	0.8(0)	0.8(0)	0.8(0)
B=50	0(0)	0.9(0)	0.9(0)	0.9(0)
B=100	0 (0)	0.91 (0)	0.91 (0)	0.93 (0)
B=500	0 (0)	0.928 (0)	0.946 (0)	0.946 (0)

Table 2 now provides conflicting results in comparison with Table 1 in the lags from 15 to 50 of LB_P. Consistency of this result with different sample sizes indicates what are the best lags of LB_p for time series trend detection. Furthermore, the rate of MDH rejection at only 60% with small sample size is enough for this method of testing.

From two experiments for simulated data, it can be confirmed that the method of WSMDH test proposed by this paper is completely suitable for testing the predictability of the trend of a time series.

4.4. Application for actual data

The actual data used for testing the MDH and WSMDH in this paper consists of 5 daily exchange rate returns on the Canadian Dollar (CAN), the Pound GBP(£), the Euro (EUR), the Japanese Yen YEN(¥) and the Vietnamese Dong

(VND) against the US dollar and 5 daily stock index returns included VN-Index VNI (Vietnam Stock Index), S&P500, DJIA (Dow Jones Industrial Average), FTSE (Financial Times Stock Exchange 100 Index) and HSI (Hong Kong Hang Seng Index). The daily data is taken from 01/01/2014 to 31/12/2017 with more than 1000 observations. All of them are available at https://vn.investing.com/ .

Now we apply the proposed method for testing the WSMDH along with testing the MDH in order to compare the predictability of the asset prices and the predictability of its trends. For the WSMDH test, we implement the multiple test for 500 random selections for each time series. The data is called reject the WSMDH if it has the rejection rate of the selections at least 60%.

Table 3: The price-predictability of returns via MDH test

	Statistics (p-value)			
	LB_5	LB_{15}	LB_{25}	LB_{50}
EUR	3.9566 (0.5557)	17.392 (0.296)	26.446 (0.3841)	55.575 (0.2729)
GBP(£)	8.6122 (0.1256)	14.581 (0.482)	22.304 (0.6182)	43.679 (0.7236)
CAN	6.3193 (0.2764)	17.067 (0.3149)	23.76 (0.5333)	48.815 (0.521)
YEN(¥)	2.3764 (0.795)	11.781 (0.6955)	29.272 (0.2527)	57.225 (0.2247)
VND	40.744 (1.057E-7)	64.394 (4.36E-8)	75.949 (4.854E-7)	112.34 (1.08E-6)
S&P500	6.0638 (0.3001)	8.7196 (0.8917)	22.724 (0.5937)	51.757 (0.4051)
DJIA	5.7192 (0.3345)	15.635 (0.4067)	32.37 (0.1476)	57.51 (0.217)
FTSE	22.439 (0.0004)	32.726 (0.0051)	54.493 (0.00057)	90.213 (0.00042)
HSI	4.0438 (0.5431)	33.226 (0.0043)	50.162 (0.002)	75.195 (0.0121)
VNI	11.471 (0.0428)	24.982 (0.0501)	34.159 (0.1045)	65.083 (0.0743)

Table 3 reports the LB_p test for the lag $p = 5, 15, 25, 50$. We can see that VND, FTSE, HSI reject the MDH at $p = 15, 25, 50$ while others support the

Table 4: The trend-predictability of the returns via WSMDH test

$B = 500$	Rejections (FDR)				
Stock	S&P500	DJIA	FTSE	HSI	VNI
LB_5	0.652 (0003)	0.132 (0.06)	0.938 (0)	0(0)	0.196 (0.03)
LB_{15}	0.068 (0.11)	0.148 (0.04)	0.946 (0)	0.94 (0)	0.232 (0.02)
LB_{25}	0.082 (0.09)	0.15 (0.05)	0.944 (0)	0.946 (0)	0.052 (0.12)
LB_{50}	0 (0)	0 (0)	0.944 (0)	0.932 (0)	0.07 (0.09)
Exchange	EUR	GBP	CAN	YEN	VND
LB_5	0(0)	0.466 (0)	0.33 (0.01)	0(0)	0.158 (0.1)
LB_{15}	0 (0)	0.056 (0.1)	0.144 (0.06)	0 (0)	0.1 (0.13)
LB_{25}	0 (0)	0.132 (0.07)	0.208 (0.035)	0.098 (0.09)	0.042 (0.18)
LB_{50}	0 (0)	0.076 (0.133)	0.37 (0.0135)	0.152 (0.05)	0.272 (0.0348)

MDH.

In WSMDH test given in Table 4, we can see FTSE and HSI also reject WMDH while the VND and VNI have turned from rejecting the MDH to supporting the WSMDH at the lag 15, 25 and 50. Interestingly, at the lag 5 then S&P500 has turned from support the MDH to reject the WMSDH. These results, in some confident level of statistics, one can believe that some assets such as VNI, VND or S&P500 might difficult to forecast its prices but not its trends and vice versa. Other assets, the consistence between MDH and WSMDH results reinforces the confidence of their unpredictability in both the prices and its trends. It means that the proposed method of testing the WMSDH provides more evidences for an efficient market.

The comparison between MDH and WMDH test is summarized in Table 5. From the table, we are easier to see the above comments and put more our faith in efficient market on the EUR, GBP, CAN, YEN and DJIA. Otherwise, the

Table 5: Comparison between testing of MDH and WSMDH

	MDH-WSMDH				
Stock	S&P500	DJIA	FTSE	HSI	VNI
LB_5	✓-✗	✓-✓	✗-✗	✓-✓	✗-✓
LB_{15}	✓-✓	✓-✓	✗-✗	✗-✗	✓-✓
LB_{25}	✓-✓	✓-✓	✗-✗	✗-✗	✓-✓
LB_{50}	✓-✓	✓-✓	✗-✗	✗-✗	✓-✓
Exchange	EUR	GBP	CAN	YEN	VND
LB_5	✓-✓	✓-✓	✓-✓	✓-✓	✗-✓
LB_{15}	✓-✓	✓-✓	✓-✓	✓-✓	✗-✓
LB_{25}	✓-✓	✓-✓	✓-✓	✓-✓	✗-✓
LB_{50}	✓-✓	✓-✓	✓-✓	✓-✓	✗-✓

✓: support, ✗: reject, left: MDH - right: WSMDH

inconsistency between testing the MDH and WMDH might be an explanation for its predictability in some trend-based forecasting models [27, 9, 18, 26].

5. Conclusion

From the following facts that:

- Lack of the definition of set-valued martingale difference,
- The MDH tests are only the necessary condition for the martingale difference holds,
- The state-of-the-art of the trend-based time series forecasting models while the stock markets are noticed to be efficient,

this paper reintroduces the definition of weak set-valued martingale difference in [30] and proves its characteristic properties in the general condition. From its meaning in trend-based forecasting models, we propose a method of testing the weak set-valued martingale difference where testing the trend-predictability of a time series is performed. By applying the method on actual data we have showed that the predictability of trend and the price of some financial and economic indicators is inconsistent. Such this inconsistency recommends that the proposed method of testing the WSMDH should be used for testing the efficient market hypothesis as well as for testing its trend-predictability before performing a trend-based forecasting model.

For some future works, we intend to study on set-valued martingale difference for the generalized Hukuhara difference, vector-valued martingale difference and find out more its real life applications.

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