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ON THE DIRECT PRODUCT OF NEARLY S-PERMUTABLE SUBGROUPS

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Abstract: A subgroup H is said to be S-permutable in G if it permutes with all Sylow subgroups of G. A subgroup H of G is called nearly S-permutable in G if for every prime p such that $\gcd(p, |H|) = 1$ and for every subgroup K of G containing H, the normalizer $N_K(H)$ contains some Sylow p-subgroup of K. The main aims of this article is to classify the family of all nearly S-permutable subgroups for certain groups and study the direct product of their subgroups. Moreover, we prove that the direct product of certain nearly S-permutable subgroups is necessary nearly S-permutable.

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Key Words: Sylow subgroups; normal subgroup; subnormal subgroup; S-permutable subgroup; nearly S-permutable subgroup; direct product of subgroups

1. Introduction

One of the earliest results about permutable subgroups found in [7] by Isaacs,

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stated that every permutable subgroup is subnormal. The term S-permutable subgroup introduced by Ore in [12] as a subgroup H is said to be S-permutable in G if it permutes with all Sylow subgroups of G. In [9], Kegel proved that S-permutable subgroups are necessarily subnormal. In [2], Al-Sharo introduced the notion nearly S-permutable subgroup, as a subgroup H of G is called nearly S-permutable in G if for every prime p such that $\gcd(p, |H|) = 1$ and for every subgroup K of G containing H, the normalizer $N_K(H)$ contains some Sylow p-subgroup of K. Then he showed that the nearly S-permutable subgroup need not be subnormal in general. Next, Ikram in [1] studied the lattices of nearly S-permutable subgroups, in particular an example was constructed to show that the set of all nearly S-permutable subgroups of a group need not distributive nor modular lattices.

In this article, we study the direct product of nearly S-permutable subgroups and discuss some algebraic properties of this product.

2. Preliminaries

Our notions are fairly standard, all groups in this research are finite. The next definitions and results elaborate the terms: permutable, S-permutable and nearly S-permutable subgroups. These terms will be studied in the next sections.

Definition 1. ([11]) Let G be a group, and let H and K be two subgroups of G. We say that H permutes with K if $HK = KH$.

Definition 2. ([11]) Let G be a group and let H be a subgroup of G. Then H is said to be permutable if it satisfies the following equivalence conditions:

1. It permutes (commutes) with every subgroup of G.
2. Its product with every subgroup of G is a subgroup.
3. It permutes with every cyclic subgroup.

Definition 3. ([5]) A subgroup H of a group G is called p-subgroup of G, if every element in H has order a power of p.

Theorem 4. ([5]) *Let G be a finite group and p be a prime that divides $|G|$. Then G has an element of order p and, consequently, a subgroup of order p.*

Corollary 5. ([5]) *Let H be a finite subgroup. Then H is a p-subgroup if and only if $|H|$ is a power of p.*

Remark 6.

- The set of all Sylow p-subgroups of a group G is denoted by $Syl_p(G)$.
- Every Sylow p-subgroup is p-subgroup, but the converse not necessary true.
- Let p be a prime. If $p \nmid |G|$, then the only Sylow p-subgroup of G is the trivial subgroup $\{e\}$.
- If $|G| = p^n,\ n \in \mathbb{N}$, then G itself is the only Sylow p-subgroup of G.

Definition 7. ([9]) Let H be a subgroup of a group G. Then H is Sylow permutable (S-permutable) if it permutes with all Sylow p-subgroups of G for all primes p.

Remark 8. Every subgroup of an abelian group is S-permutable.

Proof. Let H be a subgroup of an abelian group G and K be a Sylow subgroup of G. Then:

$$\begin{aligned} HK &= \{hk \mid h \in H\ ,\ k \in K\} \\ &= \{kh \mid h \in H\ ,\ k \in K\} \quad (G \text{ is abelian}) \\ &= KH \end{aligned}$$

Hence, H is S-permutable. □

Remark 9. Every normal subgroup of a group is permutable.

For more about permutable subgroups, one may refer to [3].

Remark 10. For a group G there are at least two S-permutable subgroups, the trivial subgroup and the group itself.

Proof. Consider a group G, and let $\{e\}$ be the trivial subgroup. Then $\{e\}H = H\{e\} = H$ and $GH = GH = G$ for any Sylow p-subgroup H of G. □

Remark 11. S-permutability does not imply permutability.

Example 12. For any p-group G and a non permutable subgroup H, the only Sylow p-subgroup of G is G itself, which is permute with every subgroup. Let $G = D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ and $H = \{e, rs\}$. If $K = \{e, s\}$, $K \leq G$, then $HK = \{e, s, rs, r\}$ and $KH = \{e, s, rs, r^3\}$. Therefore $(HK \neq KH)$, so H is not permutable. But H is S-permutable since $HG = GH = G$ and G is the only Sylow p-subgroup of G.

Remark 13. ([9]) Every S-permutable subgroup is subnormal.

Corollary 14. *Every permutable subgroup is subnormal.*

Proof. The proof is straight forward from Remark 13. □

Definition 15. ([2]) Let H be a subgroup of a group G. We say that H is nearly S-permutable in G if for every prime p with $\gcd(p, |H|) = 1$ and for every subgroup K of G containing H, the normalizer $N_K(H)$ contains some Sylow p-subgroup of K.

Remark 16. Nearly S-permutability does not implies S-permutability.

Example 17. For the symmetric group S_4, consider the subgroup H as

$$H = \{(\ ), (1, 2), (1, 4), (2, 4), (1, 4, 2), (1, 2, 4)\}.$$

Certainly, H is a nearly S-permutable subgroup but not permutable.

Theorem 18. ([1]) *Every subgroup of nilpotent group is nearly S-permutable.*

Lemma 19. ([2]) *Let H be a S-permutable subgroup of G. Then H is S-permutable subgroup of K, wherever $H \leq K \leq G$.*

Lemma 20. ([10]) *Let H be a nearly S-permutable subgroup of G. Then H is nearly S-permutable subgroups of K, wherever $H \leq K \leq G$.*

Now, we state the following result.

Theorem 21. *Every S-permutable subgroup of a group G is nearly S-permutable.*

Proof. Let H be a S-permutable subgroup of G. Suppose on the contrary that H is not nearly S-permutable subgroup in G. So, there exists a prime p such that $\gcd(p, |H|) = 1$ and there exists a subgroup K of G such that $H \leq K \leq G$, but $P \nleq N_K(H)$, for all $P \in Syl_p(K)$. Using Lemma 19, H is S-permutable subgroup of K, that is $HP = PH$, for all $P \in Syl_p(K)$. So, P is a subgroup of $N_K(H)$, which is a contradiction. Hence, H is S-permutable subgroup in G. □

Remark 22. ([10]) Let us denote by $\pi(n)$ the set of all prime divisors of n. If H is subgroup of G such that $\pi(|H|) = \pi(|G|)$, then H is nearly S-permutable in G.

Remark 23.

- Permutability implies S-permutability, and S-permutability implies nearly S-permutability.
- The trivial subgroup and the improper subgroup are nearly S-permutable.

Remark 24. Every subgroup of an abelian group is nearly S-permutable.

Proof. Let H be a subgroup of an abelian group G. Then H is normal, and so H is permutable, implies that H is nearly S-permutable. □

Remark 25. Let G be a p-group. Then all subgroups of G are nearly S-permutable.

Remark 26. The intersection of two nearly S-permutable subgroups not necessary nearly S-permutable, as the next example shows.

Example 27. Consider the symmetric group on 4 letters $G = S_4$, and the nearly S-permutable subgroups: $H_1 = \langle(1,4),(1,4,3)\rangle$ and $H_2 = \langle(3,4),(2,4,3)\rangle$. Note that $\pi(G) = \pi(H_1) = \pi(H_2)$, which implies that H_1 an H_2 are nearly S-permutable subgroups of G. But, $H_1 \cap H_2 = \langle(3,4)\rangle$, which is not nearly S-permutable.

We denote the set of all S-permutable (nearly S-permutable) subgroups of a group G by $SP(G)$ ($NSP(G)$). We denote the number of S-permutable (nearly S-permutable) subgroups of a group G by $\#SP(G)$ ($\#NSP(G)$).

Example 28. The following table shows some groups and their $\#SP(G)$ and $\#NSP(G)$:

Table 1: Number of S-permutable and nearly S-permutable subgroups of some groups

G	#SP(G)	#NSP(G)
$\mathbb{Z}_3$	2	2
$\mathbb{Z}_9$	3	3
S_3	3	3
A_4	3	3
S_4	4	8
D_8	10	10
D_{10}	3	3
D_{12}	7	7
D_{18}	4	7

Remark 29. Clearly, $\mathbb{Z}_n$ is an abelian group, so by Theorem 24 we have that all subgroups of $\mathbb{Z}_n$ are nearly S-permutable subgroups. In this case $\#NSP(G)$ is number of subgroups of G. For $\mathbb{Z}_p$, p-prime, we have $\#NSP(\mathbb{Z}_p)$ $= 2$. In general, $\#NSP(\mathbb{Z}_n) = \tau(n)$, where $\tau(n)$ is the number of prime divisors of the integer n.

Remark 30. ([10]) Nearly S-permutable subgroups need not to be subnormal.

Example 31. Let $G = D_{18}$, which contains 3 subgroups of order 6, these subgroups are nearly S-permutable in G but not subnormal.

In the next figure, we show the relation between permutability, S-permutability, nearly S-permutability, normality and subnormality.

3. Direct product of nearly S-permutable subgroups

In this part, we study the permutability of the direct product of S-permutable subgroups. This can be shown by the next theorem.

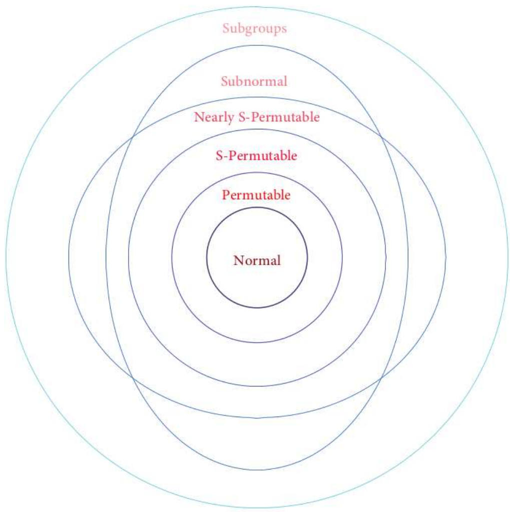

Figure 1: The relation between: Permutability, S-permutability, Nearly S-permutability, Normality and Subnormality using Venn Diagram.

Theorem 32. *Let G_1 and G_2 be two groups with relatively prime orders. Then the direct product of nearly S-permutable subgroups of G_1 and G_2 is nearly S-permutable subgroup.*

Proof. Let A_1 be a nearly S-permutable subgroup in G_1 and A_2 be a nearly S-permutable subgroup in G_2. Suppose on the contrary that $A = A_1 \times A_2$ is not nearly S-permutable subgroup in $G = G_1 \times G_2$. Then there is a prime p such that $\gcd(p, |A|) = 1$ and a subgroup K in G such that $A \leq K \leq G$ and $N_K(A)$ does not contain $P = P_1 \times P_2$ where P is the Sylow p-subgroup of K. Since K contains $A_1 \times A_2$, then K must factor as a product of two subgroups $K_1 \times K_2 = K$, where $A_1 \leq K_1 \leq G_1$ and $A_2 \leq K_2 \leq G_2$.
Moreover, $N_{K_1 \times K_2}(A_1 \times A_2) = N_{K_1}(A_1) \times N_{K_2}(A_2)$ and $P = P_1 \times P_2$, where $P_1 \in Syl_p(K_1)$ and $P_2 \in Syl_p(K_2)$. Since $P \nleq N_K(A)$, then either $P_1 \nleq N_{K1}(A_1)$ or $P_2 \nleq N_{K_2}(A_2)$. But this implies that either A_1 is not nearly S-permutable subgroup in G_1 or A_2 is not nearly S-permutable subgroup in G_2, which is a contradiction. Hence the claim. □

Remark 33. The direct product of non nearly S-permutable subgroups could be nearly S-permutable subgroup.

Example 34. Consider the symmetric group S_4, and the non nearly S-permutable subgroups $H_1 = \langle(1,2)\rangle$ and $H_2 = \langle(1,2,3)\rangle$. It is clear that $H_1 \times H_2 = \langle(1,2),(1,2,3)\rangle$ is nearly S-permutable subgroups of $S_4 \times S_4$, since it is of order $2 \cdot 3$ and $S_4 \times S_4$ of order $2^6 \cdot 3^2$.

Theorem 32 indicates that, the direct product of nearly S-permutable subgroups is always nearly S-permutable subgroup. So we conclude that the number of nearly S-permutable subgroups in the direct product is greater than or equal the product of the numbers of nearly S-permutable subgroups of each group.

Conjecture 35. Consider the groups G_1 and G_2. The number of nearly S-permutable subgroups of $G_1 \times G_2$ is equal to the product of the numbers of nearly S-permutable subgroups of the groups G_1 and G_2 if and only if $|G_1|$ and $|G_2|$ are relatively prime.

Problem. The restriction that $\gcd(|G_1|,|G_2|) = 1$ in Theorem 32 is not necessary in general. Is there a proof for the general case?

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