

**DYNAMICS OF FRACTIONAL-ORDER LORENZ
SYSTEM APPLYING DIFFERENT NUMERICAL METHODS**

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Abstract: In this paper we numerically study the chaotic behaviors of the fractional-order Lorenz system, considered as a highly simplified model for the wether, comparing Euler's method, Fourth Order Runge-Kutta method (RK4) and Vectorial Fourth Order Runge-Kutta method with an semi-numerical method named as Fractional Multi-step Differential Transformation method (FMDTM), that exploits the power series representation of the solution. The system is shown to display interesting chaotic behavior depending on the third parameter $c = 13.93$ (homoclinic orbit) and $c = 24.74$ (formed strange attractor) for same initial conditions and fractional order $\alpha = 0.998$, which are analyzed by comparing system phase portraits with each other and $\alpha = 1$. The fractional derivatives are described in the Caputo sense. The results demonstrate reliability and efficiency of the algorithm developed using Mathematica Package.

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1. Introduction

Not until 1884, when the theory of generalized operators achieved such a level

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in its development, had been extended to include D^α operators, where α could be rational or irrational, positive or negative, real or complex. At present, the number of applications of fractional calculus rapidly grows. These mathematical phenomena allow us to describe and model a real object more accurately than the classical “integer” methods [4, 5, 6]. The mathematical modeling and simulation of systems, naturally leads to differential equation of fractional order [7]. Exact solutions of the most of the fractional nonlinear differential equations cannot be expressed in a closed form, thus analytical and numerical methods are used almost without exceptions. In this paper we numerically study the chaotic behaviors of the fractional-order Lorenz system, considered as a highly simplified model for the wether, comparing Euler’s method, RK4 method and Vectorial RK4 method [5] with an semi-numerical method named as FMDTM [1, 2, 3], that exploits the power series representation of the solution.

Definition 1. A fractional integral of order $\alpha > 0$ for a function $f(t)$ is usually defined (in Riemann-Liouville sense) as follows:

$${}_a I_t^\alpha f(t) = {}_a D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \quad (1)$$

Definition 2. The Riemann-Liouville (R-L) definition of a fractional derivative of order $\alpha > 0$ reads as follows:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau \quad (2)$$

for $(n - 1 < \alpha < n)$, where a and t are the limits of the operation ${}_a D_t^\alpha f(t)$.

For the case $(0 < \alpha < 1)$ and $f(t)$ being a causal function of t , that is $f(t) = 0$ for $t < 0$, the expression for the R-L fractional order derivative reduces to:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} f(\tau) d\tau. \quad (3)$$

Definition 3. The so-called Caputo (C) definition of fractional derivative of order $\alpha > 0$ is:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{\alpha-1} f^{(n)}(\tau) d\tau \quad (4)$$

with same integer n as in Definition 2.

When the Caputo fractional derivative (Definition 3) is chosen to be used, because it allows initial and boundary conditions to be included in the formulation of the problem [7] and in the Lorenz system of fractional-order, an 3-dimensional dynamical system that exhibits chaotic flow is known as “butterfly effect”. The flapping wing represents a small change in the initial conditions of the system.

For the considered fractional dynamical system of order $0 < \alpha \leq 1$,

$$D^\alpha y_i = f(y_1, y_2, y_3) \quad (i = 1, 2, 3), \tag{5}$$

$y^* = (y_1^*, y_2^*, y_3^*)$ is called an equilibrium point if $f_i(y^*) = 0$, for $i = 1, 2, 3$. Let $y^* = (y_1^*, y_2^*, y_3^*)$ be a equilibrium point of the system (5) and $\xi_i = y_i - y_i^*$, $i = 1, 2, 3$, be a small perturbation from a fixed point:

$$D^\alpha \xi_i \approx \xi_1 \frac{\partial f_i y^*}{\partial y_1} + \xi_2 \frac{\partial f_i y^*}{\partial y_2} + \xi_3 \frac{\partial f_i y^*}{\partial y_3}, \quad i = 1, 2, 3. \tag{6}$$

System (6) is equivalent to $[D^\alpha \xi_1 \ D^\alpha \xi_2 \ D^\alpha \xi_3] = J[\xi_1 \ \xi_2 \ \xi_3]$, where J is the Jacobian matrix evaluated at point y^* . An equilibrium point p of the system (5) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix evaluated at y^* satisfy the following condition: $|arg(eig(J))| > \alpha \frac{\pi}{2}$. An equilibrium point y^* is defined as a non-hyperbolic equilibrium point if $|arg(eig(J))| \neq \alpha \frac{\pi}{2}$ for every eigenvalue the Jacobian matrix J at y^* , see [7]. An equilibrium point y^* is called a saddle point if the Jacobian matrix at y^* has at least one stable and one unstable eigenvalue, [4, 6, 7].

Definition 4. A Lorenz system is described as:

$$\begin{aligned} D_t^\alpha x(t) &= a(y(t) - x(t)), \\ D_t^\alpha y(t) &= x(t)(c - z(t)) - y(t), \\ D_t^\alpha z(t) &= x(t)y(t) - bz(t), \end{aligned} \tag{7}$$

where $x(0) = x_0, y(0) = y_0, z(0) = z_0, a, b, c > 0, a = 10, b = 8/3$ and with varied c .

Lorenz observed that the system with $\alpha = 1$ behaves chaotically whenever $c > 24.74$. It has solutions that remain bounded but never converge to a fixed point or a periodic orbit, [8]. We use numerical algorithms to see the behavior of the system for order closer to 1. Generally, for different fractional orders $\alpha_1, \alpha_2, \alpha_3$, which in our case are taken equal, the chaos exists for the system

order $\alpha_1 + \alpha_2 + \alpha_3 < 3$, [7]. Equilibrium points of the system according to the Jacobian matrix are: $C_0 = (0, 0, 0)$, $C_1 = (\sqrt{b(c-1)}, \sqrt{b(c-1)}, c-1)$ and $C_2 = (-\sqrt{b(c-1)}, -\sqrt{b(c-1)}, c-1)$.

2. Numerical methods

The study of fractional chaotic systems has applications in secure communications, which can be made using the systems which are difficult to break. Fractional-order derivative acts as additional parameter which works as a key. In this section we present several numerical methods for fractional order systems, starting with Euler's and Runge-Kutta methods [4, 5], adding some modifications like interpolation and vectorial form, results compared with FMDTM [2], an semi-numerical method, that exploits the power series representation of the solution.

2.1. Euler's method

We introduce an generalized Euler method to the case of differential equation for fractional order $0 < \alpha < 1$, of the type:

$$D^\alpha y(t) = f(t, y(t)), \quad y(t_0) = y_0. \quad (8)$$

We assume that $y(t)$, $D^\alpha y(t)$, $D^{2\alpha} y(t)$ are continuous on $[t_0, a]$ and apply the generalized Taylor's formula:

$$y(t) = y(t_0) + D^\alpha y(t_0) \frac{t^\alpha}{\Gamma(\alpha + 1)} + D^{2\alpha} y(t_0) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad t_0 < t \leq t, \quad (9)$$

to obtain the iterative Euler's formula:

$$y(t) \approx y(t_0) + \frac{h^\alpha}{\Gamma(\alpha + 1)} f(t_0, y(t_0)),$$

or expressed in the recurrent form for the Lorenz system, it is:

$$y_{n+1} \approx y_n + \frac{h^\alpha}{\Gamma(\alpha + 1)} f(t_n, y_n). \quad (10)$$

2.2. Fourth order Runge-Kutta method

In the case of fourth order Runge-Kutta method, we start with the fractional differential equation (8), where $y \in C^{p+1}([t_0, t_0 + T])$. In the neighborhood of

y_0 we suppose that $D^\alpha E(0) = D^{2\alpha} E(0) = 0$, $E(h) = y(t+h) - t(t)$. The approximate solution can be obtained from the expansion

$$y_{n+1} = y_n + \frac{h^\alpha}{6\Gamma(\alpha + 1)}(K_1 + 2K_2 + 2K_3 + K_4), \tag{11}$$

where

$$\begin{aligned} K_1 &= f(t_n, y_n), \\ K_2 &= f\left(t_n + \frac{h^\alpha}{\Gamma(2\alpha + 1)}, y_n + \frac{1}{2} \frac{h^\alpha}{\Gamma(2\alpha + 1)}\right), \\ K_3 &= f\left(t_n + \frac{1}{2} \frac{h^\alpha}{\Gamma(2\alpha + 1)}, y_n + \frac{1}{2} \frac{h^\alpha}{\Gamma(2\alpha + 1)}\right), \\ K_4 &= f\left(t_n + \frac{h^\alpha}{\Gamma(2\alpha + 1)}, y_n + \frac{h^\alpha}{\Gamma(2\alpha + 1)}\right). \end{aligned} \tag{12}$$

2.3. Fractional multi-step differential transform method (FMDTM)

The DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, [1]. The method, however, has some drawbacks: the series solution always converges in a very small region and it has slow convergent rate in the wider region. To overcome this shortcoming, we present in this section the FMDTM that was originally developed for the numerical solutions of ordinary differential equations [2], [3]. For this purpose, we consider the following nonlinear initial value problem (8).

Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (8). In actual applications of the DTM, the approximate solution of the initial value problem (8) can be expressed by the finite series: $y(t) = \sum_{n=0}^N a_n t^n$, $t \in [0, T]$. The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0, T]$ is divided into M sub-intervals $[t_{m-1}, t_m]$, $m = 1, 2, \dots, M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_m = mh$. The main idea of the FMDTM is in the following. First, we apply the DTM to (8) over the interval $[0, t_1]$, we will obtain the following approximate solution:

$$y_1(t) = \sum_{n=0}^N a_{1n} t^n, \quad t \in [0, t_1], \tag{13}$$

using the initial conditions $y_1^{(k)}(0) = c_k$. For $m \geq 2$ and at each sub-interval $[t_{m-1}, t_m]$ we will use the initial conditions $y_m^{(k)}(t_{m-1}) = y_{m-1}^{(k)}(t_{m-1})$ and apply

the DTM to (8) over the interval $[t_{m-1}, t_m]$, where t_0 in $F(k) = \frac{1}{k!} [\frac{d^k f(t)}{dt^k}]_{t=t_0}$ is replaced by t_{m-1} .

The process is repeated and generates a sequence of approximate solutions $y_m(t)$, $m = 1, 2, \dots, M$, for the solution $y(t)$: $y_m(t) = \sum_{n=0}^k a_{mn}(t - t_{m-1})^n$, $t \in [t_m, t_{m+1}]$, where $N = KM$. In fact the FMDTM assumes the following solution:

$$y(t) = \begin{cases} y_1(t), & t \in [0, t_1] \\ y_2(t), & t \in [t_1, t_2] \\ \vdots \\ y_M(t), & t \in [t_{M-1}, t_M] \end{cases}. \quad (14)$$

The new algorithm, FMDTM, is simple for computational performance for all values of h . It is easily observed that if the step size $h = T$, then the FMDTM reduces to the classical DTM.

3. Numerical simulations

In this section we present simulation results of the Lorenz's system (7) on three dimensional plot, taking standard parameters $a = 10$, $b = 8/3$, and varied c , to analyze its chaotic behaviour especially for $c = 13.93$ and $c = 24.74$. We aim to show how Euler's method, RK4 method and FMDTM will agree with each other in the numerical approximation of the solutions of the system. Modification of RK4 using interpolation and vectorial form, will be shown as special numerical methods with intervention on standard above mention methods.

If we consider the standard form with the parameters $(a, c, b) = (10, 28, 8/3)$, the minimal order for which the Lorenz system is chaotic is $\alpha > 0.9941$, [6, 7, 8]. For $c = 13.93$, eigenvalues of matrix J are $(-12.8706, -0.398058 + 7.30653i, -0.398058 - 7.30653i)$, the $|\arg(\lambda_1)| = \pi > 0.998\frac{\pi}{2}$ and $|\arg(\lambda_{2/3})| = 1.62522 > 0.998\frac{\pi}{2}$, the equilibrium points $C_1 = (5.87, 5.87, 12.93)$ and $C_2 = (-5.87, -5.87, 12.93)$ are asymptotically stable, the system enters (7) a state of transient chaos Figure 1.

For $c = 24.74$, eigenvalues of matrix J are:

$$(-13.6789, 0.00612186 + 9.66131i, 0.00612186 - 9.66131i),$$

the $|\arg(\lambda_1)| = \pi > 0.998\frac{\pi}{2}$ and $|\arg(\lambda_{2/3})| = 1.57016 > 0.998\frac{\pi}{2}$, the equilibrium points

$$C_1 = (7.99, 7.99, 23.74) \quad \text{and} \quad C_2 = (-7.99, -7.99, 23.74)$$

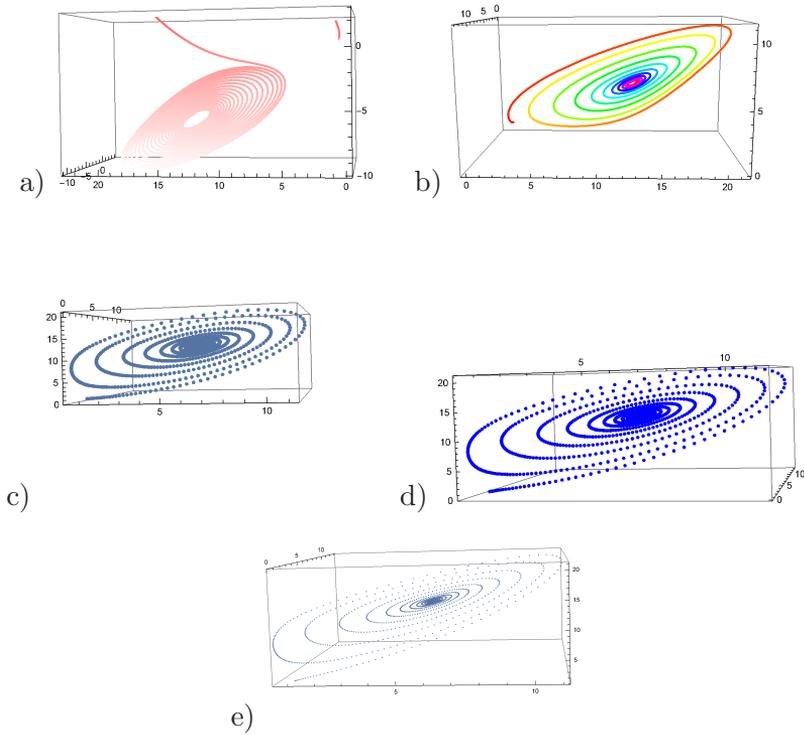


Figure 1: 3D phase portrait of system (7) with $\alpha = 0.998$, initial conditions $(x_0, y_0, z_0) = (1, 1, 1)$, $h = 0.01$, $t = 100s$ and $c = 13.93$. a) Euler method; b) RK4 with Interpolation; c) RK4; d) Vectorial RK4; e) FMDTM.

are asymptotically stable. The trajectory is not periodic, remains on the attractor forever, general form is independent of initial conditions and has fractal structure shown on Figure 2. Instability measure is 0.342941, therefore the system (7) satisfies the necessary condition for exhibiting a double scroll attractor. In all simulations are taken the same initial conditions $(x_0, y_0, z_0) = (1, 1, 1)$ and fractional order closer to 1, $\alpha = 0.998$. System shows to be chaotic for $c = 24.74$ and stable for $c = 13.93$. Time-series solution $x(t)$ of the system (7) using FMDTM, represent the same conclusions, shown on Figure 3, a) for $c = 13.93$ and $c = 24.74$.

The system shows approximately the same behavior like the differential Lorenz system Figure 4, dynamical system of order $\alpha = 1$.

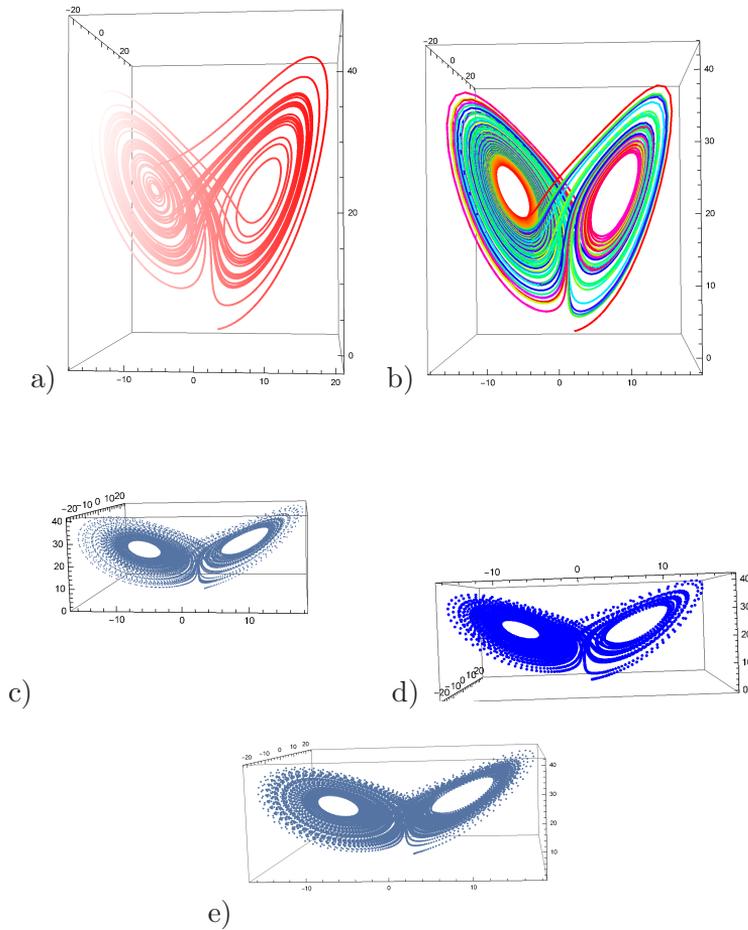


Figure 2: 3D phase portrait of system (7) with $\alpha = 0.998$, initial conditions $(x_0, y_0, z_0) = (1, 1, 1)$, $h = 0.01$, $t = 100s$ and $c = 24.74$. a) Euler method; b) RK4 with Interpolation; c) RK4; d) Vectorial RK4; e) FMDTM.

4. Conclusions

This present analysis exhibits the applicability of the three numerical methods (Euler, RK4 and FMDTM) to show chaotic behaviour of the fractional order Lorenz system (7). The work emphasizes our belief that the methods are reliable techniques to handle nonlinear fractional differential systems. There we use Euler's method and RK4, two numerical methods and FMDTM, an modification of DTM, which has two possible uses: as an analytic tool or as a

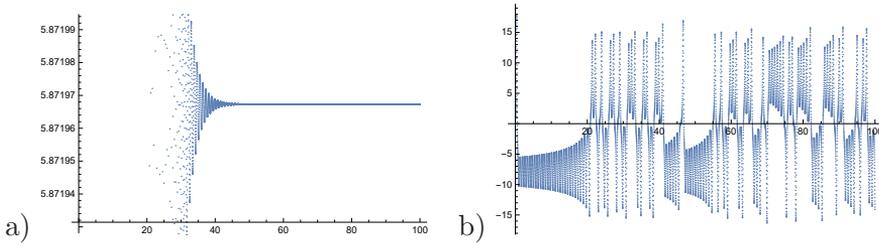


Figure 3: Time-series of $x(t)$ numerical solution of system (7) with $\alpha = 0.998$, initial conditions $(x_0, y_0, z_0) = (1, 1, 1)$, $h = 0.01$, $t = 100s$ and a) $c = 19.93$ and b) $c = 24.74$ using FMDTM.

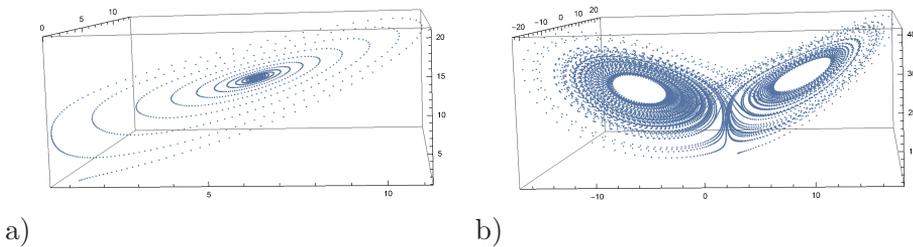


Figure 4: 3D phase portrait of system (7) with $\alpha = 1$, initial conditions $(x(0), y(0), z(0)) = (1, 1, 1)$, $h = 0.01$, $t = 100s$ a) $c = 13.93$ using RK4; b) $c = 24.74$ using Vectorial RK4.

numerical tool, even that it has most often been used as an analytic tool, so that some authors have even forgotten the initial aim and proposed to reinvent the method.

The system is described using 3D phase portrait Figure 1 and Figure 2, the numerical results are given for both non-chaotic $c = 13.93$ and chaotic $c = 24.74$ cases. It is found that numerical methods: Euler, RK4 and its modifications agree with each other compared with FMDTM. The results agree with the first order Lorenz dynamical system Figure 4, its behavior varying with third parameter c . Numerical methods have different speed of computing, Euler's with $0.703125s$, RK4 with $0.71875s$, and FMDTM with $19.9531s$, according to their nature, primitive with modified multi-step computations. FMDTM shows more approximated solutions to the exact ones, work in progress, with results that will be studied in the future with their comparison with exact solutions of related small number of fractional-order systems.

The fractional order $\alpha = 0.998$ is taken according to the fact $\alpha > 0.9941$, mentioned in Section 3, some results are achieved for $\alpha < 0.9941$ and standard

parameters $a = 10$, $b = 8/3$ and $c = 28$ compared with the varied $c = 13.93$ and $c = 24.74$ with unexpected behavior of the system according the methods, left to be analyzed and presented in the future.

References

- [1] C. Berviller, Status of the differential transformation method, *App. Math. and Comp.*, **20** (2012), 10158-10170.
- [2] Z.M. Odibat, C. Bertelle, M. Aziz Alaoui, G.H.E. Duchamp, A multi-step differential transform method and application to non-chaotic or chaotic systems, *Computer and Mathematics with Applications*, **59** (2010), 1462-1472.
- [3] V.S. Erturk, Sh. Momani, Solving systems of fractional differential equations using differential transform method, *Journal of Computational and Applied Mathematics*, **215** (2008), 142-151.
- [4] C. Milici, Ch. Draganescu, J.A. Tenreiro Machado, *Introduction to Fractional Differential Equations*, Springer Nature, Switzerland (2019).
- [5] Ch. Li, F. Zeng, *Numerical Methods for Fractional Calculus*, CRC Press, Taylor and Francis Group, New York (2015).
- [6] R. Caponetto, G. Dongola, L. Fortuna, I. Petras, *Fractional Order Systems, Modeling and Control Applications*, World Scientific Publ. Ser. on Nonlinear Science Series A, USA (2011).
- [7] I. Petras, *Fractional-Order Nonlinear Systems Modeling, Analysis and Simulation*, Springer-Verlag, Berlin (2011).
- [8] S. Luch, *Dynamical Systems with Applications using Mathematica*, New York, USA (2007).