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STUDY THE NONEXISTENCE OF RADIAL POSITIVE SOLUTIONS FOR A CLASS OF NONPOSITONE SEMILNEAR ELLIPTIC SYSTEMS

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Abstract: In this article, we are concerned with the nonexistence of radial positive solutions for a class of nonpositone semilinear elliptic systems in an annulus when the nonlinearities have more than one zero.

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Key Words: nonpositone problem; radial positive solutions

1. Introduction

The purpose of this work is to study the nonexistence of radial positive solutions for the following system

$$\begin{cases} -\Delta u(x) = \lambda f(v(x)), & x \in \Omega, \\ -\Delta v(x) = \mu g(u(x)), & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where λ, $\mu \geq \varepsilon_0 > 0$, Ω is an annulus in $\mathbb{R}^N$: $\Omega = C(0, R, \widehat{R}) = \{x \in \mathbb{R}^N : R < |x| < \widehat{R}\}$. $(0 < R < \widehat{R},\ N \geq 2)$, f and g are the smooth functions, nonpositone and have more than one zero. This study can be done by using energy analysis and comparison methods.

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The existence result for positive solutions for classes of superlinearities satisfying some conditions, see [6] and [7]. In the single equations case, see [1], [2], [10] for nonexistence results and [1], [4], [9] for existence results.

Remark 1. Let us note that when Ω is a ball and $N \geq 2$, by [3] all nonnegative solutions are positive componentwise. Hence by [12] solutions are radially symmetric and decreasing.

In the case when f and g have only one zero, the problem (1) has been studied by Hai, Shivaji and Oruganti in a ball [3], and by Hakimi in an annulus, [11].

The nonexistence of radial positive solutions of (1) is equivalent of the nonexistence of positive solutions of the following

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1} f(v), & R < r < \widehat{R}, \\ -(r^{N-1}v')' = \mu r^{N-1} g(u), & R < r < \widehat{R}, \\ u(R) = u(\widehat{R}) = 0 = v(R) = v(\widehat{R}). \end{cases} \tag{2}$$

Our goal is to assure the result of the nonexistence of radial positive solutions u ($u(x) = u(r)$, $r = \|x\|$) of (1) in the case when the nonlinearities f and g have more than one zero and increasing from the last zero. More precisely, we assume the following conditions:

(H_1) f, $g : [0, +\infty) \longrightarrow \mathbb{R}$ are continuous, $f(0) < 0$, $g(0) < 0$, and f (resp. g) increasing on $(\beta_1, +\infty)$ (resp. $(\beta_2, +\infty)$), where β_1 (resp. β_2) is the greatest zero of f (resp. of g).

(H_2) There exist two positive real numbers a_i and b_i, $i = 1, 2$ such that

$$f(z) \geq a_1 z - b_1,$$

$$g(z) \geq a_2 z - b_2,$$

for all $z \geq 0$.

2. The main result

The main result in this paper is the following theorem.

Theorem 2. *Assume that the hypotheses (H_1), (H_2) are satisfied. Then there exists a positive real number σ such that the problem* (1) *has no radial positive solution for $\lambda\mu > \sigma$.*

To prove Theorem 2, we will use, as pointed out in the introduction, the energy analysis and comparison methods following the work and used similar ideas of Hai, Shivaji and Oruganti [8]. For this, we need the next three technical lemmas. We note that the proofs of the first and second lemmas are analogous to [8, Lemma 3.1 and Lemma 3.2]. On the opposite, the proof of the third lemma is different from that [8, Lemma 3.3]. This is due to that in our case f and g may have more than one zero and are not increasing entirely $[0, +\infty)$.

Lemma 3. *There exists a positive constant C such that for $\lambda\mu$ large,*

$$u(R_0) + v(R_0) \leq C,$$

where $R_0 = \frac{R+\widehat{R}}{2}$.

Proof. Multiplying the first equation in (2) by a positive eigenfunction say ϕ corresponding to λ_1 and using (H_1) we obtain

$$-\int_R^{\widehat{R}} (r^{N-1}u')'\phi dr \geq \int_R^{\widehat{R}} \lambda\left(a_1 v - b_1\right)\phi r^{N-1} dr,$$

that is,

$$\int_R^{\widehat{R}} \lambda_1 u r^{N-1}\phi dr \geq \int_R^{\widehat{R}} \lambda\left(a_1 v - b_1\right)\phi r^{N-1} dr. \tag{3}$$

Similarly, using the second equation in (2) and (H_2), we obtain

$$\int_R^{\widehat{R}} \lambda_1 v r^{N-1}\phi dr \geq \int_R^{\widehat{R}} \mu\left(a_2 u - b_2\right)\phi r^{N-1} dr. \tag{4}$$

Combining (3) and (4), we obtain

$$\int_R^{\widehat{R}} \left[\lambda_1 - \lambda\mu\frac{a_1 a_2}{\lambda_1}\right] v\Phi r^{N-1} dr \geq \int_R^{\widehat{R}} \mu\left[-\lambda\frac{a_2 b_1}{\lambda_1} - b_2\right]\Phi r^{N-1} dr.$$

Now, if $\frac{\lambda\mu}{2}a_1 a_2 \geq \lambda_1^2$, then

$$\int_R^{\widehat{R}} \mu\left[-\lambda a_2 b_1 - b_2\lambda_1\right]\Phi r^{N-1} dr \leq \int_R^{\widehat{R}} -\frac{\lambda\mu}{2} a_1 a_2 v\Phi r^{N-1} dr,$$

that is,

$$\int_R^{\widehat{R}} \frac{a_1 a_2}{2} v\Phi r^{N-1} dr \leq \int_R^{\widehat{R}} \left[a_2 b_1 + \frac{b_2\lambda_1}{\varepsilon_0}\right]\Phi r^{N-1} dr \tag{5}$$

(because $\lambda \geq \varepsilon_0$).

Similarly,

$$\int_R^{\widehat{R}} \frac{a_1 a_2}{2} u \Phi r^{N-1} dr \leq \int_R^{\widehat{R}} \left[a_1 b_2 + \frac{b_1 \lambda_1}{\varepsilon_0} \right] \Phi r^{N-1} dr. \tag{6}$$

Adding (5) and (6), we obtain the following inequality

$$\int_R^{\widehat{R}} (u+v) \Phi r^{N-1} dr \leq \frac{2}{a_1 a_2} \int_R^{\widehat{R}} \left[a_1 b_2 + \frac{b_1 \lambda_1}{\varepsilon_0} + a_2 b_1 + \frac{b_2 \lambda_1}{\varepsilon_0} \right] \Phi r^{N-1} dr.$$

Then

$$\begin{aligned} &(u+v)(R_0) \int_{\overline{t}}^{R_0} \Phi r^{N-1} dr \leq \int_{\overline{t}}^{R_0} (u+v) \Phi r^{N-1} dr \\ &\leq \int_R^{\widehat{R}} (u+v) \Phi r^{N-1} dr \\ &\leq \frac{2}{a_1 a_2} \int_R^{\widehat{R}} \left[a_1 b_2 + \frac{b_1 \lambda_1}{\varepsilon_0} + a_2 b_1 + \frac{b_2 \lambda_1}{\varepsilon_0} \right] \Phi r^{N-1} dr, \end{aligned}$$

where $\overline{t} = \max(\overline{t}_1, \overline{t}_2)$ with $\overline{t}_1$ and $\overline{t}_2$ are such that
$\overline{t}_1 = \max \left\{ r \in \left(R, \widehat{R} \right) \ \mid \ u'(r) = 0 \right\}$
and $\overline{t}_2 = \max \left\{ r \in \left(R, \widehat{R} \right) \ \mid \ v'(r) = 0 \right\}$.
The proof is complete. $\square$

We remark that $\overline{t}_i \leq R_0$, for $i = 1, 2$ was shown in [5]. Now, assume that there exists $z > 0$ on $\overline{I}$, where $I = (\alpha, \beta)$, and a constant γ such that

$$-\left(r^{N-1} z' \right)' \geq \gamma r^{N-1} z \,, \quad r \in I. \tag{7}$$

Let $\lambda_1 = \lambda_1(I) > 0$ denote the principal eigenvalue of

$$\begin{cases} -\left(r^{N-1} \Phi' \right)' = \lambda r^{N-1} \Phi, & r \in (\alpha, \beta) \\ \Phi(\alpha) = 0 = \Phi(\beta), \end{cases} \tag{8}$$

where $0 < \alpha < \beta \leq 1$.

We shall prove the following lemma.

Lemma 4. *Let* (7) *hold. Then* $\gamma \leq \lambda_1(I)$.

Proof. Multiplying (7) by Ψ (>0), an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$, and integrating by parts (twice) we obtain

$$\int_\alpha^\beta [\gamma - \lambda_1(I)]\, r^{N-1} z\Psi dr \le \beta^{N-1}\Psi'(\beta)z(\beta) - \alpha^{N-1}\Psi'(\alpha)z(\alpha), \tag{9}$$

but $\Psi'(\beta) < 0$ and $\Psi'(\alpha) > 0$. Hence the right hand side of (9) is ≤ 0 and thus $\gamma \le \lambda_1(I)$. The proof is complete. ☐

Now, consider $\underline{R}$ and $\overline{R}$ in $(R_0, \widehat{R})$ such that $R_0 < \underline{R} < \overline{R} < \widehat{R}$.

Lemma 5. *For $\lambda\mu$ sufficiently large, $u(\overline{R}) \le \beta_2$ or $v(\overline{R}) \le \beta_1$, respectively.*

Proof. We argue by contradiction. Suppose that $u(\overline{R}) > \beta_2$ and $v(\overline{R}) > \beta_1$.

Case 1: $u(\underline{R}) > \rho_2$ or $v(\underline{R}) > \rho_1$, where $\rho_1 = \frac{\beta_1+\theta_1}{2}$ and $\rho_2 = \frac{\beta_2+\theta_2}{2}$ (θ_1 and θ_2 are the greatest zeros of F and G respectively, where $F(x) = \int_0^x f\,(t)dt$ and $G(x) = \int_0^x g\,(t)dt$).

If $u(\underline{R}) > \rho_2$, then

$$\begin{aligned} -\left(r^{N-1}v'\right)' &= \mu r^{N-1} g(u) \\ &\ge \varepsilon_0 r^{N-1} g(\rho_2) \text{ in } J = (R_0, \underline{R}) \end{aligned}$$

and $v(r) \ge \beta_1$ on $\overline{J}$.

Let ω be the unique solution of

$$\begin{aligned} -\left(r^{N-1}\omega'\right)' &= \varepsilon_0 r^{N-1} g(\rho_2) \text{ in } J \\ \omega &= \beta_1 \text{ in } \partial J. \end{aligned}$$

Then by comparison arguments, $v(r) \ge \omega(r) = \varepsilon_0 g(\rho_2)\omega_0(r) + \beta_1$ on $\overline{J}$, where ω_0 is the unique (positive) solution of

$$\begin{aligned} -\left(r^{N-1}\omega_0'\right)' &= r^{N-1} \text{ in } J \\ \omega_0 &= 0 \text{ on } \partial J. \end{aligned}$$

In particular, there exists $\overline{\beta}_1 > \beta_1$ (we choose $\overline{\beta}_1$ such that $f(\overline{\beta}_1) \ne 0$) such that

$$\begin{aligned} v\left(R_0 + \frac{2(\underline{R} - R_0)}{3}\right) &\ge \omega\left(R_0 + \frac{2(\underline{R} - R_0)}{3}\right) \\ &\ge \overline{\beta}_1 \text{ in } J^* = \left(R_0 + \frac{\underline{R} - R_0}{3}, R_0 + \frac{2(\underline{R} - R_0)}{3}\right). \end{aligned}$$

Then

$$\begin{aligned} -\left(r^{N-1}(u-\beta_2)'\right)' &= \lambda r^{N-1} f(v) \\ &\geq \lambda r^{N-1} f(\overline{\beta}_1) \\ &\geq \left(\frac{\lambda f(\overline{\beta}_1)}{C}\right) r^{N-1}(u-\beta_2) \text{ on } J^* \end{aligned}$$

(where C is as in Lemma 3).

Since $u-\beta_2>0$ on $\overline{J}^*$, it follows that

$$\frac{\lambda f(\overline{\beta}_1)}{C} \leq \lambda_1(J^*), \tag{10}$$

where $\lambda_1(J^*)$ is the principal value of (8) (with $(\alpha,\beta)=J^*$).

Next consider

$$\begin{aligned} \left(r^{N-1}(v-\beta_1)'\right)' &= \mu r^{N-1} g(u) \\ &\geq \mu r^{N-1} g(\rho_2) \\ &\geq \left(\frac{\mu g(\rho_2)}{C}\right) r^{N-1}(v-\beta_1) \text{ on } J. \end{aligned}$$

Since $v-\beta_1>0$ on $\overline{J}$, then

$$\frac{\mu g(\rho_2)}{C} \leq \lambda_1(J), \tag{11}$$

where $\lambda_1(J)$ is the principal value of (8) (with $(\alpha,\beta)=J$).

Combining (10) and (11), we obtain

$$\frac{\lambda\mu f(\overline{\beta}_1) g(\rho_2)}{C^2} \leq \lambda_1(J^*)\lambda_1(J),$$

but $f(\overline{\beta}_1)$, $g(\rho_2)$ and C are fixed positive constants.

This is a contradiction for $\lambda\mu$ large.

A similar contradiction can be reached for the case when $v(\underline{R})>\rho_1$.

Case 2: $u(\underline{R})\leq\rho_2$ and $v(\underline{R})\leq\rho_1$. Then $\beta_2<u\leq\rho_2$ and $\beta_1<v\leq\rho_1$ on $J_1=[\underline{R},\overline{R}]$. Then by the mean value theorem, there exist c_1, c_2 $\in(\underline{R},\overline{R})$ such that

$$\left|u'(c_2)\right| \leq \frac{\rho_2}{\overline{R}-\underline{R}},$$

$$\left|v'(c_1)\right| \leq \frac{\rho_1}{\overline{R}-\underline{R}}.$$

Since $-\left(r^{N-1}u'\right)' \geq 0$ on $[\underline{R},\overline{R})$, then

$$-r^{N-1}u'(r) \leq -c_2^{N-1}u'(c_2) \text{ on } J_2=[\underline{R},c_2),$$

thus

$$\begin{aligned}\left|u'(r)\right| &\leq \frac{c_2^{N-1}}{r^{N-1}}\left|u'(c_2)\right| \\ &\leq \left(\frac{\overline{R}}{\underline{R}}\right)^{N-1}\frac{\rho_2}{\overline{R}-\underline{R}} \text{ in } J_2.\end{aligned}$$

Similarly, we obtain

$$\left|v'(r)\right| \leq \left(\frac{\overline{R}}{\underline{R}}\right)^{N-1}\frac{\rho_1}{\overline{R}-\underline{R}} \text{ in } J_3=[\underline{R},c_1).$$

Hence there exists $r_0 \in (\underline{R},\overline{R})$ such that

$$\left|u'(r_0)\right| \leq \widetilde{c} \text{ and } \left|v'(r_0)\right| \leq \widetilde{c},$$

where $\widetilde{c}=\frac{1}{\overline{R}-\underline{R}}\left(\frac{\overline{R}}{\underline{R}}\right)^{N-1}\max(\rho_2,\rho_1)$. Now, we define the energy function

$$E(r)=u'(r)v'(r)+\lambda F(v(r))+\mu G(u(r)).$$

Then

$$E'(r)=-\frac{2(N-1)}{r}u'(r)v'(r) \leq 0,$$

and hence $E \geq 0$ on $[R,\widehat{R}]$, (because $u'(\widehat{R})v'(\widehat{R}) \geq 0$). However,

$$E(r_0) \leq \widetilde{c}^2+\lambda F(\rho_1)+\mu G(\rho_2), \tag{12}$$

and $F(\rho_1)<0$ and $G(\rho_2)<0$. Hence $E(r_0)<0$ for $\lambda\mu$ large which is a contradiction. The proof is complete. □

Proof of Theorem 2. Assume $\lambda\mu$ is large enough so that both Lemmas 3 and 5 hold true. We take the case when $u(\overline{R}) \leq \beta_2$. Then

$$\begin{aligned}-\left(r^{N-1}v'\right)' &= \mu r^{N-1}g(u) \leq 0 \text{ on } J_3=(\overline{R},\widehat{R}) \\ v(\overline{R}) &\leq C,\ v(\widehat{R})=0,\end{aligned}$$

hence, by comparison argument $v(r) \leq \widetilde{\omega}(r)$, where $\widetilde{\omega}$ is the solution of

$$-\left(r^{N-1}\widetilde{\omega}'\right)' = 0 \text{ on } J_3$$

$$\widetilde{\omega}(\overline{R}) \;=\; C,\ \widetilde{\omega}(\widehat{R}) = 0.$$

However, $\widetilde{\omega}(r) = \frac{C}{\int_{\overline{R}}^{\widehat{R}} s^{1-N}ds}\int_r^{\widehat{R}} s^{1-N}ds$ decreases from C to 0 on $[\overline{R}, \widehat{R}]$, hence there exists $r_1 \in (\overline{R}, \widehat{R})$ (independent of $\lambda\mu$) such that $\widetilde{\omega}(r_1) = \frac{\beta_1}{2}$. (Here we assume that $\frac{\beta_1}{2} < C$, unless we can choose N_0 such that $\frac{\beta_1}{N_0} < C$).

Hence $v(r_1) \le \frac{\beta_1}{2}$, and

$$\begin{aligned} -\left(r^{N-1}(\beta_2 - u)'\right)' &= -\lambda r^{N-1} f(v) \\ &\ge -\lambda r^{N-1} f(\frac{\beta_1}{2}) \\ &\ge \lambda\left(-f(\frac{\beta_1}{2})\right) r^{N-1}\frac{\beta_2 - u}{\beta_2} \quad \text{on} \quad J_4 = (r_1, \widehat{R}). \end{aligned}$$

Since $\beta_2 - u > 0$ on $\overline{J}_4$, then

$$\frac{\lambda\widetilde{K}_1}{\beta_2} \le \lambda_1(J_4), \tag{13}$$

where $\widetilde{K}_1 = -f(\frac{\beta_1}{2})$ et $\lambda_1(J_4)$ is the principal eigenvalue of (4) (with $(\alpha, \beta) = J_4$).

Similarly, there exists $r_2 \in (r_1, \widehat{R})$ (independent of $\lambda\mu$) such that

$$v(r_2) < \frac{\beta_1}{2}.$$

Hence

$$\begin{aligned} -\left(r^{N-1}u'\right)' &= \mu r^{N-1} f(v) \le 0 \quad \text{on} \quad J_5 = (r_2, \widehat{R}) \\ u(r_2) &\le C,\ \ u(\widehat{R}) = 0, \end{aligned}$$

then, by comparison argument we obtain

$$u(r) \le \omega_1(r) = \frac{C}{\int_{r_2}^{\widehat{R}} s^{1-N}ds}\int_r^{\widehat{R}} s^{1-N}ds,$$

thus

$$\begin{aligned} -\left(r^{N-1}\omega_1'\right)' &= 0,\ \text{in } J_5, \\ \omega_1(r_2) &= C,\ \ \omega_1(\widehat{R}) = 0. \end{aligned}$$

Arguing as before, there exists $r_3 \in (r_2, \widehat{R})$ (independent of $\lambda\mu$) such that

$$u(r_3) \leq \omega_1(r_3) \leq \frac{\beta_2}{2} < C.$$

Hence

$$\begin{aligned} -\left(r^{N-1}(\beta_1 - v)'\right)' &= -\mu r^{N-1} g(v) \\ &\geq -\mu r^{N-1} g(\frac{\beta_2}{2}) \\ &\geq \mu\left(-g(\frac{\beta_2}{2})\right) r^{N-1} \frac{\beta_1 - v}{\beta_1} \quad \text{on } J_6 = (r_3, \widehat{R}). \end{aligned}$$

Since $\beta_1 - v > 0$ on $\overline{J}_6$, it follows that

$$\frac{\mu\widetilde{K}_2}{\beta_1} \leq \lambda_1(J_6), \tag{14}$$

where $\widetilde{K}_2 = -g(\frac{\beta_1}{2})$ and $\lambda_1(J_6)$ is the principal eigenvalue of (7) (with $(\alpha,\beta) = J_6$).

Combining (13) and (14), we obtain

$$\frac{\lambda\mu\widetilde{K}_1\widetilde{K}_2}{\beta_1\beta_2} \leq \lambda_1(J_4)\lambda_1(J_6),$$

which is a contradiction with $\lambda\mu$ large.

A similar contradiction can be reached for the case $v(R_2) \leq \beta_1$.

Hence Theorem 2 is proven. □

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