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PERFORMANCE ANALYSIS OF LI-HYPERBOLIC SINE ACTIVATION FUNCTION FOR TIME-VARYING COMPLEX PROBLEMS

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Abstract: A new nonlinear activation function is formulated to minimize the time taken to converge to the solution of the complex-valued time-varying problems such as matrix inversion, linear equation, Sylvester equation, Lyapunov equation and Moore-Penrose inversion. These problems play a vital role in real-time systems in which decisions are taken based on the solution of the equations. In the case of real-valued problems, numerous nonlinear activation functions were proposed and investigated. The question of convergence becomes difficult when various activation functions are applied to complex-valued problems. The error in the computation process of the problems can be reduced by using monotonically increasing odd activation functions in which the problem is modeled as a complex-valued Zhang neural network. A new nonlinear activation function called Li-hyperbolic sine is proposed. Many other activation functions are applied to test the superior performance of the proposed Li-hyperbolic sine activation function. Theoretical results show the stability of convergence. The network is expanded component-wise using the Kronecker product and solved numerically. Several examples are used to measure the superior convergence of the Li-hyperbolic sine over other activation functions on complex-valued problems.

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1. Introduction

Nonlinear equations have paved the way in solving many problems encountered in scientific and engineering applications. Time-varying matrix inversion problem plays a vital role in robotic motion planning [1], signal processing [13], image processing [12], and in areas of physics [4] and statistics [14]. The matrix inversion problem can be expressed as, $MN = I$, where coefficient matrix is $M \in R^{n\times n}$, $N \in R^{n\times n}$ is the inverse matrix to be calculated and $I \in R^{n\times n}$ is the identity matrix. The time taken to compute the inverse of a constant matrix is $O(n^3)$ [6]. Thus having large computational time, these algorithms might not be effective on a real-time process. To get a real-time response, parallel schemes with neural networks are used.

Recurrent neural networks (RNN) are versatile tools for implementing high-speed parallel processing with hardware. A natural example of RNN is the human brain with interconnections of neurons and feedback connections which are capable of studying the aspects of behavior, classification of items, faster recognition, and so on. Artificial neural networks are studied to emulate the learning capability of the human brain. RNN is a particular category of artificial neural networks, which is used in applications like robotics, identification of a human voice, handwritings, patterns and music, and other challenging problems in prediction and analysis.

Gradient neural network (GNN) is a type of RNN [15] that has been proposed for solving real matrix inversion problems. Frobenius norm of $\|MN^* - I\|_F$ was used to measure the error. It has been shown that GNN is pragmatic for solving constant matrix inversion problem [25]. Still, the performance of GNN in the case of a time-varying matrices is not satisfactory with significant computational errors [19].

Considering the time-varying matrix for inversion, Getz and Marsden [2, 3] investigated a dynamic system that converges to the theoretical inverse $M^{-1}(t)$ exponentially. But, the system could converge when the parameters take tremendous values and the initial conditions supplied is to be closer enough to the theoretical inverse $M^{-1}(0)$.

Taking into account the disadvantages of GNN and the Getz - Marsden system, Zhang et al. [22] formulated the zeroing neural network, also known as the Zhang Neural network (ZNN). Unlike GNN, ZNN follows implicit dynamics. The network allows activation functions to change the speed at which error goes to zero as time increases. Some activation functions used in the literature are given in [21] for real time-varying matrix. On a comparative note, the power sigmoidal activation function proves its superiority in a real-valued time-varying matrix.

Table 1 presents resent papers on complex-valued ZNN models. In [24] a comparison between complex-valued Zhang Neural Network (CVZNN) and complex-valued gradient neural network (CVGNN) model is investigated for matrix inversion with a linear activation function. Sign bi power activation function was proposed for complex Sylvester equation in [7]. For generalized inverse, a complex neural network without any activation function was investigated in [8]. In [16, 10], Drazin inverse using complex neural network was studied with the application of nonlinear activation functions. To speed up the convergence of CVZNN, formulating nonlinear activation functions was stated to be an open problem [5]. A novel nonlinear activation function called the Li-hyperbolic sine function for the CVZNN model is proposed and investigated.

Table 1: Literature survey on Complex valued ZNN model

	Dynamical System	Activation Function
[24]	Matrix Inversion	—
[7]	Sylvester Equation	Sign bi Power
[8]	Generalized Inverse Matrices	—
[16]	Drazin Inverse	Linear, Bipolar Sigmoidal Power Sigmoidal Smooth Power Sigmoidal
[10]	Drazin Inverse	Li function

The main contributions of this paper are as follows:

- The behavior of complex-valued Zhang neural network for time-varying matrix inversion with nonlinear activation function which is in use for real-valued models are examined for convergence.
- A novel nonlinear activation function termed Li-hyperbolic sine activation function is proposed that converges faster than the other activation functions.
- Complex valued Zhang neural network for time-varying linear equation, Sylvester equation, Lyapunov equation and Moore-Penrose inversion is also examined along with matrix inversion problem.
- The asymptotic stability of the system with Li-hyperbolic sine activation function is established.

- Time-varying complex matrices, time-varying complex Toeplitz matrices, and constant complex matrix are used to test the system's efficiency with the proposed activation function and other activation functions for CVZNN.
- The CVZNN network is component-wise enlarged using the Kronecker product and vectorization method and numerically solved using the Dormand-Prince method.
- Simulation results demonstrate the superior performance of Li-hyperbolic sine activation function.

In Section 2, the CVZNN model is described with the theoretical results. In the next section, the Li-hyperbolic sine activation function is defined for complex-valued time-varying matrix inversion. Section 4 deals with different complex-valued time-varying equations. Simulation results are shown in Section 5 for the example problems. Concluding remarks are drawn in Section 6.

2. Model description

This section deals with a glimpse of CVZNN and various activation functions used in real-valued matrix inversion problems.

2.1. Preliminaries

The time varying matrix equation with complex elements [24] are given as,

$$M(t_m)N(t_m) = I, \quad t_m \in [0, +\infty), \tag{1}$$

where $M(t_m) \in \mathbb{C}^{n\times n}$ is a given complex matrix along with the known derivative, I denotes the identity matrix and $N(t_m) \in \mathbb{C}^{n\times n}$ is the unknown complex matrix that is to be computed. Equation (1) can be rewritten as follows,

$$[M_{re}(t_m) + iM_{im}(t_m)][N_{re}(t_m) + iN_{im}(t_m)] = I, \tag{2}$$

where, $M_{re}(t_m)$ and $M_{im}(t_m)$ denotes the real and imaginary parts of $M(t_m) \in \mathbb{C}^{n\times n}$ respectively. Likewise, $N_{re}(t_m)$ and $N_{im}(t_m)$ are to $N(t_m) \in \mathbb{C}^{n\times n}$. The value $i = \sqrt{-1}$. Expanding equation (2),

$$\begin{aligned} M_{re}(t_m)N_{re}(t_m) - M_{im}(t_m)N_{im}(t_m) = I,\\ M_{re}(t_m)N_{im}(t_m) + M_{im}(t_m)N_{re}(t_m) = 0. \end{aligned} \tag{3}$$

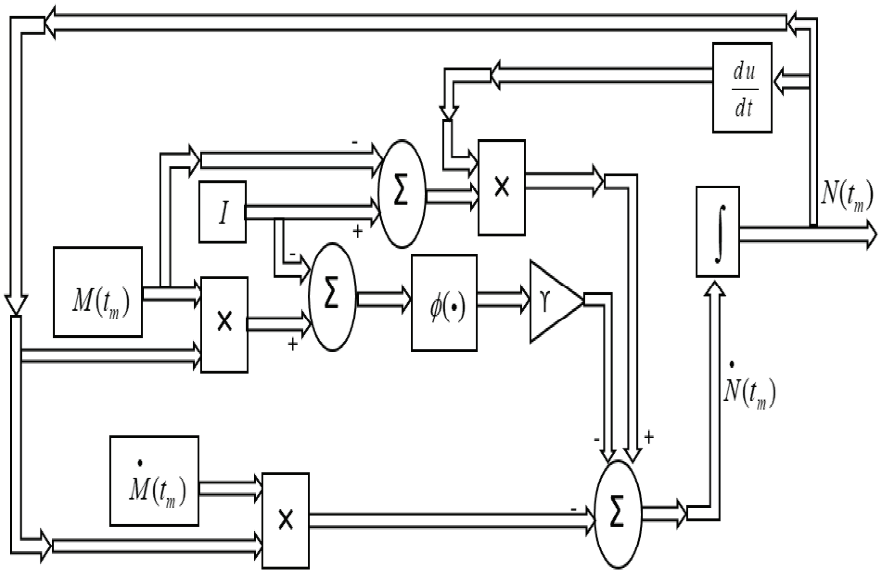

Figure 1: Block diagram of the complex Zhang Neural Network for solving the time-varying complex matrix inversion.

The error for CVZNN is described as following,

$$E(t_m) = M(t_m)N(t_m) - I, \quad t_m \in [0, +\infty). \tag{4}$$

The error function is designed in such a way that it converges to zero as time goes to infinity. According to the Zhang's design formula, the derivative of the error is given as,

$$\dot{E}(t_m) = -\gamma\Phi(E(t_m)), \tag{5}$$

where $\gamma > 0$ and $\Phi(.)$ is the activation function. Substituting equation (4) in (5),

$$M(t_m)\dot{N}(t_m) = -\dot{M}(t_m)N(t_m) - \gamma\Phi(M(t_m)N(t_m) - I), \tag{6}$$

where unknown matrix $N(t_m) \in \mathbb{C}^{n\times n}$ is computed, starting with an initial random condition $N(0) \in \mathbb{C}^{n\times n}$ which approaches to the theoretical inverse solution $N^*(t_m) = M^{-1}(t_m) \in \mathbb{C}^{n\times n}$ from (1) as time goes to infinity.

2.2. Complex-valued time varying matrix

While working with complex domains, it is unnecessary to convert the complex-valued equations to real-valued equations and then convert them back to the complex domain on the application of linear activation function. However, it is not a conventional task to apply nonlinear activation from real domain to complex domain and still guarantee global convergence. This could be the reason for choosing only linear activation function while dealing with CVZNN in [24]. Therefore, the application of nonlinear activation functions on the complex domain for time-varying matrix inversion is examined. A complex number $Z = \Re + i\Im$ can be represented by a point $(\Re, \Im)$ on the Cartesian plane. On application of an activation function to a complex number represented in Cartesian form, real part$(\Re)$ and the imaginary part$(\Im)$ is treated as two real parts,

$$\Phi_1(\Re + i\Im) = \phi(\Re) + i\phi(\Im). \tag{7}$$

Theorem 1. *Consider time varying equation $M(t_m)N(t_m) = I \in \mathbb{C}^{n\times n}$, where $M(t_m)$ is assumed to be nonsingular. If a monotonically increasing odd activation function $\Phi_1(.)$ is applied to complex valued matrix represented in Cartesian plane, then the state matrix $N(t_m)$ of CVZNN model (6), starting from a randomly-generated initial state $N(0) \in \mathbb{C}^{n\times n}$, can globally converge to the unique time-varying theoretical solution (1) $N^*(t_m)$. In addition, by choosing different nonlinear activation functions, the convergence rate of CVZNN model (6) can be accelerated accordingly.*

Proof. From equation (5), $\dot{E}(t_m) = -\gamma\Phi_1(E(t_m))$ has a set of n^2 decoupled differential equations and can be expressed as follows,

$$\dot{e}_{a,b}(t_m) = -\gamma\Phi_1(e_{a,b}(t_m)), \quad \forall\, a, b \in \{1, 2, 3, ..., n\}. \tag{8}$$

Using the definition of Φ_1, if $\dot{\Re}_{a,b}(t_m)$ and $\dot{\Im}_{a,b}(t_m)$ are the real and imaginary parts of $\dot{e}_{a,b}(t_m)$ respectively, then

$$\dot{\Re}_{a,b}(t_m) = -\gamma\phi(\Re_{a,b}(t_m)), \tag{9}$$

$$\dot{\Im}_{a,b}(t_m) = -\gamma\phi(\Im_{a,b}(t_m)). \tag{10}$$

Therefore, to analyse the ab^{th} subsystem (8), a Lyapunov function candidate $V(t_m) = e^T(t_m)e(t_m)$ is chosen as discussed in [5]. For ab^{th} subsystem, $e_{a,b}(t_m)$ is a scalar complex value, therefore $V(t_m) = e^2_{a,b}(t_m)$. $\dot{V}_{a,b}(t_m)$ is,

$$\dot{V}_{a,b}(t_m) = e_{a,b}(t_m)\dot{e}_{a,b}(t_m) = -\gamma e_{a,b}(t_m)\Phi(e_{a,b}(t_m)). \tag{11}$$

Since, $e_{a,b}(t_m)$ contain both real and imaginary parts rewriting Lyapunov function and its derivative for real and imaginary parts,

$$V\Re_{a,b}(t_m) = \Re^2_{a,b}(t_m)/2, \tag{12}$$

$$\dot{V}\Re_{a,b}(t_m) = -\gamma\Re_{a,b}(t_m)\phi(\Re_{a,b}(t_m)), \quad (13)$$

$$V\Im_{a,b}(t_m) = \Im^2_{a,b}(t_m)/2, \quad (14)$$

$$\dot{V}\Im_{a,b}(t_m) = -\gamma\Im_{a,b}(t_m)\phi(\Im_{a,b}(t_m)). \quad (15)$$

Since $\phi(\cdot)$ is a odd monotonically increasing activation function, $\phi(-\Re_{a,b}(t_m))$ $= -\phi(\Re_{a,b}(t_m))$, then

$$\Re_{a,b}(t_m)\phi(\Re_{a,b}(t_m)) \begin{cases} > 0, if \;\; \Re_{a,b}(t_m) \neq 0, \\ = 0, if \;\; \Re_{a,b}(t_m) = 0, \end{cases} \quad (16)$$

which assures the negative definiteness of $\dot{V}\Re_{a,b}(t_m)$ (i.e., $\Re_{a,b}(t_m)$ $\forall\, a,b$ globally converges to zero). Similarly, negative definiteness of $\dot{V}\Im_{a,b}(t_m)$ can be shown, and hence $\Im_{a,b}(t_m)\forall a,b$ also globally asymptotically converges to zero. This implies the error matrix $E(t_m)$ globally converges to zero. □

Remark 2. Theorem 1 states that odd monotonically increasing activation function can be applied to CVZNN. Some well established real valued nonlinear activation functions found in literature are as follows:

1. Power Function [24] : $\phi(e_i) = e_i^p$, where p is an odd integer, $p \geq 3$.
2. Power Sum Function [17] : $\phi(e_i) = \sum_{k=1}^{N} e_i^{2k-1}$, with integer $N > 1$.
3. Power Sigmoidal Function [20] :
   $\phi(e_i) = \frac{1+\exp(-\xi)}{1-\exp(-\xi)} \cdot \frac{1-\exp(-e_i\xi)}{1+\exp(-e_i\xi)}$, if $|e_i| < 1$,
   e_i^p, if $|e_i| \geq 1$, where p is an odd integer and $\xi > 0$.
4. Bipolar Sigmoidal Function [18] : $\phi(e_i) = \frac{1+\exp(-\xi)}{1-\exp(-\xi)} \cdot \frac{1-\exp(-\xi e_i)}{1+\exp(-\xi e_i)}$, where $\xi > 2$.
5. Hyperbolic Sine Function [23] : $\phi(e_i) = \frac{\exp(e_i m)}{2} - \frac{\exp(-e_i m)}{2}$, where m is an odd integer.

Remark 3. The chances of nonlinearity in the system might occur even if the linear activation function is applied to CVZNN. This could be due to a various factors, including truncation or rounding off values, as well as an irregularity in line slope [9]. Thus, the applications of various activation functions upon CVZNN give insight knowledge about the convergence rate at the network.

Remark 4. Nonlinear activation functions such as power, power sum, bipolar sigmoidal, power sigmoidal, and hyperbolic sine functions are applied to CVZNN. It is to be noted that the network converges with the hyperbolic sine activation function (5) faster compared to other activation functions. This is due to the extra parameter m that amplifies the error in the Equation (6). Due to the large gradient and absolute value of the error, the hyperbolic sine activation function produces a better convergence rate.

3. Li-hyperbolic sine activation function

Another alternative way of representing a complex number is in its polar form [7],

$$\Phi_2(\Re + i\Im) = \phi(\Gamma) \circ \exp(i\theta), \tag{17}$$

where, $\Re \in R^{n\times n}$, $\Im \in R^{n\times n}$, $\Gamma \in R^{n\times n}$, $\theta \in (-\pi, \pi]^{n\times n}$ and $\circ$ denotes element-wise multiplication. When a complex valued matrix is written in polar form, a novel nonlinear activation function called the Li-hyperbolic sine activation function is constructed, which makes the network converge to zero faster than the hyperbolic sine function:

$$\phi(e_{a,b}) = Lip(F(e_{a,b}, m), m)^m + Lip(F(e_{a,b}, m), m)^{1/m}, \tag{18}$$

$$F(e_{a,b}, m) = \frac{\exp(e_{a,b}m)}{2} - \frac{\exp(-e_{a,b}m)}{2}, \tag{19}$$

$$Lip(F(e_{a,b}, m), m) = |F(e_{a,b}, m)|^m, \tag{20}$$

where m is an odd integer and $e_{a,b}$ is a complex number. In order to prove the convergence of network with Li-hyperbolic sine activation function, consider Lyapunov function $V_e(t_m) = |e_{a,b}(t_m)|^2 = (e_{a,b}(t_m)\overline{e_{a,b}(t_m)})$, where $e_{a,b}(t_m)$ is a complex value, $|e_{a,b}(t_m)|$ represents the modulus of $e_{a,b}(t_m)$ and $\overline{e_{a,b}(t_m)}$ denotes the complex conjugate of the complex number $e_{a,b}(t_m)$. By definition of $\Phi_2(.)$, it can be written as $\Phi_2(e_{a,b}(t_m)) = \phi(|e_{a,b}(t_m)|) \cdot \exp(\sqrt{-1}. \arg(e_{a,b}(t_m)))$. The derivative of Lyapunov function can be written as,

$$\begin{aligned}
\dot{V}_e(t_m) &= -\gamma e_{a,b}(t_m)\overline{\Phi_2(e_{a,b}(t_m))} - \gamma\overline{e_{a,b}(t_m)}\Phi_2(e_{a,b}(t_m)), \\
&= -\gamma e_{a,b}(t_m)\phi(|e_{a,b}(t_m)|) \cdot \exp(-\sqrt{-1}\ \arg(e_{a,b}(t_m))) \\
&\quad - \gamma\overline{e_{a,b}(t_m)}\phi(|e_{a,b}(t_m)|) \cdot \exp(\sqrt{-1}\ \arg(e_{a,b}(t_m))), \\
&= -\gamma|e_{a,b}(t_m)| \cdot \exp(\sqrt{-1}\ \arg(e_{a,b}(t_m)))\phi(|e_{a,b}(t_m)|)
\end{aligned}$$

$$
\begin{aligned}
&\quad\quad \times\ \exp(-\sqrt{-1}\ \arg(e_{a,b}(t_m))) \\
&\quad - \gamma|e_{a,b}(t_m)| \cdot \exp(-\sqrt{-1}\ \arg(e_{a,b}(t_m)))\phi(|e_{a,b}(t_m)|) \\
&\quad\quad \times\ \exp(\sqrt{-1}\ \arg(e_{a,b}(t_m))), \\
&= \ -2\,\gamma|e_{a,b}(t_m)|\,\phi(|e_{a,b}(t_m)|).
\end{aligned}
$$

Since the absolutes of $|e_{a,b}(t_m)| \geq 0$, $\phi(|e_{a,b}(t_m)|) \geq 0$, hence $\dot{V}_e(t_m)$ is negative definite. This proves the CVZNN model is globally asymptotically stable with respect to Li-hyperbolic sine activation function.

4. Complex-valued time-varying equations

In this section, the representation of complex-valued time-varying linear, Sylvester, Lyapunov equations and Moore-Penrose matrix inversion are discussed.

4.1. Complex valued time-varying linear equation

The general form of a complex-valued time-varying linear equation is given as,

$$A(t_m)X(t_m) = B(t_m), \quad t_m \in [0, +\infty), \tag{21}$$

where $A(t_m) \in \mathbb{C}^{n\times n}$ and $B(t_m) \in \mathbb{C}^{n\times m}$ are complex valued time varying known matrices. $X(t_m) \in \mathbb{C}^{n\times n}$ is the unknown complex matrix to be obtained. Since $A(t_m)$, $B(t_m)$ and $X(t_m)$ are complex value matrices they contain both real (re) and imaginary parts (im) respectively. Equation (21) is therefore written as,

$$[A_{re}(t_m)+iA_{im}(t_m)][X_{re}(t_m) + iX_{im}(t_m)] = [B_{re}(t_m) + iB_{im}(t_m)]. \tag{22}$$

The value of $i = \sqrt{-1}$. Expanding equation (22),

$$
\begin{aligned}
A_{re}(t_m)X_{re}(t_m) - A_{im}(t_m)X_{im}(t_m) = B_{re}(t_m) \in \mathbb{R}^n, \\
A_{re}(t_m)X_{im}(t_m) + A_{im}(t_m)X_{re}(t_m) = B_{im}(t_m) \in \mathbb{R}^n.
\end{aligned} \tag{23}
$$

The error for a linear time-varying equation (21) is given below and the main objective is to minimize the error function:

$$E(t_m) = A(t_m)X(t_m) - B(t_m), \quad t \in [0, +\infty). \tag{24}$$

Time derivative of the error function according to Zhang's design is as follows,

$$\dot{E}(t_m) = -\gamma\Phi(E(t_m)). \tag{25}$$

The parameter $\gamma > 0$ and $\Phi(:)$ is the activation function. The complex valued time varying linear equation is obtained by substituting (24) in (25),

$$A(t_m)\dot{X}(t_m) = -\dot{A}(t_m)X(t_m) + \dot{B}(t_m) - \gamma\Phi(A(t_m)X(t_m) + B(t_m)), \tag{26}$$

where $X(t_m)$ is the matrix to be computed starting with an initial condition. The norm of $A(t_m)X(t_m) - B(t_m)$ is found as time goes to infinity. As time increases the calculated solution $X(t_m)$ becomes equal to the theoretical inverse $X^*(t_m) = A^{-1}(t_m) \times B(t_m) \in \mathbb{C}^n$ making the value of norm reach zero and thus the network converges to zero.

4.2. Complex valued time varying Sylvester equation

Time varying Sylvester equation is given as,

$$A(t_m)X(t_m) + X(t_m)B(t_m) = C(t_m), \quad t_m \in [0, +\infty), \tag{27}$$

where $A(t_m) \in \mathbb{C}^{m\times m}$, $B(t_m) \in \mathbb{C}^{n\times n}$ and $C(t_m) \in \mathbb{C}^{m\times n}$ are known matrices. $X(t_m) \in \mathbb{C}^{m\times n}$ is the unknown matrix. The Sylvester equation has a unique solution if and only if $A(t_m)$ and $-B(t_m)$ do not have a common eigenvalue [7]. Expanding equation (27) in complex form,

$$\begin{aligned} &[A_{re}(t_m) + iA_{im}(t_m)][X_{re}(t_m) + iX_{im}(t_m)] \\ &\qquad + [X_{re}(t_m) + iX_{im}(t_m)][B_{re}(t_m) + iB_{im}(t_m)] \\ &\qquad\qquad = [C_{re}(t_m) + iC_{im}(t_m)]. \end{aligned} \tag{28}$$

Due to the value of $i^2 = -1$, the following is obtained as expanding equation (28):

$$\begin{aligned} &A_{re}(t_m)X_{re}(t_m) - A_{im}(t_m)X_{im}(t_m) \\ &\quad + X_{re}(t_m)B_{re}(t_m) - X_{im}(t_m)B_{im}(t_m) = C_{re}(t_m) \in \mathbb{R}^n, \\ &A_{re}(t_m)X_{im}(t_m) + A_{im}(t_m)X_{re}(t_m) \\ &\quad + X_{re}(t_m)B_{im}(t_m) + X_{im}(t_m)B_{re}(t_m) = C_{im}(t_m) \in \mathbb{R}^n. \end{aligned} \tag{29}$$

The main objective is to minimize the error of the Sylvester equation and the error function is defined as

$$E(t_m) = A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m), t_m \in [0, +\infty). \tag{30}$$

The time derivative of the error function is given as

$$\dot{E}(t_m) = -\gamma\Phi(E(t_m)). \tag{31}$$

As defined earlier the parameter $\gamma > 0$ and $\Phi(:)$ denotes the activation function applied to the network. By substituting (30) in (31),

$$\begin{aligned} &A(t_m)\dot{X}(t_m) + \dot{X}(t_m)B(t_m) \\ &\quad = -\dot{A}(t_m)X(t_m) - X(t_m)\dot{B}(t_m) + \dot{C}(t_m) - \gamma\Phi(A(t_m)X(t_m)) \\ &\qquad - \gamma\Phi(X(t_m)B(t_m)) + \gamma\Phi(C(t_m)). \end{aligned} \tag{32}$$

The solution $X(t_m)$ is obtained by solving the differential equation (32) stating with an initial condition. The norm is calculated for $A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)$. The error, norm $||A(t_m)X(t_m)+X(t_m)B(t_m)-C(t_m)||_F$, reaches zero as time increases which implies that the computed $X(t_m)$ and the theoretical solution $X^*(t_m)$ becomes equal allowing the network to converge to zero.

4.3. Complex-valued time-varying Lyapunov equation

Lyapunov equation with time-varying complex values is given as

$$A^T(t_m)X(t_m) + X(t_m)A(t_m) = C(t_m), \quad t_m \in [0, +\infty), \tag{33}$$

where $A(t_m) \in \mathbb{C}^{n\times n}$ and $C(t_m) \in \mathbb{C}^{n\times n}$ are known matrices, whereas $X(t_m) \in \mathbb{C}^{n\times n}$ is the unknown matrix. $A^T(t_m)$ is the transpose of matrix A at time t_m. Lyapunov equation (33) is a special case of the Sylvester equation (27). The complex form of equation (33) is written as

$$\begin{aligned} &[A^T_{re}(t_m) + iA^T_{im}(t_m)][X_{re}(t_m) + iX_{im}(t_m)] \\ &\quad + [X_{re}(t_m) + iX_{im}(t_m)][A_{re}(t_m) + iA_{im}(t_m)] \\ &\qquad = [C_{re}(t_m) + iC_{im}(t_m)]. \end{aligned} \tag{34}$$

Since $i = \sqrt{-1}$, the expanding equation (34) is

$$\begin{aligned} &A^T_{re}(t_m)X_{re}(t_m) - A^T_{im}(t_m)X_{im}(t_m) \\ &\quad +X_{re}(t_m)A_{re}(t_m) - X_{im}(t_m)A_{im}(t_m) = C_{re}(t_m) \in \mathbb{R}^n, \\ &A^T_{re}(t_m)X_{im}(t_m) + A^T_{im}(t_m)X_{re}(t_m) \\ &\quad +X_{re}(t_m)A_{im}(t_m) + X_{im}(t_m)A_{re}(t_m) = C_{im}(t_m) \in \mathbb{R}^n. \end{aligned} \tag{35}$$

The error function for Lyapunov equation and its time derivative are given as:

$$E(t_m) = A^T(t_m)X(t_m) + X(t_m)A(t_m) - C(t_m), t_m \in [0, +\infty), \quad (36)$$

$$\dot{E}(t_m) = -\gamma\Phi(E(t_m)). \quad (37)$$

The design parameter γ is a positive integer, and $\Phi(:)$ is the activation function applied to the network. Φ can be any monotonically increasing odd activation function. To solve the Lyapunov equation, we substitute (36) in (37),

$$A^T(t_m)\dot{X}(t_m) + \dot{X}(t_m)A(t_m) = -\dot{A}^T(t_m)X(t_m) - X(t_m)\dot{A}(t_m) + \dot{C}(t_m) - \gamma\Phi(A^T(t_m)X(t_m)) - \gamma\Phi(X(t_m)A(t_m)) + \gamma\Phi(C(t_m)). \quad (38)$$

From Equation (38), $X(t_m)$ can be calculated with a given initial condition $X(0)$. The norm of $A^T(t_m)X(t_m) + X(t_m)A(t_m) - C(t_m)$ is found as time goes to infinity. As time increases the calculated solution $X(t_m)$ becomes equal to the theoretical inverse $X^*(t_m)$ from Equation (33) making the value of norm reach zero.

4.4. Complex-valued time-varying Moore-Penrose inversion problem

The Moore-Penrose pseudoinverse for a matrix is unique and is used to estimate an over or under-determined system. There exists a Moore-Penrose inverse $M(t_m)^\dagger \in C^{n\times m}$ for $M(t_m) \in C^{m\times n}$ if the the following conditions are satisfied:

1. $M(t_m)M^\dagger(t_m)M(t_m) = M(t_m)$,
2. $M^\dagger(t_m)M(t_m)M^\dagger(t_m) = M^\dagger(t_m)$,
3. $(M(t_m)M^\dagger(t_m))^T = M(t_m)M^\dagger(t_m)$,
4. $(M^\dagger(t_m)M(t_m))^T = M^\dagger(t_m)M(t_m)$,

where T denotes the transpose of the matrix. A matrix that satisfies only condition 1. is called a generalized inverse. A matrix that satisfies both conditions 1. and 2, is called generalized reflexive inverse. A generalized inverse has no unique solution. Conditions 3. and 4. guarantee a unique solution. Therefore a Moore-Penrose inversion problem has a unique solution, and the corresponding Frobenius norm is minimum than for other inverses.

If $M(t_m) \in C^{m\times n}$ with $m \neq n$, then $rank(M(t_m)) = min(m, n)$ implies $M^T(t_m)M(t_m) \in C^{n\times n}$ or $M(t_m)M^T(t_m) \in C^{m\times m}$ is non singular and thus $(M^T(t_m)M(t_m))^{-1}$ or $(M(t_m)M^T(t_m))^{-1}$ exists. Moore-Penrose inverse $M^\dagger(t_m)$ is given as,

$$M^\dagger(t_m) = \begin{cases} M^T(t_m)(M(t_m)M^T(t_m))^{-1}, & \text{if} \quad m < n, \\ (M^T(t_m)M(t_m))^{-1}M^T(t_m), & \text{if} \quad m > n. \end{cases} \tag{39}$$

Thus $M^\dagger(t_m)$ fulfils the conditions for full rank matrix $M(t_m)$.

$$M^\dagger(t_m)M(t_m)M^T(t_m) = M^T(t_m), \quad \text{if} \quad m < n, \tag{40}$$

$$M^T(t_m)M(t_m)M^\dagger(t_m) = M^T(t_m), \quad \text{if} \quad m > n. \tag{41}$$

Equations (40) and (41) are used in the formulation of Zhang's right and left Moore-Penrose inverse. The error for Equation (40) is given as,

$$E(t_m) = X(t_m)M(t_m)M^T(t_m) - M^T(t_m). \tag{42}$$

According to the derivative of error defined in Zhang neural network, $\dot{E}(t_m) = -\gamma\Phi(E(t_m))$ where $\gamma > 0$ and Φ is the activation function applied to the neural network and $\Phi : C^{n\times m} \to C^{n\times m}$. Substituting Equation (42) in $\dot{E}(t_m)$ and solving for time varying Moore-Penrose right inversion solution $X(t_m)$,

$$\begin{aligned} \dot{X}(t_m)M(t_m)M^T(t_m) &= -\gamma\Phi(X(t_m)M(t_m)M^T(t_m) - M^T(t_m)) \\ &\quad - X(t_m)(\dot{M}(t_m)M^T(t_m) + M(t_m)\dot{M}^T(t_m)) + \dot{M}^T(t_m). \end{aligned} \tag{43}$$

Similarly, the error is formulated for the left Moore-Penrose inverse using Equation (41) is formulated as,

$$E(t_m) = M^T(t_m)M(t_m)X(t_m) - M^T(t_m). \tag{44}$$

Using the definition formulated by Zhang $\dot{E}(t_m) = -\gamma\Phi(E(t_m))$, the time varying Moore-Penrose left inverse is given as,

$$\begin{aligned} M^T(t_m)M(t_m)\dot{X}(t_m) &= -\gamma\Phi(M^T(t_m)M(t_m)X(t_m) - M^T(t_m)) \\ &\quad - (\dot{M}^T(t_m)M(t_m) + M^T(t_m)\dot{M}(t_m))X(t_m) + \dot{M}^T(t_m), \end{aligned} \tag{45}$$

where the unknown matrix $X(t_m)$ is computed for right or left Moore-Penrose inversion using equation (43) and (45) respectively, starting with an initial random condition $X(0)$. The norm (i.e., $||X(t_m)$ $M(t_m)M^T(t_m) - M^T(t_m)||_F$) reaches zero as time increases, which implies that the computed $X(t_m)$ and the theoretical solution $M^\dagger(t_m)$ becomes equal. The norm (i.e., $||M^T(t_m)M(t_m)X(t_m) - M^T(t_m)||_F$) reaches zero as time increases, which implies that the computed $X(t_m)$ and the theoretical solution $M^\dagger(t_m)$ becomes equal.

5. Simulation and numerical studies

In this section, numerical simulation results are presented which show the superior performance of the Li-hyperbolic sine activation function using CVZNN model by solving complex valued time varying problems with certain selected examples. Activation functions are applied to the network to improve rate of convergence. CVZNN model for solving complex valued time varying problems are activated using various activation functions outlined in Remark 2 of Section 2 and the proposed Li-hyperbolic sine activation function as in Section 3. For all the selected example problems the following parameters are fixed with respect to corresponding activation functions: $\gamma = 10$, $p = 3$ for power function, $N = 5$ for power sum, $p = 3$ and $\xi = 3$ for power sigmoidal, $\xi = 3$ for bipolar sigmoidal, $m = 3$ for hyperbolic sine and Li-hyperbolic sine activation function.

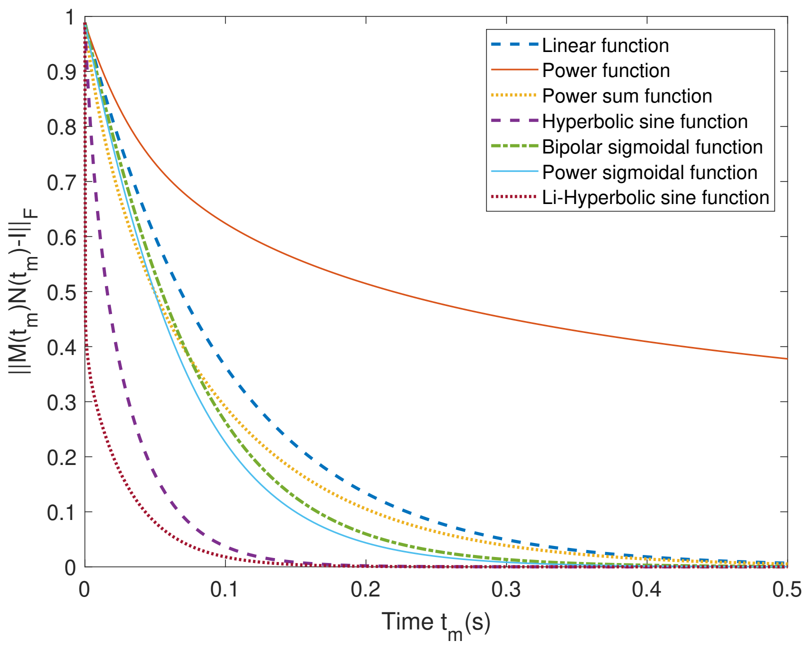

Figure 2: Comparison of error residuals $||M(t_m)N(t_m) - I||_F$ with various activation functions corresponding to Example 1 with $n = 4$.

Example 1: Toeplitz matrix inversion using CVZNN model

Let us consider a Toeplitz matrix with complex valued time varying elements $M(t_m) \in \mathbb{C}^{n\times n}$. Let, $a_1(t_m) = n + \sin(it_m)$ and $a_k(t_m) = \cos(it_m)/(k-1)$ for $k = 2, 3, ..., n$. According to [26], $M(t_m)$ is a diagonal dominant matrix $\forall t_m \geq 0$, therefore $M(t_m)$ is invertible.

$$M(t_m) = \begin{pmatrix} a_1(t_m) & a_2(t_m) & a_3(t_m) & \dots & a_n(t_m) \\ a_2(t_m) & a_1(t_m) & a_2(t_m) & \dots & a_{n-1}(t_m) \\ a_3(t_m) & a_2(t_m) & a_1(t_m) & \dots & a_{n-2}(t_m) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n(t_m) & a_{n-1}(t_m) & a_{n-2}(t_m) & \dots & a_1(t_m) \end{pmatrix}. \qquad (46)$$

For this example, the Toeplitz matrix $M(t_m)$ with $n = 4$ and $n = 6$. From equation (1), $M(t_m)$ is the known matrix, $N(t_m)$ is the unknown matrix to be obtained and I is the identity matix. The CVZNN time-varying inverse for the matrix $M(t_m)$ (i.e., $N(t_m)$) is calculated using equation (6) the application of each of the activation functions discussed in Remark 2 (Section 2) and Section 3. The Frobenius norm of $M(t_m)N(t_m) - I$ is calculated to find the error and is shown in Figure 2 and 3 for $n = 4$ and $n = 6$ respectively. From both the figures, it can be noticed that the network shows faster convergence when Li-hyperbolic sine activation function is used on comparison with linear, power, power sum, bipolar sigmoidal, power sigmoidal and hyperbolic sine activation functions.

Example 2: Solution to linear equation

Let us consider the following complex valued time varying linear matrix problem with $A(t_m)$ and $B(t_m)$ as,

$$A(t_m) = \begin{pmatrix} 10 + \sin(it_m) & 2 + \cos(it_m) & 2 + \sin(2it_m) \\ 15 - \cos(it_m) & 5 + \sin(-it_m) & \sin(it_m) \\ \cos(t_m) & \sin(2t_m) + i & \cos(3it_m) \end{pmatrix}, \qquad (47)$$

$$B(t_m) = \begin{pmatrix} \sin(t_m) + \cos(t_m) & \cos(t_m) \\ \cos(it_m) + \sin(t_m) & \sin(it_m) \\ \cos(2it_m) & \sin(it_m) \end{pmatrix}. \qquad (48)$$

The complex valued time varying linear matrix problem is represented as $A(t_m)X(t_m) - B(t_m) = 0$. For the selected example $A(t_m) \in \mathbb{C}^{3\times 3}$ and

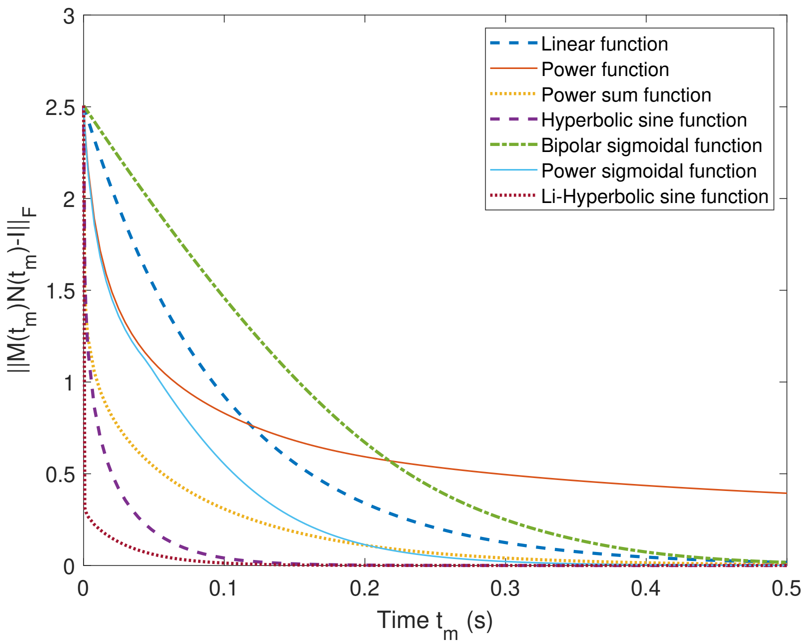

Figure 3: Comparison of error residuals $||M(t_m)N(t_m) - I||_F$ with various activation functions corresponding to Example 1 with $n = 6$.

$B(t_m) \in \mathbb{C}^{3\times 2}$. The unknown matrix $X(t_m) \in \mathbb{C}^{3\times 2}$ can be obtained by solving Equation (26). Figure 4 shows the error residuals $||A(t_m)X(t_m) - B(t_m)||_F$ with various activation functions. It is observed that Li-hyperbolic sine activation function converges to zero faster than other nonlinear activation functions mentioned in Remark 2.

Example 3: Sylvester equation

We consider complex-valued time varying Sylvester equation with two different cases:

Case 1:

Let us consider special case of Sylvester equation $A(t_m)X(t_m) + X(t_m)B(t_m) = C(t_m)$ in which $A(t_m) \in \mathbb{C}^{2\times 2}$ is a non singular matrix, $X(t_m)$ is the unknown

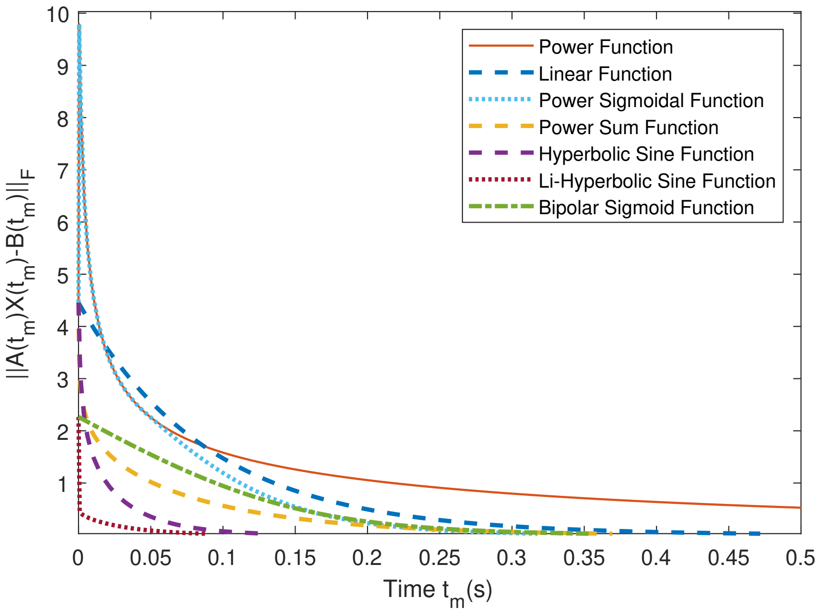

Figure 4: Comparison of error residuals $||A(t_m)X(t_m) - B(t_m)||_F$ with various activation functions corresponding to Example 2.

matrix, $B(t_m) = 0 \in \mathbb{R}^{3\times3}$ and $C(t_m) \in \mathbb{R}^{2\times3}$,

$$A(t_m) = \begin{pmatrix} \exp(10it_m) & -i\exp(-10it_m) \\ -i\exp(10it_m) & \exp(-10it_m) \end{pmatrix}, \tag{49}$$

$$C(t_m) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{50}$$

This example is a particular case of matrix inversion problem. The solution for CVZNN time-varying Sylvester equation $X(t_m)$ is calculated using equation (32) with the application of each activation function. Further, Frobenius norm of $A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)$ is calculated to find the error and is shown in Figure 5. It can be noticed that the network shows faster convergence when Li-hyperbolic sine activation function is used on comparison with other activation functions.

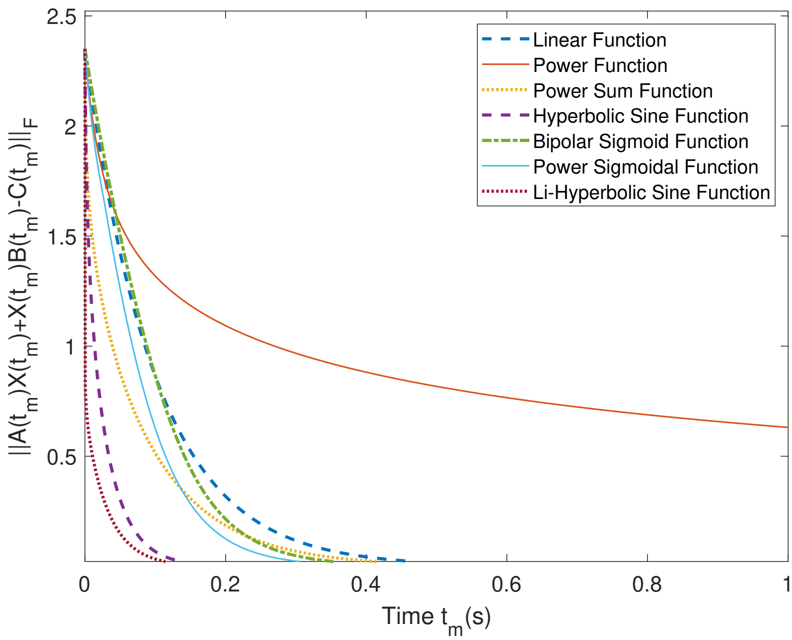

Figure 5: Comparison of error residuals $||A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)||_F$ with various activation functions corresponding to Example 3: Case 1.

Case 2:

In this example let us consider the following complex valued matrices $A(t_m)$, $B(t_m)$ and $C(t_m)$ as,

$$A(t_m) = \begin{pmatrix} 2+\cos(2it_m) & 2\sin(2it_m) \\ \sin(2it_m) & 2-\cos(2it_m) \end{pmatrix}, \tag{51}$$

$$B(t_m) = \begin{pmatrix} 2+\cos(2it_m) & \sin(2it_m) & 0 \\ \sin(2it_m) & 1 & \cos(2it_m) \\ \cos(2it_m) & \sin(2it_m) & 2 \end{pmatrix}, \tag{52}$$

$$C(t_m) = \begin{pmatrix} C(t_m)_{1,1} & C(t_m)_{1,2} & C(t_m)_{1,3} \\ C(t_m)_{2,1} & C(t_m)_{2,2} & C(t_m)_{2,3} \end{pmatrix}, \tag{53}$$

$$C(t_m)_{1,1} = -2+\cos(2it_m),\ C(t_m)_{1,2} = 1+\sin(2it_m),$$

$$C(t_m)_{1,3} = \cos(2it_m),\ C(t_m)_{2,1} = \sin(2it_m),$$
$$C(t_m)_{2,2} = -2\sin(2it_m),\ C(t_m)_{2,3} = 1 + \cos(2it_m).$$

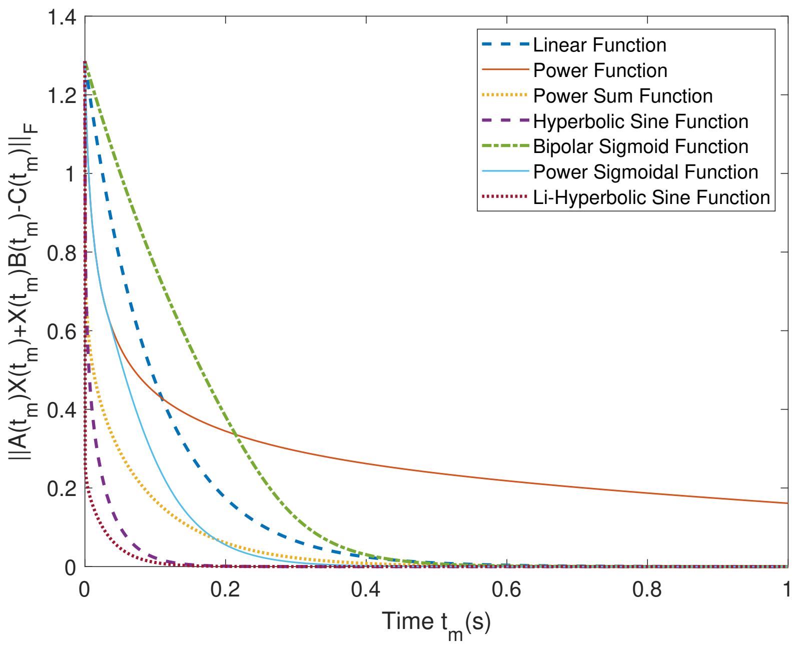

Figure 6: Comparison of error residuals $||A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)||_F$ with various activation functions corresponding to Example 3: Case 2

With the application of each activation function, the solution for the CVZNN time-varying Sylvester equation $X(t_m)$ is determined using equation (32). To estimate the error, the Frobenius norm of $A(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)$ is calculated and recorded in Figure 6. In comparison to linear, power, power sum, bipolar sigmoidal, power sigmoidal, and hyperbolic sine activation functions, the network exhibits faster convergence when Li-hyperbolic sine activation function is applied. Therefore the performance of the network has been effective with Li-hyperbolic sine activation function.

Example 4: Lyapunov equation

The efficacy of Li-hyperbolic sine activation function for a complex valued time varying Lyapunov equation is examined. Let us consider the following matrices $A(t_m)$ and $C(t_m)$,

$$A(t_m) = \begin{pmatrix} -1 - 1/2\cos(2it_m) & 1/2\sin(2it_m) \\ 1/2\sin(2it_m) & -1 + 1/2\cos(2it_m) \end{pmatrix}, \tag{54}$$

$$C(t_m) = \begin{pmatrix} \sin(2it_m) & \cos(2it_m) \\ -\cos(2it_m) & \sin(2it_m) \end{pmatrix}. \tag{55}$$

Using equation (38), $X(t_m)$ is calculated for complex valued time varying Lyapunov equation with a given initial condition. $\Phi(:)$ takes any of the nonlinear activation function discussed in Remark 2 (Section 2) or the proposed Li-hyperbolic sine activation function. The norm for $A^T(t_m)X(t_m) + X(t_m)A(t_m) - C(t_m)$ is calculated and plotted in Figure 7 for power, power sum, bipolar sigmoidal, power sigmoidal, hyperbolic sine and the Li-hyperbolic sine activation functions. It can be observed that Li-hyperbolic sine activation function converges to zero faster than the other activation function for a Lyapunov equation.

Example 5: Moore-Penrose inversion problem

We consider complex-valued time varying Moore-Penrose inversion problem with two different cases:

Case 1:

Let $M(t_m)$ be a the given matrix for which the inverse is to be obtained:

$$M(t_m) = \begin{pmatrix} \sin(2it_m) & \cos(2it_m) & -\sin(2it_m) \\ -\cos(2it_m) & \sin(2it_m) & \cos(2it_m) \end{pmatrix}. \tag{56}$$

Here $M(t_m) \in \mathbb{C}^{2\times 3}$. Since $m < n$, the problem can be solved using right Moore-Penrose inverse approach. Using equation (43), the solution $X(t_m)$ is obtained using various activation functions. To estimate the error, the Frobenius norm of $X(t_m)M(t_m)M^T(t_m) - M^T(t_m)$ is calculated and recorded in Figure 8. In comparison to linear, power, power sum, bipolar sigmoidal, power sigmoidal, and hyperbolic sine activation functions, the network exhibits faster convergence when Li-hyperbolic sine activation function is applied.

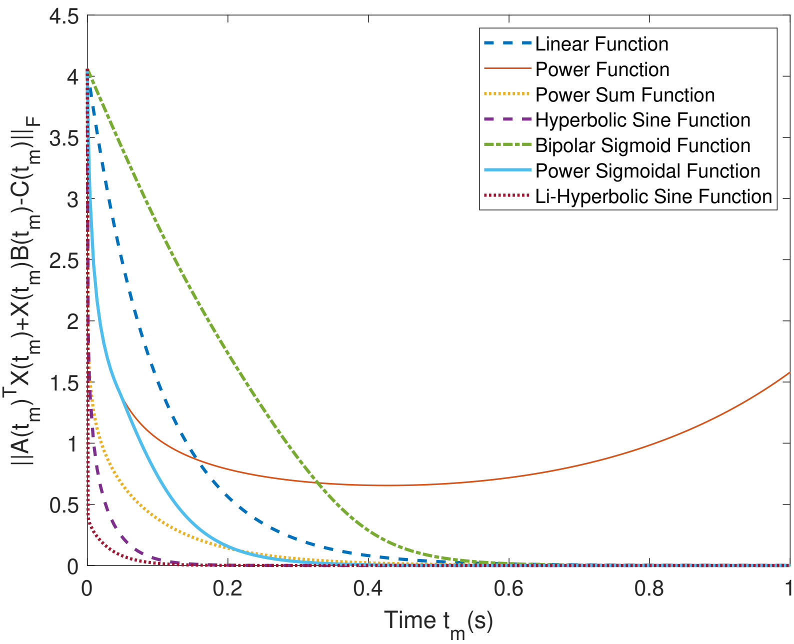

Figure 7: Comparison of error residuals $||A^T(t_m)X(t_m) + X(t_m)B(t_m) - C(t_m)||_F$ with various activation functions corresponding to Example 4.

Case 2:

Let $M(t_m)$ be a the given matrix for which the inverse is to be obtained:

$$M(t_m) = \begin{pmatrix} \sin(2it_m) & \cos(2it_m) \\ -\cos(2it_m) & \sin(2it_m) \\ \sin(2it_m) & \cos(2it_m) \end{pmatrix}, \tag{57}$$

$M(t_m) \in \mathbb{C}^{3\times 2}$ with dimensions $m = 3$ and $n = 2$. This problem can be solved using the left Moore-Penrose inverse method since $m > n$. For various activation functions, the solution $X(t_m)$ is obtained using equation (45). The Frobenius norm of $M^T(t_m)M(t_m)$
$X(t_m) - M^T(t_m)$ is calculated and recorded in Figure 9 to estimate the error. It is found, when Li-hyperbolic sine activation function is used the network converges faster compared to other activation functions.

Table 2 shows the time taken by the CVZNN model for all the examples

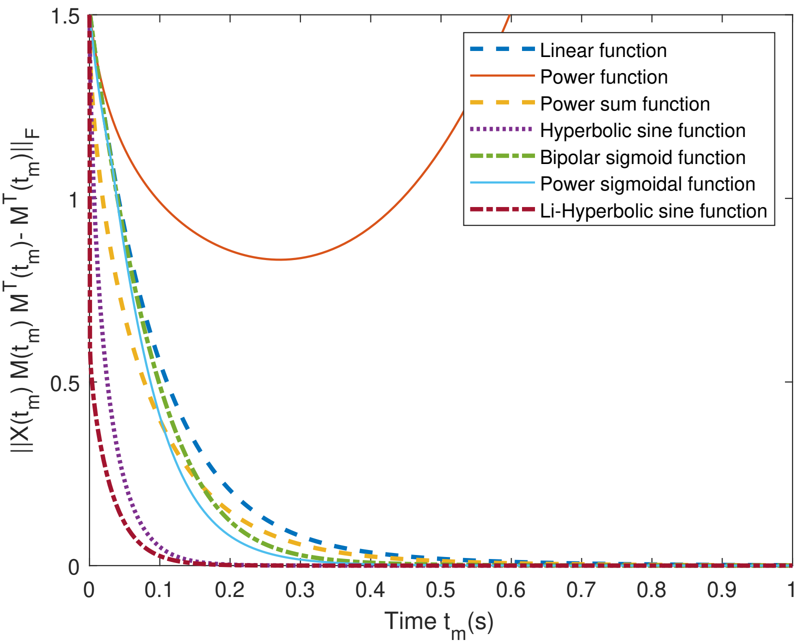

Figure 8: Comparison of error residuals $||X(t_m)M(t_m)M^T(t_m) - M^T(t_m)||_F$ with various activation functions corresponding to Example 5: Case 1.

with various activation functions to reach zero (i.e., when the error between the actual solution and the calculated solution approaches zero). The network does not converge with power activation function until one second, as seen in the Figure 2,3,...,9 for all the considered examples. From Table 2 it can also be observed that the network with the proposed Li-hyperbolic sine activation takes the least time to converge compared to the other activation functions.

6. Conclusions

In this paper, a new nonlinear activation function called the Li-hyperbolic sine is proposed and investigated with the complex-valued Zhang neural network model for complex-valued time-varying problems. Theoretical analysis proves that the network converges when the Li-hyperbolic sine activation function is used. The network's performance is compared to various nonlinear activation

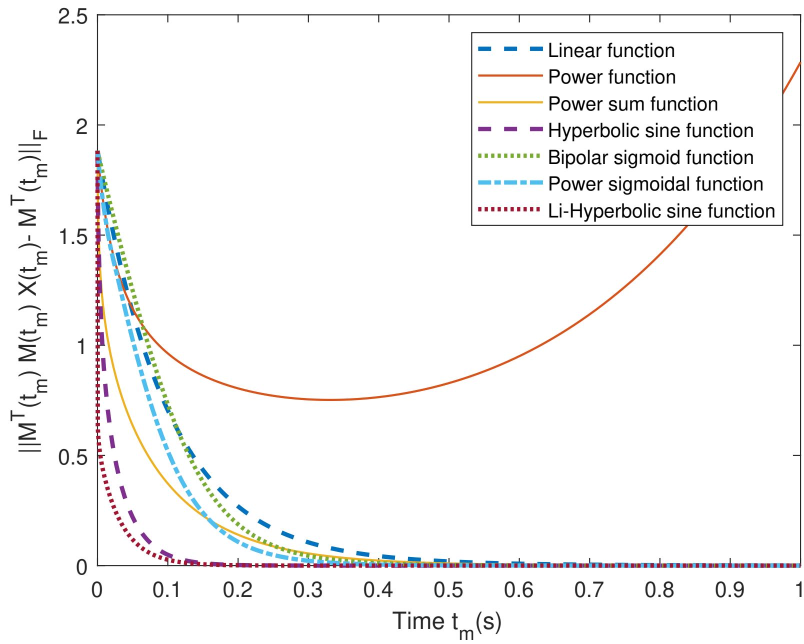

Figure 9: Comparison of error residuals $||M^T(t_m)M(t_m)X(t_m) - M^T(t_m)||_F$ with various activation functions corresponding to Example 5: Case 2.

functions to see how well it performs with the proposed activation function. Numerical simulation results have justified the efficiency and superiority of the network with Li-hyperbolic sine activation function over other activation functions for time-varying complex problems such as matrix inversion, linear equation, Sylvester equation, Lyapunov equation and Moore-Penrose inversion.

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Table 2: Time taken in seconds for convergence of a CVZNN model

	Linear	Power sum	Hyperbolic sine	Bipolar sigmoidal	Power sigmoidal	Li-hyperbolic sine
Example 1						
$n = 4$	0.4908	0.5978	0.2002	0.4268	0.3887	0.1737
$n = 6$	0.7143	0.6061	0.2179	0.6474	0.4473	0.1666
Example 2	0.7502	0.6231	0.2201	0.5488	0.4736	0.1847
Example 3						
case 1	0.7097	0.6581	0.2198	0.5137	0.4518	0.1940
case 2	0.6747	0.5996	0.2011	0.5135	0.4289	0.1668
Example 4	0.8328	0.6874	0.2119	0.7375	0.4816	0.1764
Example 5						
case 1	0.8926	0.8020	0.2146	0.5279	0.4532	0.1910
case 2	0.8018	0.7194	0.2893	0.5302	0.4578	0.2685

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