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A GENERALIZATION OF TRIPLE STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

Carlos Granados[1 §], Ajoy Kanti Das[2]
[1]Estudiante de Doctorado en Matemáticas
Magister en Ciencias Matemáticas
Universidad de Antioquia
Medellín, COLOMBIA
[2] Department of Mathematics
Bir Bikram Memorial College
Agartala-799004
Tripura, INDIA

Abstract: In this paper, we introduce a class of summability methods that can be applied to λ-triple statistical convergence in topological groups and we show some interesting results.

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1. Introduction and definitions preliminaries

Looking through historically at statistical convergence of single sequences, we shall recall that the notion of statistical convergence of sequences was first studied by Fast [3]. The notion of statistical convergence of a sequence (x_a) in a locally convex Hausdorff topological linear space X was presented recently by Maddox [8], where it was shown that the slow oscillation of (s_a) was a Tauberian condition for the statistical convergence of (s_a). In [7], statistical convergence to normed spaces was extended by Kolk. Further in [1] and [2], Cakalli extended

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§Correspondence author

this notation to topological Hausdorff groups. The study of triple sequence in different fields of sequences spaces has grown in the last decade (see [4, 5, 6]).

By the convergence of a triple sequence, we mean the convergence in Pringsheim's sense [9]. A triple sequence $x = (x_{asd})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{asd} - \psi| < \varepsilon$ whenever $a, s, d \geq N$. ψ is called the Pringsheim limit of x. Furthermore, A triple sequence $(x = x_{asd})$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{pql} - x_{asd}| < \varepsilon$ for all $p \geq a \geq N, q \geq s \geq N$ and $l \geq d \geq N$. In a topological group E, the above definitions become as in the following: a triple sequence $x = (x_{asd})$ in E is said to be convergent to ψ in E in the Pringsheim's sense if for every neighbourhood V of 0 there exists $N \in \mathbb{N}$ such that $x_{asd} - \psi \in V$ whenever $a, s, d \geq N$. ψ is called the Pringsheim limit of x. A triple sequence $x = (x_{asd})$ is said to be a Cauchy sequence if for every neighbourhood V of 0 there exists $N \in \mathbb{N}$ such that $x_{pql} - x_{asd} \in V$ for all $p \geq a \geq N, q \geq s \geq N$ and $l \geq d \geq N$.

By E, we will denote an Abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset B of E, $s(B)$ will denote the set of all sequences (x_a) such that (x_a) is in B for $a = 1, 2, 3, ...$ $c(E)$ will denote the set of all convergent sequences. On the other hand, a sequence (x_a) in E is called statistically convergent to an element ψ of E if for each neighbourhood V of 0, (see [2]) $\lim_{a\to\infty} \frac{1}{a}|z \leq a : x_a - \psi \notin V\}| = 0$, and is called statistically Cauchy in E if for each neighbourhood V of 0 there exists a positive integer $a_0(V)$, depending on the neighbourhood V, such that $\lim_{a\to\infty} \frac{1}{a}|z \leq a : x_a - x_{a_0(V)} \notin V\}| = 0$ where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in E is denoted by $S(E)$ and the set of all statistically Cauchy sequences in E is denoted by $SC(E)$. It is known that $SC(E) = S(E)$ if E is complete. Additionally, those notions and the notion λ-statistical convergent were extended for double sequences by Savas [11].

On the other hand, let $\lambda = (\lambda_p, \mu = (\lambda_q)$ and $\phi = (\phi_l)$ be three non-decreasing sequences of positive real numbers, three of them of which tends to ∞ as p, q and l approach ∞, respectively. Besides, let $\lambda_{p+1} \leq \lambda_p + 1, \lambda_1 = 1$, $\mu_{q+1} \leq \mu_q + 1, \mu_1 = 1$ and $\phi_{l+1} \leq \phi_l + 1, \phi_1 = 1$. The collection of such sequence will be denoted by Δ. We write the generalized double de la Valée-Poussin mean by

$$t_{p,q,l}(x) = \frac{1}{\lambda_p \mu_q \phi_l} \sum_{a \in I_p, s \in J_q, d \in W_l} x_{asd},$$

where $I_p = [p - \lambda_p + 1, p], J_Q = [Q - mu_q + 1, q]$ and $W_l = [l - \phi_l + 1, l]$.

Throughout this paper we shall denote $\lambda_p \mu_q \phi_l$ by λ_{pql} and $(a \in I_p, s \in J_q, d \in W_l)$ by $(a, s, d) \in I_{pql}$.

The aim of this paper is to introduce the λ-triple statistical convergence of triple sequences in topological groups and to prove some useful theorems.

2. λ-Triple statistical convergence

Let $J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a three-dimensional set of positive integers and let $J(q, w, e)$ be the numbers of (a, s, d) in J such that $a \leq q, s \leq w$ and $d \leq e$. Then, the three-dimensional analogue of natural density can be defined as follows. The lower asymptotic density of a set $J \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_3}(J) = \lim_{q,w,e} \inf \frac{J(q, w, e)}{qwe},$$

In case that the sequence $(\frac{J(q, w, e)}{qwe})$ has a limit in Pringsheim's sense, then we say that J has a triple natural density and is defined as

$$\delta_3(J) = \lim_{q,w,e} \frac{J(q, w, e)}{qwe}.$$

Sahiner and Tripahy [10] called a real triple sequence $x = (x_{asd})$ statistically convergent to the number ψ if for each $\varepsilon > 0$, the set $\{(a, s, d), a \leq q, s \leq w$ and $d \leq e : |x_{asd} - \psi| \geq \varepsilon\}$ has triple natural density zero. In this case, we write $S_3\text{-}\lim_{a,s,d} x_{asd} = \psi$ and we denote the set of all statistically convergent triple sequences by S_3. Now, we define statistical convergence of triple sequences $x = (x_{asd})$ in a topological group in the following.

Definition 1. A triple sequence $x = (x_{asd})$ is statistically convergent to a point ψ of E if for each neighbourhood V of 0 the set

$$\{(a, s, d), a \leq q, s \leq w \text{ and } d \leq e : x_{asd} - \psi \notin V\}$$

has a triple natural density zero. In this case, we write $S_3(E)\text{-}\lim_{a,s,d} x_{asd} = \psi$ and we write the set of all statistically convergent triple sequences by $S_3(E)$.

Definition 2. A triple sequence $x = (x_{asd})$ is said to be S^3_λ-convergent to ψ of E (or λ-triple statistically convergent to ψ of E) of for each neighbourhood V of 0, the set

$$\{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin V\}$$

has triple natural density zero. In this case we write S^3_λ-$\lim_{a,s,d\to\infty} x_{asd} = \psi$ or $x_{asd} \to \psi(S^3_\lambda)$, and we write the set of all λ-statistically convergent triple sequences by $S^3_\lambda(E)$.

Remark 3. A λ-statistically convergent triple sequence has a unique limit, i.e. if x is λ-statistically convergent to elements ψ_1 and ψ_2 of E, then $\psi_1 = \psi_2$.

Theorem 4. *A triple sequence $x = (x_{asd})$ in E is λ-triple statistically convergent to ψ if and only if there exists a subset $J \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\delta^3_\lambda(J) = 1$ and $\lim_{a,s,d\to\infty} x_{asd} = \psi$ where limit is being taken over the set E, i.e. $(a,s,d) \in E$.*

Proof. Necessity: Let us suppose that x be λ-triple statistically convergent to ψ, and (V_r) be a base of nested closed neighbourhood of 0. Now, write $J_r = \{(a,s,d) \in I_{pql} : x_{asd} - \psi \notin V_r\}$ and $Q_r = \{(a,s,d) \in I_{pql} : x_{asd} - \psi \in V_r\}$ where $r = 1,2,3,....$ Then, $\delta^3_\lambda(J_r) = 0$ and

$$Q_1 \supset Q_2 \supset ... \supset Q_a \supset Q_{a+1} \supset ... \tag{1}$$

and

$$\delta^3_\lambda(Q_r) = 1, r = 1,2,3,... \tag{2}$$

Now, we have to show that for $(a,s,d) \in Q_r$, (x_{asd}) is λ-triple statistically convergent to ψ. Now, consider that (x_{asd}) is not λ-triple statistically to ψ so that there is a neighbourhood V of 0 such that $x_{asd} - \psi \notin V$ for in finitely many terms. Let $V_r \subset V$ where $r = 1,2,3,...$ and $Q_V = \{(a,s,d) : x_{asd} - \psi \in V\}$. Then, $\delta^3_\lambda(Q_V) = 0$ and by (1), $Q_r \subset Q_V$. Therefore, $\delta^3_\lambda(Q_r) = 0$ which is a contradiction to (2). Hence, (x_{asd}) is λ-triple statistically convergent to ψ.

Sufficiency: Consider that there exists a subset $J = \{(a,s,d) \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}\}$ such that $\delta^3_\lambda(J) = 1$ and $\lim_{a,s,d} x_{asd} = \psi$, i.e. there exists an $r_0 \in \mathbb{N}$ such that for each neighbourhood V of 0, $x_{asd} - \psi \in V$ for every $a,s,d \geq r_0$. Now,

$$\begin{aligned} J_V &= \{(a,s,d) : x_{asd} - \psi \notin V\} \\ &\subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} - \{(a_{r_0+1}, s_{r_0+1}, d_{r_0+1}), (a_{r_0+2}, s_{r_0+2}, d_{r_0+2}), ...\}. \end{aligned}$$

Therefore, $\delta^3_\lambda(J_V) \leq 0$. It follows that x is λ-triple statistically convergent to ψ. □

Corollary 5. *If a triple sequence (x_{asd}) is λ-triple statistically convergent to ψ. Then, there exists a triple sequence (y_{asd}) such that $\lim_{a,s,d} y_{asd} = \psi$ and $\delta^3_\lambda\{(a,s,d) : x_{asd} = y_{asd}\} = 1$, i.e. $x_{asd} = y_{asd}$ for almost all a,s,d.*

Definition 6. In a topological group, triple sequence $x = (x_{asd})$ is called λ-triple statistically Cauchy if for each neighbourhood V of 0 there exists $G = G(V), H = H(V)$ and $Q = Q(V)$ such that for all $a, q \geq G, s, w \geq H$ and $d, e \geq Q$ the set $\{(a, s, d) \in I_{pql} : x_{asd} - x_{qwe} \notin V\}$ has triple natural density zero. In this case, we denote the set of all statistically Cauchy triple sequences by $S^3_\lambda C(E)$.

Theorem 7. *Let E be complete. A triple sequence $x = (x_{asd})$ in E is λ-triple statistically convergent if and only if x is λ-triple statistically Cauchy.*

Proof. Let $x = (x_{asd})$ be λ-triple statistically convergent to ψ. Let V be any neighbourhood of 0. Then, we can choose a symmetric neighbourhood W of 0 such that $W + W \subset V$. Then for this neighbourhood W of 0, the set $\{(a, s, d) \in I_{pql} : x_{asd} - \psi \in W\}$ has triple natural λ-density 0. For each neighbourhood V of 0, the set $\{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin V\}$ has triple natural λ-density zero- Then, we can choose numbers G, H and Q such that $x_{GHQ} - \psi \notin V$. Now, we write $T_V = \{(a, s, d) \in I_{pql} : x_{asd} - x_{GHQ} \notin V\}$, $L_W = \{(a, s, d) \in I_{pql} : x_{asd} - \psi \notin W\}$ and $K_W = \{(N, M, J) \in I_{pql} : x_{GHQ} - \psi \notin W\}$. Then, $T_V \subset L_W \cup K_W$ and hence $\delta^3_\lambda(T_V) \leq \delta^3_\lambda(L_W) + \delta^3_\lambda(K_W) = 0$. Therefore, we obtain that x is λ-triple statistically Cauchy. To prove the converse suppose that there is a λ-triple statistically Cauchy sequence x but it is not λ-triple statistically convergent. Then we can find natural numbers G, H and Q such that the set T_V has triple natural λ-density zero. It follows from this that the set $Z_V = \{(a, s, d) \in I_{pql} : x_{asd} - X - GHQ \in V\}$ has triple natural density 1. Now, we can choose a neighbourhood W of 0 such that $W + W \subset V$. Now, take any fixed non-zero element ψ of E. Let $x_{asd} - X_{GHQ} = x_{asd} - \psi + \psi - X_{GHQ}$. It follows from this equality that $x_{asd} - x_{GHQ} \in V$ if $x_{asd} - \psi \in W$. Since x is not λ-triple statistically convergent to ψ, the set L_W has triple natural density 1, i.e. the set $\{(a, s, d) a \leq q, s \leq w, d \leq e : x_{asd} - \psi \notin W\}$ has triple natural density 0. Hence the set $\{(a, s, d) a \leq q, s \leq w, d \leq e : x_{asd} - x_{GHQ} \in W\}$ has triple natural density 0, i.e. the set T_V has triple natural density 1 which is a contradiction. □

Taking into account Theorems 4 and 7, we can state the following theorem and the proof is following directly by the previous results.

Theorem 8. *If E is complete, then the following conditions are equivalent:*

1. *x is λ-triple statistically convergent to ψ,*

2. x is λ-triple statistically Cauchy,

3. there exists a subsequence y of x such that $\lim_{a,s,d} y_{asd} = \psi$.

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