International Journal of Applied Mathematics

Volume 35 No. 1 2022, 39-48
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v35i1.3

NUMERICAL SOLUTION OF VOLTERRA NONLINEAR INTEGRAL EQUATION BY USING LAPLACE ADOMIAN DECOMPOSITION METHOD

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Abstract: In this paper we introduce a new approach for numerical solution of nonlinear Volterra integral equation of second kinds. This numerical solution is based on the Laplace transform and Adomian Decomposition Method by using He's polynomials. Then comparative study is made on the exact solution, approximate solution and estimated error.

All calculations are made by Matlab 13 Version and illustrated by example.

AMS Subject Classification: 44A30, 45D05, 49M27, 65R10
Key Words: Volterra integral equation; Adomian decomposition method; Laplace transform; He's polynomials

1. Introduction

Nonlinear Volterra integral equations are appearing in many branches of Mathematics, Physics, Chemistry, Biology, Engineering problems, etc. and the Adomian decomposition method and Laplace transformation methods are used, [1, 2, 3, 4, 5]. Nonlinear Volterra integral equation arise in many fields, e.g. as Dirichlet problems, electrostatics, potential theory, astrophysics, reactor theory, diffusion problems and heat transfer problems [6, 7, 8]; numerical analysis, mathematical economics, vector analysis, chemical kinetics, Laplace transformations. One of the important methods used is the Laplace transformation by using Adomian decomposition methods, [9, 10]. Adomian decomposition

Received: May 10, 2021

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methods lead to several modifications in various fields of research and improve the accuracy and expand the original methods. This numerical method basically clarifies how the Laplace transform may be used for approximate solutions and exact solutions of the nonlinear Volterra integral equations in view of the Adomian decomposition methods, [11].

Adomian Decomposition Method:

The Adomian Decomposition Method (ADM) was popularized and advanced by George Adomian. It consists of decay in unknown function $y(x)$ in an infinite number of decomposition series,

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \ldots\ldots .$$

Then we establish the recurrence relation and substitute the relation to get

$$\sum_{n=0}^{\infty} y_n(x) = f(x) + \lambda \int_0^x k(x,t) \left(\sum_{n=0}^{\infty} y_n(x) \right) dt.$$

Or equivalently,

$$y_0(x) + y_1(x) + y_2(x) + \ldots$$
$$= f(x) + \lambda \int_0^x k(x,t) \left[y_0(t) + y(t) + y_2(t) + \ldots \right] dt.$$

Then, comparing the recurrence relation in above equation, we get

$$y_0(x) = f(x),$$
$$y_{n+1} = \lambda \int_0^x k(x,t)\, y_n(t)\, dt.$$

Nonlinear Volterra integral equations of the second kind

Consider a nonlinear Volterra integral equation of the second kind

$$y(x) = f(x) + \int_0^x k(x-t)\, y(t)\, dt. \tag{1}$$

Here $y(t)$ is a nonlinear function of $y(x)$ and $f(x)$ is a known real valued function. We apply the Laplace transform to both sides of (1), and by using the linear property, we have

$$L\left[y\left(x\right)\right]=L\left[f\left(x\right)\right]+L\left[k\left(x-t\right)\right]L[y\left(t\right)]. \tag{2}$$

The approximate solution (1) is given as an infinite series:

$$y\left(x\right)=\sum_{n=0}^{\infty}y_{n}. \tag{3}$$

And the nonlinear term $y(t)$ is as the following decomposition:

$$y\left(t\right)=\sum_{n=0}^{\infty}A_{n}, \tag{4}$$

where $A'_n s$ are Adomian polynomials of $y_{0},\ y_{1},\ y_{2},\ y_{3},...,$ given by the formula

$$A_{n}=\ \frac{d^{n}}{d\lambda^{n}}[N(\sum_{n=0}^{\infty}\lambda^{i}y_{i})]\lambda=0,\quad n\geq0,\quad i=0,\ 1,\ 2,\ 3,\ \ \ldots. \tag{5}$$

Substituting (3) and (4) into (2), we get

$$L[\sum_{n=0}^{\infty}y_{n}(x)]=L[f(x)]+L[k(x-t)]\sum_{n=0}^{\infty}L[A_{n}\left(x\right)].$$

By the Laplace transform by using linearity property, we get

$$\sum_{n=0}^{\infty}L\left[y_{n}\left(x\right)\right]=L\left[f\left(x\right)\right]+\ L[k\left(x-t\right)]\sum_{n=0}^{\infty}L[A_{n}\left(x\right)]. \tag{6}$$

If $y_{0},\ y_{1},\ y_{2},\ y_{3},\ldots$ can be determined by infinite series and we compare on both side in (6), then we have to use in following iteration formula

$$L\left[y_{0}\left(x\right)\right]=L\left[f\left(x\right)\right]. \tag{7}$$

By the general relation (6), we have

$$L\left[y_{n+1}\left(x\right)\right]=L\left[k\left(x-t\right)\right]L\left[A_{n}\left(x\right)\right]. \tag{8}$$

Applying the inverse Laplace transform to the first part of (9) gives y_0 defined as A_0, and using A_0 enables to determine of y_1. Then the evaluation of

A_1 will always lead to determination to y_2 and using in (3), we have the series for Volterra integral equation.

A more effective modified technique for solving Volterra nonlinear integral equation is shown with the help of examples. Then an adequacy technique is proposed to fins the maximum absolute error so given as:

$$e = \max \left| y_{exs} - y_{app} \right|.$$

Let us denote by e be the maximum absolute error in the some given interval.

Example 1. Suppose the following Volterra nonlinear integral equation

$$y(x) = 2 - e^x + \int_0^x e^{x-t} y^2(t)\, dt, \tag{9}$$

has exact solution $y(x) = 1$.

Solution. Applying the Laplace transform to both sides of (9) and using the linearity property of the Laplace transform, we have

$$L[y(x)] = L[2 - e^x] + L[\int_0^x e^{x-t} y^2(t)\, dt]. \tag{10}$$

Then by the method by series solution, the function $y(x)$ is

$$L\left[\sum_{n=0}^{\infty} y_n(x)\right] = \frac{2}{p} - \frac{1}{p-1} + \frac{1}{p} L\left[e^x\right] L[y^2(x)]. \tag{11}$$

The nonlinear term $y^2(x)$ is decomposed and we use the formula (5). If modified Adomian polynomials are used, it follows

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_0y_1, \\ A_2 &= y_1^2 + 2y_0y_2, \\ A_3 &= 2y_1y_2 + 2y_0y_3, \\ A_4 &= y_2^2 + 2y_1y_3 + 2y_0y_4. \end{aligned}$$

Comparing both sides of (11), we get

$$L\left[y_0\right] = \frac{2}{p} - \frac{1}{p-1}.$$

Applying the inverse Laplace transform on both sides, we get

$$y_0 = 2 - e^x.$$

Using again in general relation, and so on, leads to

$$L[y_1] = \left\{ \frac{1}{p-1} - \frac{1}{p} \right\} L[A_0],$$

$$y_1 = L^{-1}[\left\{ \frac{1}{p-1} - \frac{1}{p} \right\} L\left\{y_0^2\right\}],$$

$$y_1 = 7e^x + \frac{1}{2}e^{2x} - 4xe^x - 4x - \frac{15}{2},$$

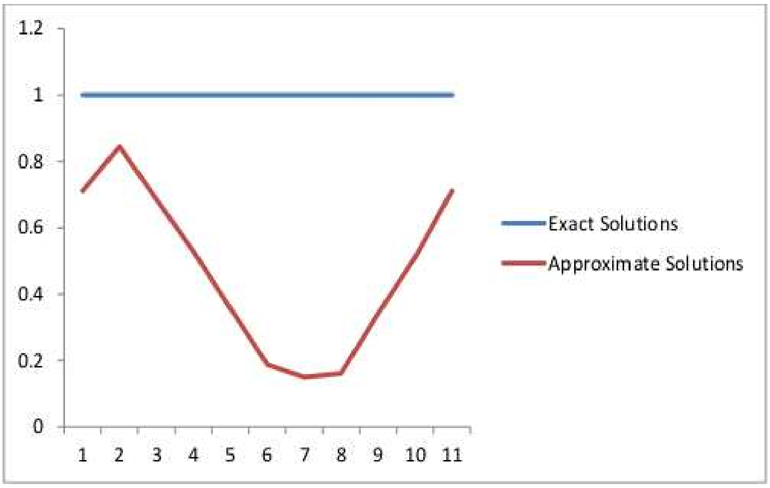
$$y_2 = 2[\frac{55}{3}xe^x - 8e^{2x} - \frac{89}{4}e^x + 7x - 4x^2e^x + 6xe^{2x} - \frac{1}{12}e^{3x} - 8x^2 + \frac{91}{3}].$$

Then the approximate solution becomes

$$y(x) = y_0 + y_1 + y_2 + y_3 + \ldots.$$

$$y(x) = \frac{331}{6} - \frac{77}{2}e^x - \frac{31}{2}e^{2x} + 10x + \frac{98}{3}xe^x - 8x^2e^x + 12xe^{2x} - \frac{1}{6}e^{3x} - 16x^2. \quad (12)$$

Nodes	Exact Solutions	Approximate Solutions	Absolute Error
0.00	1.00000000000	0.7116382600	0.2883617400
0.01	1.00000000000	0.8448550200	0.1551498000
0.02	1.00000000000	0.6860366782	0.3139633218
0.03	1.00000000000	0.5285717684	0.4714282316
0.04	1.00000000000	0.3576339300	0.6423660700
0.05	1.00000000000	0.1880944468	0.8119055532
0.06	1.00000000000	0.1503868026	0.8496131974
0.07	1.00000000000	0.1614988207	0.8385011793
0.08	1.00000000000	0.3414824563	0.6585175437
0.09	1.00000000000	0.5099943600	0.4900056400
0.10	1.00000000000	0.7116382570	0.2883617430



Example 2.

$$y\left(x\right)=2x-\frac{x^{4}}{12}+0.25\int_{0}^{x}\left(x-t\right)y^{2}\left(t\right)dt.$$

Exact solution: $y\left(x\right)=2x$.

Solution. Taking the Laplace transform on both sides and using the modified decomposition method, we get

$$L\left[y\left(x\right)\right]=L\left[2x-\frac{x^{4}}{12}\right]+0.25L\left[x\right]L[y^{2}\left(x\right)].$$

Then the series solution for the function $y(x)$ assumes

$$L\left[\sum_{n=0}^{\infty}y_{n}\left(x\right)\right]=L[2x-\frac{x^{4}}{12}\Bigg]+\frac{1}{4p^{2}}L[\sum_{n=0}^{\infty}A_{n}(x)]. \tag{13}$$

Comparing both sides in iterative form and using Laplace inverse transform, we get:

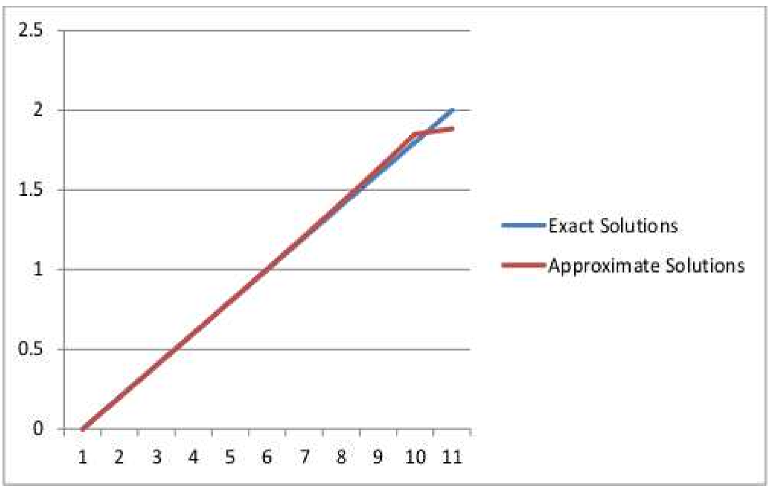
$$y_{0}\left(x\right)=2x-\frac{x^{4}}{12},$$
$$y_{1}\left(x\right)=\frac{x^{4}}{12}-\frac{x^{7}}{126}+\frac{x^{10}}{51840},$$
$$y_{2}\left(x\right)=\frac{x^{7}}{504}-\frac{23x^{10}}{181440}+\frac{127x^{13}}{56609280}-\frac{x^{16}}{298598400},$$

$$y_3(x) = \frac{x^4}{12} - \frac{x^7}{504} + \frac{x^{10}}{2492} - \frac{19x^{13}}{14152320} + \frac{71x^{16}}{2264371200} - \frac{7893x^{19}}{575787643000000}.$$

Then the approximate solution becomes

$$y(x) = y_0 + y_1 + y_2 + y_3 + \ \ldots$$
$$y(x) = 2x + \frac{x^4}{12} - \frac{x^7}{126} - \frac{x^{10}}{362880} + \frac{51x^{13}}{56609280} + \ \ldots.$$

Nodes	Exact Solutions	Approximate Solutions	Absolute Error
0.0	0.0	0.0000000000	0.0000000000
0.1	0.2	0.2000083325	0.0000083325
0.2	0.4	0.4001332317	0.0001332317
0.3	0.6	0.6006732643	0.0006732643
0.4	0.8	0.8021203322	0.0021203322
0.5	1.0	1.0051463480	0.0051463480
0.6	1.2	1.2105779450	0.0105779450
0.7	1.4	1.4193552730	0.0193552730
0.8	1.6	1.6328034650	0.0328034650
0.9	1.8	1.8508857190	0.0508857190
1.0	2.0	1.8833526250	0.1166473750



Example 3.

$$y(x) = x + \int_0^x y^2(t)\,dt.$$

Exact solution: $y(x) = \tan(x)$.

Solution. Taking the Laplace transform on both sides and the using modified decomposition method, we get

$$L[y(x)] = L[x] + L[\int_0^x y^2(t)\,dt].$$

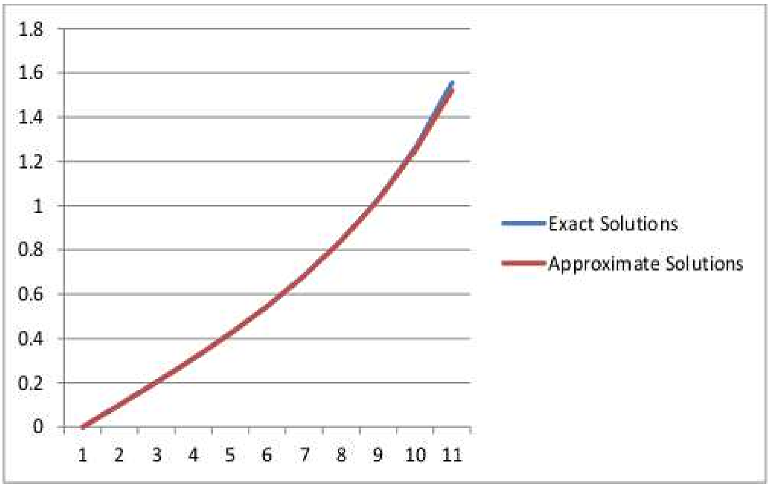
Using (13), we have

$$\begin{aligned} y_0 &= x, \\ y_1 &= \frac{x^3}{3}, \\ y_2 &= \frac{2x^5}{15}, \\ y_3 &= \frac{17x^7}{315}. \end{aligned}$$

Then the approximate solution becomes

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + y_3 + \ \ldots \\ y(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ \ldots . \end{aligned}$$

Nodes	Exact Solutions	Approximate Solutions	Absolute Error
0.0	0.00000000000	0.0000000000	0.0000000000
0.1	0.1002940335	0.1003346721	0.0000406386
0.2	0.2026262629	0.2027100241	0.0000837612
0.3	0.3092040035	0.3093358029	0.0001317994
0.4	0.4226035289	0.4227870883	0.0001835594
0.5	0.5460413117	0.5462549603	0.0002136486
0.6	0.6837824776	0.6838787656	0.0000962880
0.7	0.8418070516	0.8411871844	0.0006198672
0.8	1.0289756740	1.0256752970	0.0033003770
0.9	1.2592215210	1.2475448490	0.0116766720
1.0	1.5560303730	1.5206349210	0.03539545198



Conclusions

The results obtained in this work show some new technique to find numerical solution of Volterra nonlinear integral equation of second kind, by using the Adomian decomposition method and Laplace transform. All results for the exact solutions and approximate solutions are compared and errors estimated.

Acknowledgements

Manuscript communication No (MCN): IU/R & amp; D/2020 - MCN 000843, Office of Research and Development Integral University, Lucknow.

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