International Journal of Applied Mathematics
Volume 34 No. 6 2021, 1187-1203
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v34i6.10

DEGREE PROPERTIES IN HETEROGENEOUS NETWORKS

Diego Ruiz
Department of Mathematics
University of Cauca
Popayán – 190001, COLOMBIA

Abstract: In social networks it is usual to find nodes that tend to establish relationships based on common attributes like gender and age. To represent this behavior, past models consider a partition of the set of nodes in types and introduce an *affinity level*, representing the tendency of a node to connect with other nodes of the same type. The partition of the set of nodes in types give rise to a class of networks called *heterogeneous*. In this work we characterize mathematical expressions for the dynamics and convergence of the probability and complementary cumulative degree distribution functions in a model of heterogeneous networks. We show that the degree distribution of each type of nodes follows a power law characterizing its scaling exponent. Furthermore, using the stability in the sense of Lyapunov of the expected average degree for each type, we propose an approach to detect instants at which the formation of new edges does not follow the mechanisms of the proposed network.

AMS Subject Classification: 05C82; 91D30; 91G30
Key Words: complex networks, social networks, stochastic models

1. Introduction

Understanding the dynamics of topological measures in social networks has increased the importance of determining the way how nodes establish relationships between them [2, 3, 11]. Centrality measures often help us to understand these dynamics, being the degree a measure that allows us to explain real world phenomena represented by social networks (e.g., the spreading of information [13, 10, 18]). In particular, the model given in [2] introduces mechanisms in

Received: July 3, 2021 © 2021 Academic Publications

which new edges are established in proportion to the degree centrality of existing nodes. Although the model helps to understand some real world phenomena, it is known its limitations to describe some social networks [4, 1, 17].

Based on attractive parameters and non-linear functions, some works explain the dynamics of empirical networks in which the edge formation is affected by node attributes [7, 5, 8, 14]. In particular, considering a partition of the set of nodes in two groups, the work in [8] introduces a model to recreate heterogeneous networks in which the probability of establishing new edges is based on a combination of the degree centrality and an affinity level between existing nodes. The affinity allows to new nodes to prefer to connect nodes with common characteristics at a higher rate than between dissimilar ones [12]. The preference for establishing new edges is called *homophily*. Based on [8], the work in [14] characterizes the dynamics and convergence of three homophily measures at node, group, and network level. Moreover, the work also characterizes the convergence of network modularity and show that the formation of community structures can be expressed as a function of network homophily.

Using a discrete analysis of the heterogeneous network model given in [8, 14], in this work we derive expressions for the dynamics and the limit values of the probability degree distribution function (pdf) and the complementary cumulative degree distribution function (ccdf) for nodes of each group. In particular, using Stirling's formula, we show that the asymptotic value of the ccdf follows a power law and establish the scaling exponent of the distribution for large degrees. Using similar arguments as in [16, 15], we also characterize the expected average degree for any proportion of groups as an invariant set and show, for equal proportion of groups, that the set is stable in the sense of Lyapunov, which is the key for detecting anomalies in the formation of edges between nodes in the network. The anomalous event detection we present in this work is an extension of the approach given in [15].

The reminder of this paper is organized as follows. Section 2 describes the model and shows that the expected sum of the degrees of the nodes of any group follows a linear function. Using a discrete analysis, Section 3 characterizes the dynamics and the limit value of the pdf and the ccdf of nodes of each group and for the entire network. Furthermore, it also establishes the stability of the expected average degree for any proportion of groups of nodes and uses the stability for detecting instants at which anomalies in the edge formation occur in the model. Section 4 presents simulations that illustrate the theoretical results given in Section 3. Finally, Section 5 draws some conclusions and future research directions.

2. The model

In this section we characterize some properties of the network model given in [8]. At any discrete time $t \geq 0$, consider a simple undirected network $G(t)$ with set of nodes $V(t)$, partitioned in two groups, and set of edges $E(t)$. Let $n(t)$ denote the cardinality of $V(t)$. We will refer to a node of type i as a node belonging to the group i, for $i = 1, 2$. Let $V_i(t)$ represent the set of nodes of type i at t. We denote by $k_u(t)$ the degree of a node $u \in V(t)$, and by π_j^i the affinity between nodes of types j and i defined by

$$\pi_j^i = \begin{cases} q & \text{if } i = j, \\ 1-q & \text{otherwise,} \end{cases}$$

with $0 \leq q \leq 1$. We write p_i for the proportion of nodes of type i. Note that at any time $t \geq 0$, the expected number of nodes of type i is $p_i n(t)$. The following mechanisms define the evolution of $G(t)$:

(M1) *New nodes*: A new node w of type i attaches to the network with m undirected edges.

(M2) *New edges*: The probability of selecting a node $u \in V(t-1)$ of type j to establish a new edge with the node w is given by

$$\Pi_u^{ij} = \frac{\pi_j^i k_u(t-1)}{\sum_{v \in V(t-1)} \pi_{\tau(v)}^i k_v(t-1)}, \tag{1}$$

where $\tau(v)$ denotes the type of v. Note that mechanism M1 implies that the network grows by the addition of new nodes and new edges. Note also that mechanism M2 implies that a new node tends to establish new edges with nodes of its same type and a high degree.

From eq. (1), let $s_i(t) = \sum_{v \in V_i(t)} k_v(t)$ denote the expected sum of the degrees of the nodes of type i at time t. Moreover, let -i denote the type different than type i (i.e., -$i = 1$ if and only if $i = 2$). Based on [14], we know

$$\begin{aligned} s_i(t) &= s_i(t-1) + mp_i + \frac{mp_i q s_i(t-1)}{q s_i(t-1) + (1-q) s_{\text{-}i}(t-1)} \\ &\quad + \frac{m(1-p_i)(1-q) s_i(t-1)}{q s_{\text{-}i}(t-1) + (1-q) s_i(t-1)} \\ &= s_i(t-1) + mp_i + \frac{mp_i q}{2q - 1 + (1-q)\frac{d(0)+2m(t-1)}{s_i(t-1)}} \end{aligned}$$

$$+\frac{m(1-p_i)(1-q)}{1-2q+q\frac{d(0)+2m(t-1)}{s_i(t-1)}}, \tag{2}$$

where $d(0)$ denotes the sum of the degrees of the nodes in $V(0)$. For $q = 1$, we know by eq. (2) that $s_i(t) = p_i d(0) + 2mp_i t$. We next show that for $q \neq 1$, the expression $s_i(t)$ behaves like a linear function for a large t. Indeed, if $\lim_{t\to\infty} \frac{t}{s_i(t)} = 0$, then in eq. (2) we would have

$$s_i(t) = s_i(t-1) + mp_i + \frac{mp_i q}{2q-1} + \frac{m(1-p_i)(1-q)}{1-2q},$$

implying that $s_i(t)$ is a linear function in terms of t, which is not possible. On the other hand, if $\lim_{t\to\infty} \frac{t}{s_i(t)} = \infty$, then in eq. (2) we would have $s_i(t) = s_i(t-1) + mp_i$, i.e., $s_i(t)$ would be a linear function in terms of t, and again this is not possible. Thus

$$s_i(t) \sim \beta_i t \tag{3}$$

for a non-zero real number β_i. According to the work in [9], we know that β_i is a root of the third degree equation

$$r(x) = x - \frac{mp_i q}{(2q-1)x + 2m(1-q)}x - \frac{m(1-p_i)(1-q)}{(1-2q)x + 2mq}x - mp_i.$$

Note that β_i represents the expected value of the slope of $s_i(t)$, that is, $s_i(t)$ can be written as $s_i(t) = p_i d_0 + \beta_i t$ and so

$$\begin{aligned}\sigma_i(t) := \sum_{v\in V(t)} \pi^i_{\tau(v)} k_v(t) &= q s_i(t) + (1-q)s_{\text{-}i}(t)\\ &= (2q-1)s_i(t) + (1-q)(d(0) + 2mt).\end{aligned}$$

3. Degree properties

3.1. Dynamics of the pdf and the ccdf

Using a discrete analysis, this section characterizes the dynamics and limit value of the probability degree distribution function (pdf) and of the complementary cumulative degree distribution function (ccdf) for each of both groups of nodes. It also provides the limit value of the ccdf for the entire network. Consider the following notation. Let $P_{j,k}(t)$ denote the probability that the degree of a node of type j selected uniformly at random at time t is k, and put

$$\delta_{j,k} = \frac{mkp_j q}{q\beta_j + (1-q)\beta_{\text{-}j}} + \frac{mkp_{\text{-}j}(1-q)}{q\beta_{\text{-}j} + (1-q)\beta_j} \tag{4}$$

for all $k \geq m$.

Theorem 1 (Pdf for nodes of type j)**.** *The limit value of the probability degree distribution for nodes of type j is given by*

$$\lim_{t\to\infty} P_{j,k}(t) = \frac{m^2\Gamma(k)\Gamma\left(\frac{m}{\delta_{j,m}} + m\right)}{\delta_{j,m}\Gamma\left(k+1+\frac{m}{\delta_{j,m}}\right)\Gamma(m+1)},$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proof. Let $n_j(t)$ denote the number of nodes of type j at time t. For a large t we know that $P_{j,k}(t) = P_{j,k}(t-1)$ for all $k \geq m$. In particular, note that

$$P_{j,k}(t) = \frac{|V_{j,k}(t)|}{n_j(t)},$$

where $|V_{j,k}(t)|$ denotes the number of nodes of type j and degree k at time t. Since the number of new edges established by the new node is m, the number of nodes of type j and degree k at time t is given by $|V_{j,k}(t-1)| + a$ for some $a \in \pm\{0, 1, \ldots, m\}$, and so

$$P_{j,k}(t) = \frac{|V_{j,k}(t-1)| + a}{n_j(t)} = \frac{|V_{j,k}(t-1)|}{n_j(t-1)\frac{n_j(t)}{n_j(t-1)}} + \frac{a}{n_j(t)}.$$

Moreover, since $\lim_{t\to\infty} \frac{n_j(t)}{n_j(t-1)} = 1$ and $\lim_{t\to\infty} \frac{a}{n_j(t)} = 0$, we have $P_{j,k}(t) = P_{j,k}(t-1)$ for a large t.

Now, note that the probability that the new node connects to a node of type j and degree k at time t is given by

$$\mathbb{P}_{j,k}(t) = \frac{mkp_j q}{\sigma_j(t-1)} + \frac{mkp_{\text{-}j}(1-q)}{\sigma_{\text{-}j}(t-1)}. \tag{5}$$

The first and second terms in eq. (5) represent the probability that the new node of type j and type -j, respectively, connects to a node with degree k and type j. We use eq. (5) to characterize the expected number of nodes of type j and degree $k \geq m$.

Observe that the expected number of nodes of type j with degree $k = m$ at time t is

$$n_j(t)P_{j,m}(t) = n_j(t-1)P_{j,m}(t-1) - n_j(t-1)P_{j,m}(t-1)\mathbb{P}_{j,m}(t) + p_j. \quad (6)$$

Because $n_j(t) = p_j n(t)$ for a large enough t, we have

$$P_{j,m}(t) = \frac{1}{1 + n(t-1)\mathbb{P}_{j,m}(t)}.$$

Since $\lim_{t\to\infty} n(t-1)\mathbb{P}_{j,m}(t)$ exists, it follows that $\lim_{t\to\infty} P_{j,m}(t)$ also exists and according to eq. (4) we get

$$\lim_{t\to\infty} P_{j,m}(t) = \frac{1}{1 + \delta_{j,m}}.$$

Now, the expected number of nodes with degree $k > m$ and type j is given by

$$\begin{aligned} p_j n(t)P_{j,k}(t) &= p_j n(t-1)P_{j,k}(t-1) - p_j n(t-1)P_{j,k}(t-1)\mathbb{P}_{j,k}(t) \\ &\quad + p_j n(t-1)P_{j,k-1}(t-1)\mathbb{P}_{j,k-1}(t). \end{aligned} \quad (7)$$

Using mathematical induction and the fact that $\lim_{t\to\infty} P_{j,m}(t)$ exists, we can show that $\lim_{t\to\infty} P_{j,k}(t)$ exists for all k and then

$$\begin{aligned} \lim_{t\to\infty} P_{j,k}(t) &= \frac{\lim_{t\to\infty} n(t-1)\mathbb{P}_{j,k-1}(t)}{1 + \lim_{t\to\infty} n(t-1)\mathbb{P}_{j,k}(t)} \lim_{t\to\infty} P_{j,k-1}(t) \\ &= \frac{\delta_{j,k-1}}{1 + \delta_{j,k}} \lim_{t\to\infty} P_{j,k-1}(t). \end{aligned}$$

From the above we get

$$\lim_{t\to\infty} P_{j,k}(t) = \begin{cases} \dfrac{\delta_{j,k-1}}{1 + \delta_{j,k}} \lim_{t\to\infty} P_{j,k-1}(t) & \text{if } k > m, \\ \dfrac{1}{1 + \delta_{j,m}} & \text{if } k = m. \end{cases} \quad (8)$$

Solving the recurrence given in eq. (8), we have

$$\lim_{t\to\infty} P_{j,k}(t) = \frac{m^2 \Gamma(k)\Gamma\left(\frac{m}{\delta_{j,m}} + m\right)}{\delta_{j,m}\Gamma\left(k + 1 + \frac{m}{\delta_{j,m}}\right)\Gamma(m+1)}.$$

□

According to eqs. (6) and (7), it is possible to express the dynamics of the probability degree distribution for nodes of type j as an infinite dimensional time-varying affine linear model as in [6]. In particular, eq. (6) characterizes the dynamics of $P_{j,m}(t)$. Furthermore, from eq. (7), for $k > m$ we know that

$$(P_{j,m+1}(t), P_{j,m+2}(t), \dots)^\top = A(t)(P_{j,m}(t), P_{j,m+1}(t), \dots)^\top$$

represents the dynamics of the degree distribution, and for $i, \ell \geq 1$, the element in the i-th row and ℓ-th column of the matrix $A(t)$ is given by

$$A(t)_{i\ell} = \frac{n(t-1)}{n(t)} \begin{cases} \mathbb{P}_{j,m+i-1}(t) & \ell = i, \\ 1 - \mathbb{P}_{j,m+i}(t) & \ell = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for all t we have

$$\begin{pmatrix} P_{j,m+1}(t) \\ P_{j,m+2}(t) \\ \vdots \end{pmatrix} = \frac{n(t-1)}{n(t)} \begin{pmatrix} \mathbb{P}_{j,m}(t) & 1-\mathbb{P}_{j,m+1}(t) & 0 & 0 & 0 & \cdots \\ 0 & \mathbb{P}_{j,m+1}(t) & 1-\mathbb{P}_{j,m+2}(t) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} P_{j,m}(t-1) \\ P_{j,m+1}(t-1) \\ P_{j,m+2}(t-1) \\ P_{j,m+3}(t-1) \\ \vdots \end{pmatrix}.$$

Theorem 2 (Ccdf for nodes of type j)**.** *The limit value of the complementary cumulative degree distribution for nodes of type j is given by*

$$\lim_{t\to\infty} F_{j,k}(t) = \frac{\Gamma(k)\Gamma\left(\frac{m}{\delta_{j,m}} + m + 1\right)}{\Gamma\left(k + 1 + \frac{m}{\delta_{j,m}}\right)\Gamma(m)},$$

where $F_{j,k}(t)$ denotes the probability that a randomly selected node of type j has a probability greater than or equal to k at time t.

Proof. Because $k_u(t) \geq m$ for all $t \geq 0$ and all $u \in V(t)$, we note that $F_{j,k}(t) = 1$ for all $k \leq m$. Note also that the expected number of nodes of type j with degree $k > m$ is given by

$$p_j n(t) F_{j,k}(t) = p_j n(t-1) F_{j,k}(t-1) + p_j n(t-1) \mathbb{P}_{j,k-1}(t) P_{j,k-1}(t-1).$$

Since $P_{j,k-1}(t-1) = F_{j,k-1}(t-1) - F_{j,k}(t-1)$, we obtain

$$p_j n(t) F_{j,k}(t) = p_j n(t-1) F_{j,k}(t-1) \\ + p_j n(t-1)\mathbb{P}_{j,k-1}(t)(F_{j,k-1}(t-1) - F_{j,k}(t-1)). \tag{9}$$

For a large t we know that $F_{j,k}(t) = F_{j,k}(t-1)$. Since $F_{j,m}(t) = 1$, using mathematical induction we can show that $\lim_{t\to\infty} F_{j,k}(t)$ exists for all k and

$$\lim_{t\to\infty} F_{j,k}(t) = \frac{\lim_{t\to\infty} n(t-1)\mathbb{P}_{j,k-1}(t)}{1 + \lim_{t\to\infty} n(t-1)\mathbb{P}_{j,k}(t)} \lim_{t\to\infty} F_{j,k-1}(t) \\ = \frac{\delta_{j,k-1}}{1+\delta_{j,k}} \lim_{t\to\infty} F_{j,k-1}(t).$$

In summary

$$\lim_{t\to\infty} F_{j,k}(t) = \begin{cases} \dfrac{\delta_{j,k-1}}{1+\delta_{j,k}} \lim_{t\to\infty} F_{j,k-1}(t) & \text{if } k > m, \\ 1 & \text{if } k = m. \end{cases} \tag{10}$$

From eq. (10), it follows that

$$\lim_{t\to\infty} F_{j,k}(t) = \frac{\Gamma(k)\Gamma\left(\frac{m}{\delta_{j,m}} + m + 1\right)}{\Gamma\left(k + 1 + \frac{m}{\delta_{j,m}}\right)\Gamma(m)}.$$

□

Using Stirling's formula, for a large enough k we know that the complementary cumulative degree distribution of nodes of type j follows a power law, that is

$$F_{j,k}(\infty) \sim k^{-\left(1+\frac{m}{\delta_{j,m}}\right)}.$$

Now, based on eq. (9), it is possible to express the dynamics of the complementary cumulative degree distribution for nodes of type j as an infinite dimensional time-varying affine linear model as follows

$$(F_{j,2}(t), F_{j,3}(t), \ldots)^\top = B(t)(F_{j,1}(t), F_{j,2}(t), \ldots)^\top,$$

where for $i, \ell \geq 0$, the element in the i-th row and ℓ-th column of the matrix $B(t)$ is given by

$$B(t)_{i\ell} = \frac{n(t-1)}{n(t)} \begin{cases} \mathbb{P}_{j,i}(t) & \ell = i, \\ 1 - \mathbb{P}_{j,i}(t) & \ell = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all t we have

$$\begin{pmatrix} F_{j,2}(t) \\ F_{j,3}(t) \\ \vdots \end{pmatrix} = \frac{n(t-1)}{n(t)} \begin{pmatrix} \mathbb{P}_{j,1}(t) & 1-\mathbb{P}_{j,1}(t) & 0 & 0 & 0 & \cdots \\ 0 & \mathbb{P}_{j,2}(t) & 1-\mathbb{P}_{j,2}(t) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} F_{j,1}(t-1) \\ F_{j,2}(t-1) \\ F_{j,3}(t-1) \\ \vdots \end{pmatrix}.$$

Corollary 3 (Ccdf for the entire network)**.** *If $k \geq m$, then the limit value of the expected complementary cumulative degree distribution is*

$$\lim_{t\to\infty} F_k(t) = \sum_{j=1}^{2} p_j \lim_{t\to\infty} F_{j,k}(t).$$

Proof. A finite mixture of the complementary cumulative degree distributions given in Theorem 2 provides the value for $\lim_{t\to\infty} F_k(t)$. □

3.2. Stability of the expected average degree and anomalous event detection

This section characterizes the expected average degree as an invariant stable set for each type of nodes and establishes conditions to detect instants at which the deletion of edges in the proposed model take place. Note that the average degree of nodes of type j at any time t is given by

$$\bar{K}_j(t) = \frac{\sum_{v\in V_j(t)} k_v(t)}{n_j(t)}.$$

Based on eq. (3), we know that the expected value of $\bar{K}_j(t)$ for a large t satisfies

$$\begin{aligned} \mathrm{E}[\bar{K}_j(t)] &= \mathrm{E}\left[\frac{\sum_{v\in V_j(t)} k_v(t)}{n_j(t)}\right] \\ &= \frac{p_j d_0 + \beta_j t}{n_j(t)}. \end{aligned} \tag{11}$$

Because $n(t) = n(0) + t$, from eq. (11) we have

$$\lim_{t\to\infty} \mathrm{E}[\bar{K}_j(t)] = \frac{\beta_j}{p_j}.$$

Now, note that the expected value of $\bar{K}_j(t+1)$ for a large t satisfies

$$\mathrm{E}[\bar{K}_j(t+1)] = \frac{n_j(t)\mathrm{E}[\bar{K}_j(t)] + \beta_j}{n_j(t+1)}.$$

In particular, if $\mathrm{E}[\bar{K}_j(t)] = \frac{\beta_j}{p_j}$, then we obtain that

$$\begin{aligned}\mathrm{E}[\bar{K}_j(t+1)] &= \frac{p_j n(t)\frac{\beta_j}{p_j} + \beta_j}{p_j n(t+1)} \\ &= \frac{\beta_j}{p_j},\end{aligned}$$

where we have used that $n_j(t) = p_j n(t)$. From the above, it follows that the limit value of the expected average degree for nodes of type j is invariant. Now, for $p_j = 0.5$, note that the expected value of $\bar{K}_j(t)$ for all t is given by

$$\begin{aligned}\mathrm{E}[\bar{K}_j(t)] &= \frac{p_j d_0 + mt}{p_j n(t)} \\ &= \frac{d_0 + 2mt}{n(t)}.\end{aligned}$$

Using a similar argument as in [16, 15], it can be proved that for $p_j = 0.5$, the limit value of $\mathrm{E}[\bar{K}_j(t)]$, denoted by $x^e = \mathrm{E}[\bar{K}_j(\infty)]$, is stable in the sense of Lyapunov.

Theorem 4. *If $p_j = 0.5$, then the invariant set $\{x^e\}$ is asymptotically stable in the sense of Lyapunov.*

Proof. Let $x(t) = \mathrm{E}[\bar{K}_j(t)]$ represent the state of the model at time t. Let $\mathcal{V} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, defined by $\mathcal{V}(x(t)) = |x(t) - x^e|$, be a Lyapunov candidate function. To verify the theorem it is sufficient to show that $\mathcal{V}$ is non-increasing over time and that $\mathcal{V}(x(t)) \to 0$ as $t \to \infty$. Using eq. (11) we deduce that

$$\begin{aligned}\mathcal{V}(x(t+1)) - \mathcal{V}(x(t)) &= |x(t+1) - x^e| - |x(t) - x^e| \\ &= \left|\frac{d_0 + 2m(t+1)}{n(0)+t+1} - 2m\right| - \left|\frac{d_0 + 2mt}{n(0)+t} - 2m\right| \\ &= \left|\frac{d_0 - 2mn(0)}{n(0)+t+1}\right| - \left|\frac{d_0 - 2mn(0)}{n(0)+t}\right| < 0,\end{aligned}$$

which implies that $\mathcal{V}$ is a non-increasing function. By the definition of $x(t)$, we know that $x(t) \to x^e$ as $t \to \infty$, that is, $\lim_{t\to\infty} \mathcal{V}(x(t)) = 0$. Thus, the invariant set $\{x^e\}$ is asymptotically stable in the sense of Lyapunov. □

Now, using the stability properties of the expected average degree, we extend the approach given in [15] to detect instants at which the formation of edges does not follow the mechanisms of the proposed model. Let $M_j(t)$ denote the number of new edges established to nodes of type j at time t and write $w_j(t) = M_j(t) - \mathrm{E}[M_j(t)]$. Note that $w_j(t)$ represents the difference between the number of new edges established to nodes of type j and its expected value at time t. Because the number of new edges is m, we know that there exist non-negative constants k_1 and k_2 such that $k_1 \leq M_j(t) \leq k_2$, giving that

$$k_1 - \mathrm{E}[M_j(t)] \leq \omega_j(t) \leq k_2 - \mathrm{E}[M_j(t)].$$

Let $\gamma_j = \max\{k_2 - \mathrm{E}[M_j(t)], \mathrm{E}[M_j(t)] - k_1\}$, and define

$$\alpha_j(t) = \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] + \frac{2\gamma_j}{n(t)},$$
$$\beta_j(t) = \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] - \frac{2\gamma_j}{n(t)}.$$

At each instant of time t, consider the closed set

$$D_j(t) = \begin{cases} [\beta_j(t), \alpha_j(t)] & \text{if } \bar{K}(0) \leq \mathrm{E}[\bar{K}_j(\infty)], \\ -[\alpha_j(t), \beta_j(t)] & \text{otherwise.} \end{cases}$$

Now, let

$$f(w_j(t)) = \left| \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] - \frac{2w_j(t)}{n(t)} \right|. \tag{12}$$

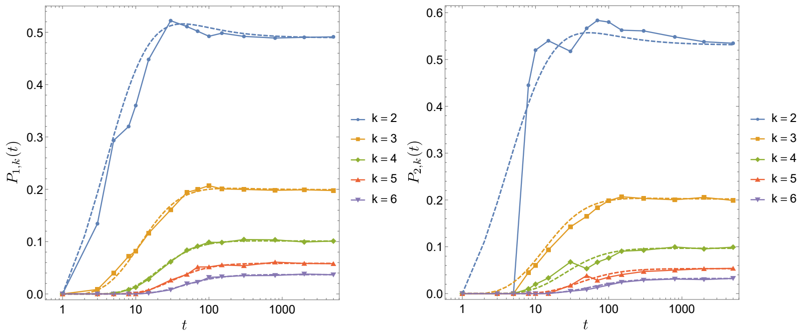
By using Theorem 6 in [15], we can show that if $|w_j(t)| \leq \gamma_j$, then $f(w_j(t)) \in D_j(t)$. In particular, if we assume that $\bar{K}(0) \leq \mathrm{E}[\bar{K}_j(\infty)]$, then we obtain that $D_j(t) = [\beta_j(t), \alpha_j(t)]$. Since

$$\mathrm{E}[\bar{K}_j(t)] = \frac{d_0 + 2mt}{n(t)} = \frac{\bar{K}(0)n(0) + 2mt}{n(t)}$$

we have that $f(w_j(t)) \in D_j(t)$ if and only if $\beta_j(t) \leq f(w_j(t)) \leq \alpha_j(t)$, that is, if and only if

$$\begin{aligned} \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] - \tfrac{2\gamma_j}{n(t)} &\leq \left| \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] - \tfrac{2w_j(t)}{n(t)} \right| \\ &\leq \mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] + \tfrac{2\gamma_j}{n(t)}. \end{aligned}$$

Note that the left-hand side of the above inequality is true if and only if $w_j(t) \leq \gamma_j$ or $\mathrm{E}[\bar{K}_j(\infty)] - \mathrm{E}[\bar{K}_j(t)] - \frac{w_j(t)}{n(t)} \leq \frac{\gamma_j}{n(t)}$, which clearly holds. Now, the right-hand side is true if and only if $w_j(t) \geq -\gamma_j$ and $n(0)(2m - \bar{K}(0)) + (\gamma_j - w_j(t)) \geq$



(a) (b)

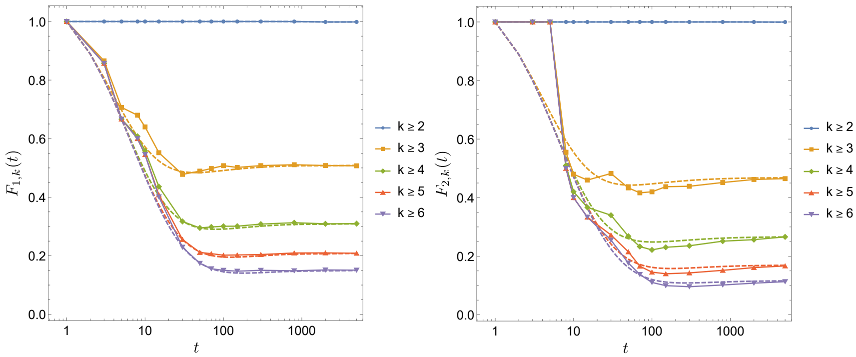
Figure 1: Dynamics of the pdf for nodes of (a) type 1; and (b) type 2, with affinity level $q = 0.8$, and $p_1 = 0.8$. Dashed curves represent the theoretical predictions obtained from Theorem 1.

0, which also holds. So our approach establishes that an anomaly is detected at time t, if for some $j \in \{1, 2\}$, the image of $w_j(t)$ under the function f does not belong to the set $D_j(t)$, that is, if $f(w_j(t)) \notin D_j(t)$.

4. Simulations

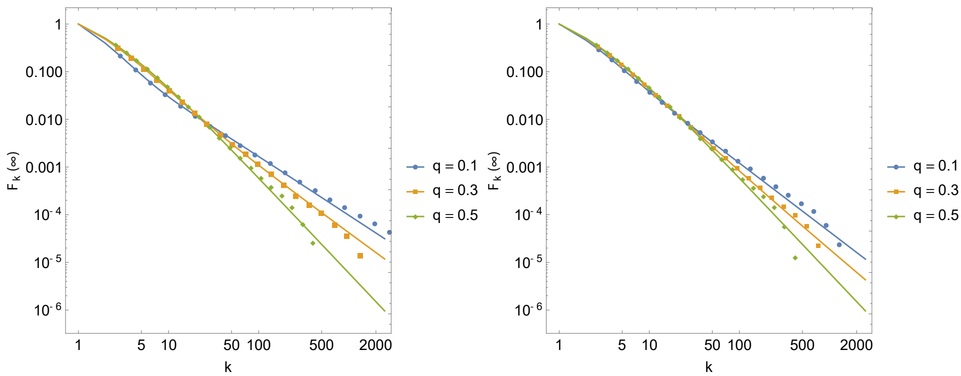
4.1. Degree properties

In this section we illustrate the dynamics of the pdf and ccdf given in Section 3.1. Let $m = 2$. Consider an initial network with $n(0) = 5$ and two types of nodes for which $p_1 = 0.8$. For degrees $k \in \{2, 3, 4, 5, 6\}$, Figures 1 and 2 illustrate the dynamics of the pdf and the ccdf for $q = 0.8$. Note that the predictions of the dynamics given in Theorems 1 and 2 are a better fit for a large values of t than for small values. Figure 3, which illustrates the dynamics of the ccdf for the entire network with affinities $q \in \{0.1, 0.3, 0.5\}$ and proportions $p_1 = 0.8$ (a) and $p_1 = 0.7$ (b), shows that the shape of the degree distribution not only depends on q, but also on the proportions of groups. According to Theorem 3, the complementary cumulative degree distribution of the entire network is a weighted sum of two power laws. For large enough k, note that the tail of the distribution follows a power law with the scaling exponent equals to the smaller of the both power laws, as we illustrate it in Figure 4 for proportions $p_1 = 0.8$ with affinity $q = 0.1$ (a), and for $p_1 = 0.7$ with affinity $q = 0.7$ (b).



(a) (b)

Figure 2: Dynamics of the ccdf for nodes of (a) type 1; and (b) type 2, with affinity level $q = 0.8$, and $p_1 = 0.8$. Dashed curves represent the theoretical predictions from Theorem 2.

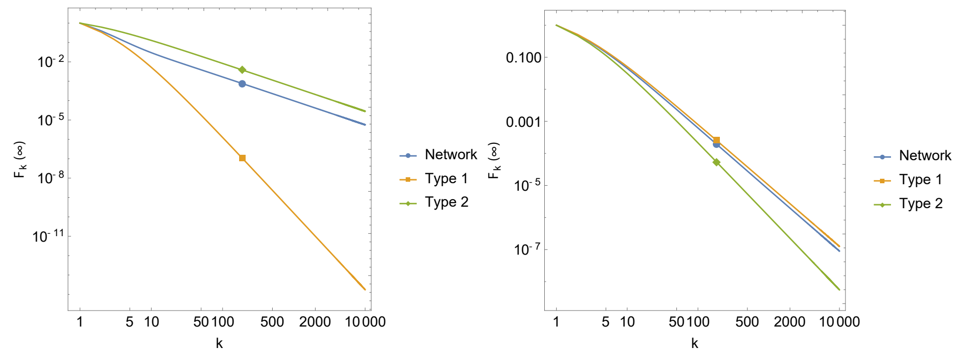


(a) (b)

Figure 3: Complementary cumulative degree distribution of the network for affinities $q \in \{0.1, 0.3, 0.5\}$ and proportions (a) $p_1 = 0.8$; and (b) $p_1 = 0.7$. Solid curves represent the theoretical expressions given in Corollary 3 and the dotted curves the simulated distributions.

4.2. Anomalous event detection

Here we apply the approach given in Section 3.2 to detect instants at which anomalous events take place in the proposed model. Consider a network evo-



(a) (b)

Figure 4: Theoretical complementary cumulative degree distributions (a) for $p_1 = 0.8$ with $q = 0.1$; and (b) for $p_1 = 0.7$ with $q = 0.7$.

lution with $m = 2$ and two types of nodes equally proportioned, i.e., $p_1 = 0.5$. Assume that at the instants of time $t \in \{100, 700, 3000\}$, a total of 8, 10, and 3 edges are removed of the network, respectively. Figure 5(a) shows the image of the average degree by type under the Lyapunov function $\mathcal{V}$. Because it is not possible to identify how the deletion of edges affects the Lyapunov function, we use the function f given in eq. (12) to map the average degree inside two fringes around the image of the expected average degree under $\mathcal{V}$. We note in Figure 5(b)-(c) that our approach detects the instants at which anomalous events occur in the the homophily model. In particular, the behavior of the transformation of the average degree of nodes of type 1 helps us to detect two instants at which the anomalies take place, and the transformation of the average degree of nodes of type 2 allows us to detect the three instants at which we remove existing edges in the network.

5. Conclusions

In this work we consider a heterogeneous network model in which the set of nodes is partitioned into two types of nodes. We characterize the degree distribution for each group of nodes and show that it follows a power law, providing the scaling exponent for each group. By using a mixture of the distributions for nodes of the same type, we also determine the degree distribution for the entire network. Our simulations suggest that the tail of the complementary cumulative degree distribution of the entire network follows a power law with a

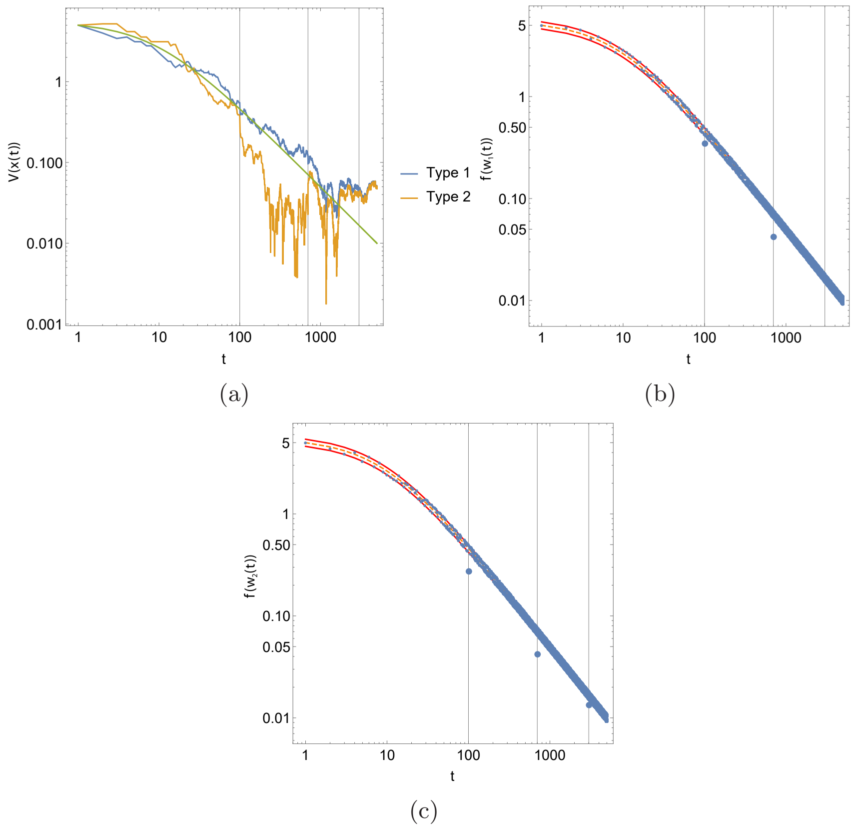


Figure 5: (a) Image of the average degree by type of nodes under the Lyapunov function $\mathcal{V}$ given in Theorem 4; and Image of the average degree under the function f given eq. (12) for nodes of (b) type 1; and (c) type 2.

scaling exponent equal to the smaller exponent of the degree distributions for the groups. Moreover, based on the stability properties of the expected average degree, we apply an approach to detect instants at which the deletion of edges occurs in the model. Establishing the stability of the expected average degree for other proportions of groups, remains as a future research direction.

Acknowledgments

This paper was supported by the Universidad of Cauca under research project VRI-ID 5365.

References

[1] A. Abbasi, L. Hossain, L. Leydesdorff, Betweenness centrality as a driver of preferential attachment in the evolution of research collaboration networks, *Journal of Informetrics*, **6**, No 3 (2012), 403-412.

[2] A. Barabási, R. Albert, Emergence of scaling in random networks, *Science*, **286**, No 5439 (1999), 509-512.

[3] G. Bianconi, A. Barabási, Competition and multiscaling in evolving networks, *Europhysics Letters*, **54**, No 3 (2001), 436-442.

[4] R. Burt, Attachment, decay, and social network, *Journal of Organizational Behavior*, **22**, No 6 (2001), 619-643.

[5] G. Caldarelli, A. Capocci, P. De Los Rios, M. Muñoz, Scale-free networks from varying vertex intrinsic fitness, *Physical Review Letters*, **89**, No 25 (2002), Art. 258702.

[6] I. Fernandez, K. Passino, J. Finke, Dynamics of degree distributions of social network, In: *Proceeding of the Conference on Decision and Control*, Melbourne, Australia, (2017), 0743-1546.

[7] S. Fortunato, A. Flammini, F. Menczer, Scale-free network growth by ranking, *Physical Review Letters*, **96**, No 21 (2006), Art. 218701.

[8] F. Karimi, M. Génois, C. Wagner, P. Singer, M. Strohmaier, Homophily influences ranking of minorities in social networks, *Scientific Reports*, **8**, No 11077 (2018), 1-12.

[9] E. Lee, F. Karimi, C. Wagner, H. H. Jo, M. Strohmaier, M. Galesic, Homophily and minority-group size explain perception biases in social networks, *Nature Human Behavior*, **3**, No 10 (2019), 1078-1087.

[10] Y. Liu, B. Wei, Y. Du, F. Xiao, Y. Deng, Identifying influential spreaders by weight degree centrality in complex networks, *Chaos, Solitons & Fractals*, **86** (2016), 1-7.

[11] A. Mantzaris, D. J. Highman, A model for dynamic communicators, *European Journal of Applied Mathematics*, **23**, No 6 (2012), 659668

[12] M. McPherson, L. Smith-Lovin, J. M. Cook, Birds of a feather: homophily in social networks, *Annual Review of Sociology*, **27**, No 1 (2001), 415-444.

[13] R. Pastor-Satorras, A. Vespignani, Epidemic spreading in scale-free networks, *Physical Review Letters*, **86**, No 14 (2001), 3200-3203.

[14] D. Ruiz, J. Campos, J. Finke, Dynamics in affinity-weighted preferential attachment networks, *Journal of Statistical Physics*, **181**, No 2 (2020), 673-689.

[15] D. Ruiz, J. Finke, Lyapunov-based anomaly detection in highly-clustered networks, *Journal of Statistical Physics*, **172**, No 4 (2018), 1127-1146.

[16] D. Ruiz, J. Finke, Lyapunov-based anomaly detection in preferential attachment networks, *International Journal of Applied Mathematics and Computer Science*, **29**, No 2 (2019), 1-11.

[17] M. Sun, Q. Liu, Analysis of dynamical behaviors for a computer virus spreading model, *International Journal of Applied Mathematics*, **28**, No 3 (2015), 237-252; doi: 10.12732/ijam.v28i3.5.

[18] Q. Wu, S. Chen, Mean field theory of epidemic spreading with effective contacts on networks, *Chaos, Solitons & Fractals*, **81** (2015), 359-364.